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Research Article

Numerical Solution of Boundary Layer Problem Using Second Order Variable Mesh Method

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Abstract. In this paper, a second-order finite difference method on non-uniform grid is proposed for the solution of singularly perturbed boundary value problems. Replace the derivatives of the problem with highorder finite differences on a non-uniform grid to get a discrