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Research Article

Solution of Fractional Telegraph Equations by Conformable Double Convolution Laplace Transform

Waleed M. Osman[1], Tarig M. Elzaki*[2] and Nagat A. A. Siddig[1]

[1] *Department of Mathematics, College of Sciences and Arts, King Kalied University, Dhahran Aljonoub, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Sciences and Arts, University of Jeddah, Alkamil, Jeddah, Kingdom of Saudi Arabia*
***Corresponding author:** tarig.alzaki@gmail.com

Abstract. This paper covers both conformable double Laplace transform and conformable double convolution, including their definitions, theorems and properties. The purpose of this research is to solve a fresh case of fractional telegraph equations with conformable double convolution by conformable double Laplace transform.

Keywords. Conformable double Laplace transform; Fractional telegraph equation; Conformable double convolution

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1. Introduction

Subject part of double conformable Laplace transforms and its properties are still new-fangled [16], some properties and definitions of fractional calculus were considered in [1, 2, 13], some fractional partial differential equations were also solved [14, 15], many researchers have given their attention to examining the solutions of fractional linear and nonlinear partial differential equations using different methods. Along with these approaches are the variational iteration method [18], Homotopy perturbation method [7], Laplace transforms [18], double

Laplace transforms [4–6, 9, 16] and the non-homogeneous equations with constant coefficients illuminated by implies of the double Laplace transform [12] and operational calculus [3].

The primary aim of this paper is to study the conformable double Laplace transform to solve a novel case of fractional telegraph equations with conformable double convolution with non-constant coefficients by exchanging the nonhomogeneous terms by conformable double convolution functions.

Definition 1.1. If $\Omega : [0,\infty) \to R$ is a function of t, then the conformable fractional derivative with respect to t, then, σ^{th} order conformable fractional derivative is defined with:

$$tD_{\sigma}\Omega(t) = \lim_{\tau\to 0} \frac{\Omega(t+\tau t^{1-\sigma}) - \Omega(t)}{\tau}, \quad \forall\, t \ge 0, \sigma \in (0,1). \tag{1}$$

Definition 1.2 ([16])**.** If $\Omega(x,t)$, $t,x \in R^+$is function of two variables then:

(i) the conformable single Laplace transform with respect to x, designated by the operator $\ell_x^{\theta}[\Omega(x,t)]$ is defined by:

$$\ell_x^{\theta}[\Omega(x,t)] = \Phi(r,t) = \int_0^{\infty} e^{-r\frac{x^{\theta}}{\theta}} \Omega(x,t)\, d_{\theta}x. \tag{2}$$

(ii) the conformable single Laplace's transform with respect to t of $\Omega(x,t)$ is

$$\ell_t^{\sigma}[\Omega(x,t)] = \Phi(x,v) = \int_0^{\infty} e^{-v\frac{t^{\sigma}}{\sigma}} \Omega(x,t)\, d_{\sigma}t. \tag{3}$$

(iii) the conformable double Laplace's transform of $\Omega(x,t)$ is

$$\ell_t^{\sigma}\ell_x^{\theta}[\Omega(x,t)] = \Phi(r,v) = \int_0^{\infty}\int_0^{\infty} e^{-r\frac{x^{\theta}}{\theta}-v\frac{t^{\sigma}}{\sigma}} \Omega(x,t) d_{\sigma}t\, d_{\theta}x. \tag{4}$$

Note that: $\ell_t^{\sigma}\ell_x^{\theta}[\Omega(x,t)] = \ell_x^{\theta}\ell_t^{\sigma}[\Omega(x,t)] = \Phi(r,v)$.

(iv) the conformable double Laplace's transform of $(\ell_t^{\sigma})^{-1}(\ell_x^{\theta})^{-1}[\Phi(r,v)] = \Omega(x,t)$ is defined by the complex double integral formula

$$(\ell_t^{\sigma})^{-1}(\ell_x^{\theta})^{-1}[\Phi(r,v)] = \Omega\left(\frac{x^{\theta}}{\theta}, \frac{t^{\sigma}}{\sigma}\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{r\frac{x^{\theta}}{\theta}} dr \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{v\frac{t^{\sigma}}{\sigma}} \Phi(r,v) dv.$$

Definition 1.3 ([16])**.** If $\Omega(x,t)$, $t,x \in R^+$ is a function of two variables, then the conformable double Laplace transform of fractional partial derivatives are:

$$\ell_t^{\sigma}\ell_x^{\theta}[xD_{\theta}\Omega(x,t)] = r\Phi(r,v) - \Phi(0,v),$$
$$\ell_t^{\sigma}\ell_x^{\theta}[tD_{\sigma}\Omega(x,t)] = v\Phi(r,v) - \Phi(r,0),$$
$$\ell_t^{\sigma}\ell_x^{\theta}[xD_{\theta}^2\Omega(x,t)] = r^2\Phi(r,v) - r\Phi(0,v) - xD_{\theta}\Phi(0,v),$$
$$\ell_t^{\sigma}\ell_x^{\theta}[tD_{\sigma}^2\Omega(x,t)] = v^2\Phi(r,v) - v\Phi(r,0) - tD_{\sigma}\Phi(r,0),$$
$$\ell_t^{\sigma}\ell_x^{\theta}[xD_{\theta}tD_{\sigma}\Omega(x,t)] = rv\Phi(r,v) - r\Phi(r,0) - v\Phi(0,v) + \Phi(0,0).$$

2. Some Properties and Theorems of Conformable Double Laplace Transform and Conformable Double Convolution

In this section, the conformable double Laplace transform and conformable double convolution and their definitions, theorems and properties with their verification, was considered.

(i) $\ell_t^{\sigma}\ell_x^{\theta}[a_1\Omega_1(x,t) + a_2\Omega_2(x,t)] = a_1\ell_t^{\sigma}\ell_x^{\theta}[\Omega_1(x,t)] + a_2\ell_t^{\sigma}\ell_x^{\theta}[\Omega_2(x,t)]$.

(ii) $\ell_t^\sigma \ell_x^\theta \left[e^{\frac{t^\sigma}{\sigma}b+\frac{x^\theta}{\theta}a}\right] = \dfrac{1}{(v-b)(r-a)}$.

(iii) $\ell_t^\sigma \ell_x^\theta \left[\cos\left(\dfrac{t^\sigma}{\sigma}b+\dfrac{x^\theta}{\theta}a\right)\right] = \dfrac{rv-ab}{(r^2+a^2)(v^2+b^2)}$.

(iv) $\ell_t^\sigma \ell_x^\theta \left[\sin\left(\dfrac{t^\sigma}{\sigma}b+\dfrac{x^\theta}{\theta}a\right)\right] = \dfrac{av+br}{(r^2+a^2)(v^2+b^2)}$.

(v) $\ell_t^\sigma \ell_x^\theta \left[e^{\frac{t^\sigma}{\sigma}b+\frac{x^\theta}{\theta}a}\Omega(x,t)\right] = \Phi(r+a,v+b)$.

(vi) $\ell_t^\sigma \ell_x^\theta \left[\left(\dfrac{t^\sigma}{\sigma}\cdot\dfrac{x^\theta}{\theta}\right)^n\right] = \dfrac{(n!)^2}{(rv)^{n+1}}$.

Proof. (i) Comes from Definition 1.2, eq. (4).

(ii)
$$\ell_t^\sigma \ell_x^\theta \left[e^{\frac{t^\sigma}{\sigma}b+\frac{x^\theta}{\theta}a}\right] = \int_0^\infty\int_0^\infty e^{-(r-a)\frac{x^\theta}{\theta}-(v-b)\frac{t^\sigma}{\sigma}}d_\sigma t\, d_\theta x$$
$$= \ell_t^\sigma\left[e^{\frac{t^\sigma}{\sigma}b};\frac{t^\sigma}{\sigma}\to v\right]\ell_x^\theta\left[e^{\frac{x^\theta}{\theta}a};\frac{x^\theta}{\theta}\to r\right]$$
$$= \frac{1}{(v-b)(r-a)}.$$

To prove (iii), (iv) we have
$$\ell_t^\sigma \ell_x^\theta \left[e^{\frac{t^\sigma}{\sigma}bi+\frac{x^\theta}{\theta}ai}\right] = \frac{1}{(v-bi)(r-ai)} = \frac{(rv-ab)+i(av+br)}{(r^2+a^2)(v^2+b^2)}$$
then
$$\ell_t^\sigma \ell_x^\theta \left[\cos\left(\frac{t^\sigma}{\sigma}b+\frac{x^\theta}{\theta}a\right)\right] = \frac{rv-ab}{(r^2+a^2)(v^2+b^2)},$$
$$\ell_t^\sigma \ell_x^\theta \left[\sin\left(\frac{t^\sigma}{\sigma}b+\frac{x^\theta}{\theta}a\right)\right] = \frac{av+br}{(r^2+a^2)(v^2+b^2)}$$
(v) Come from the definition.

(vi)
$$\ell_t^\sigma \ell_x^\theta \left[\left(\frac{t^\sigma}{\sigma}\cdot\frac{x^\theta}{\theta}\right)^n\right] = \int_0^\infty e^{-r\frac{x^\theta}{\theta}}\left(\frac{x^\theta}{\theta}\right)^n d_\theta x\int_0^\infty e^{-v\frac{t^\sigma}{\sigma}}\left(\frac{t^\sigma}{\sigma}\right)^n d_\sigma t = \frac{n!}{r^{n+1}}\cdot\frac{n!}{v^{n+1}} = \frac{(n!)^2}{(rv)^{n+1}}.$$ □

Theorem 2.1. *If* $\ell_t^\sigma \ell_x^\theta[\Omega(x,t)] = \Phi(r,v)$, *then*
$$\ell_t^\sigma \ell_x^\theta\left[\Omega\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\right] = e^{-\alpha r-\beta v}\Phi(r,v),$$
where $H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)$, *is the conformable Heaviside unit step function defined by*
$$H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right) = \begin{cases}1, & \text{if } \frac{x^\theta}{\theta}>\alpha,\ \frac{t^\sigma}{\sigma}>\beta,\\ 0, & \text{if } \frac{x^\theta}{\theta}<\alpha,\ \frac{t^\sigma}{\sigma}<\beta.\end{cases}$$

Proof. By definition of conformable double Laplace transform, we have
$$I = \ell_t^\sigma \ell_x^\theta\left[\Omega\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\right]$$
$$= \int_0^\infty\int_0^\infty e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}\Omega\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)d_\sigma t\, d_\theta x$$

$$= \int_{\alpha}^{\infty}\int_{\beta}^{\infty} e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}\Omega\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)d_\sigma t\, d_\theta x$$

if we put $\frac{\tau^\theta}{\theta}=\frac{x^\theta}{\theta}-\alpha$, $\frac{\eta^\sigma}{\sigma}=\frac{t^\sigma}{\sigma}-\beta$ then I becomes

$$I = e^{-r\alpha-v\beta}\int_0^{\infty}\int_0^{\infty} e^{-r\frac{\tau^\theta}{\theta}-v\frac{\eta^\sigma}{\sigma}}\Omega\left(\frac{\tau^\theta}{\theta},\frac{\eta^\sigma}{\sigma}\right)d_\sigma\eta\, d_\theta\tau = e^{-r\alpha-v\beta}\Phi(r,v). \qquad \square$$

Definition 2.2. If $\Omega_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$, $\Omega_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$, be integrable functions then the conformable double convolution is defined by,

$$\Omega_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\Omega_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)=\int_0^{\frac{t^\sigma}{\sigma}}\int_0^{\frac{x^\theta}{\theta}}\Omega_1\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\Omega_2(\alpha,\beta)\,d\beta\, d\alpha.$$

Theorem 2.3. *If*

$$\ell_t^\sigma\ell_x^\theta\left[\Omega_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]=\Phi_1(r,v),\quad \ell_t^\sigma\ell_x^\theta\left[\Omega_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]=\Phi_2(r,v),$$

then

$$\ell_t^\sigma\ell_x^\theta\left[(\Omega_1**\Omega_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]=\ell_t^\sigma\ell_x^\theta\left[\Omega_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]\ell_t^\sigma\ell_x^\theta\left[\Omega_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]=\Phi_1(r,v).\Phi_2(r,v).$$

Proof. Using the Definition 1.2(iii) to obtain

$$J=\ell_t^\sigma\ell_x^\theta\left[(\Omega_1**\Omega_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]$$

$$=\int_0^{\infty}\int_0^{\infty}e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}(\Omega_1**\Omega_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)d_\sigma t\, d_\theta x$$

$$=\int_0^{\infty}\int_0^{\infty}e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}\left\{\int_0^{\frac{t^\sigma}{\sigma}}\int_0^{\frac{x^\theta}{\theta}}\Omega_1\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\Omega_2(\alpha,\beta)\,d\beta\, d\alpha\right\}d_\sigma t\, d_\theta x.$$

By using the conformable Heaviside unit step function integral; J becomes

$$J=\int_0^{\infty}\int_0^{\infty}e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}\left\{\int_0^{\infty}\int_0^{\infty}\Omega_1\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\Omega_2(\alpha,\beta)\,d\beta\, d\alpha\right\}d_\sigma t\, d_\theta x$$

$$=\int_0^{\infty}\int_0^{\infty}\Omega_2(\alpha,\beta)\,d\beta\, d\alpha\left\{\int_0^{\frac{t^\sigma}{\sigma}}\int_0^{\frac{x^\theta}{\theta}}e^{-r\frac{x^\theta}{\theta}-v\frac{t^\sigma}{\sigma}}\Omega_1\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)H\left(\frac{x^\theta}{\theta}-\alpha,\frac{t^\sigma}{\sigma}-\beta\right)\right\}d_\sigma t\, d_\theta x.$$

Using Theorem 2.1 to get

$$J=\int_0^{\infty}\int_0^{\infty}\Omega_2(\alpha,\beta)\,e^{-\alpha r-\beta v}\Phi_1(r,v)\,d\beta\, d\alpha$$

$$=\Phi_1(r,v)\int_0^{\infty}\int_0^{\infty}\Omega_2(\alpha,\beta)\,e^{-\alpha r-\beta v}\,d\beta\, d\alpha=\Phi_1(r,v).\Phi_2(r,v).$$

3. Conformable Fractional Telegraph Equation with Conformable Double Convolution and Laplace Transform

In this section, the both of conformable double convolution and conformable double Laplace transform was employed to look at the main theorem and, solve two types of fractional telegraph equations.

Theorem 3.1. *If* $\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ *is a solution of fractional telegraph equation*

$$xD_\theta^2\Omega(x,t)-tD_\sigma^2\Omega(x,t)-tD_\sigma\Omega(x,t)-\Omega(x,t)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right) \tag{5}$$

with the initial condition

$$\Omega(0,t)=K_1\left(\frac{t^\sigma}{\sigma}\right),\quad \Omega(x,0)=H_1\left(\frac{x^\theta}{\theta}\right),\quad xD_\theta\Omega(0,t)=K_2\left(\frac{t^\sigma}{\sigma}\right),\quad tD_\sigma\Omega(x,0)=H_2\left(\frac{x^\theta}{\theta}\right), \tag{6}$$

and $\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ *is a solution of fractional telegraph equation with double convolution coefficients*

$$\eta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\left(xD_\theta^2\Omega(x,t)-tD_\sigma^2\Omega(x,t)-tD_\sigma\Omega(x,t)-\Omega(x,t)\right)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right) \tag{7}$$

with the same initial condition (6)*, where* $\eta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ *is a polynomial.*
Then, $\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ *is a solution of the following equation*

$$xD_\theta^2\Omega(x,t)-tD_\sigma^2\Omega(x,t)-tD_\sigma\Omega(x,t)-\Omega(x,t)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)+\zeta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right) \tag{8}$$

with the same initial condition (6)*.*

Proof. $\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ is a solution of (5), this means that:

$$xD_\theta^2\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma^2\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right). \tag{9}$$

Also, $\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ is a solution of (7) this means that:

$$\eta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\left(xD_\theta^2\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma^2\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right). \tag{10}$$

Now, we prove that: $\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ is a solution of (8)

$$xD_\theta^2(\Phi_1**\Phi_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma^2(\Phi_1**\Phi_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma(\Phi_1**\Phi_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-(\Phi_1**\Phi_2)\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$$
$$=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)+\zeta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right). \tag{11}$$

We use the following conformable partial derivatives of double convolution

$$\left(tD_\sigma^2\Phi_1\right)**\Phi_2-\left(xD_\theta^2\Phi_1\right)**\Phi_2=\Phi_1**\left(tD_\sigma^2\Phi_2\right)-\Phi_1**\left(xD_\theta^2\Phi_2\right).$$

Then eq. (11) becomes

$$\left[xD_\theta^2\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma^2\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-tD_\sigma\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)-\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)\right]**\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$$
$$=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)+\zeta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right). \tag{12}$$

Using eqs. (9) and (12), to obtain

$$\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)=\Psi\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)+\zeta\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right).$$

This means that the conformable double convolution $\Phi_1\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)**\Phi_2\left(\frac{x^\theta}{\theta},\frac{t^\sigma}{\sigma}\right)$ is a solution of eq. (8). □

To apply Theorem 3.1, we will study the following example:

Example 3.2. Consider the fractional space-time telegraph equation,

$$xD_\theta^2\Omega(x,t)-tD_\sigma^2\Omega(x,t)-tD_\sigma\Omega(x,t)-\Omega(x,t)=-2e^{\frac{x^\theta}{\theta}+\frac{t^\sigma}{\sigma}} \tag{13}$$

with the initial conditions

$$\Omega(0,t)=e^{\frac{t^\sigma}{\sigma}},\quad \Omega(x,0)=e^{\frac{x^\theta}{\theta}},\quad xD_\theta\Omega(0,t)=e^{\frac{t^\sigma}{\sigma}},\quad tD_\sigma\Omega(x,0)=e^{\frac{x^\theta}{\theta}}. \tag{14}$$

Take the conformable double Laplace transform of eq. (13), to obtain

$$r^2\Phi(r,v)-r\Phi(0,v)-xD_\theta\Phi(0,v)-v^2\Phi(r,v)+v\Phi(r,0)+tD_\sigma\Phi(r,0)-v\Phi(r,v)+\Phi(r,0)-\Phi(r,v)$$
$$=\frac{-2}{(v-1)(r-1)}. \tag{15}$$

Also, take the single conformable Laplace transform of eq. (14), to get

$$\Phi(0,v)=\frac{1}{(v-1)},\quad \Phi(r,0)=\frac{1}{(r-1)},\quad xD_\theta\Phi(0,v)=\frac{1}{(v-1)},\quad tD_\sigma\Phi.(r,0)=\frac{1}{(r-1)}. \tag{16}$$

Substituting eq. (16) in eq. (15), to get

$$\Phi(r,v)=\frac{1}{(v-1)(r-1)}.$$

Getting hold of the conformable inverse of double Laplace transform to find the exact solution in the form

$$\Phi_1(x,t)=\Omega(x,t)=e^{\frac{x^\theta}{\theta}+\frac{t^\sigma}{\sigma}}. \tag{17}$$

Now, we need to solve the following equation

$$x^2t^2 ** \left[xD_\theta^2\Omega(x,t)-tD_\sigma^2\Omega(x,t)-tD_\sigma\Omega(x,t)-\Omega(x,t)\right]=-2e^{\frac{x^\theta}{\theta}+\frac{t^\sigma}{\sigma}} \tag{18}$$

with the same initial conditions (14).
Using the same steps to obtain the solution in the form

$$\Phi_2(x,t)=\Omega(x,t)=e^{\frac{x^\theta}{\theta}+\frac{t^\sigma}{\sigma}}+\zeta(x,t). \tag{19}$$

It is easy to see the relationship between two solutions in eqs. (17) and (19).

4. Conclusion

This paper introduce the conformable double Laplace transform and conformable double convolution, and explained the effectiveness and ease of this double transform to solve a fresh case of fractional telegraph equations with conformable double convolution.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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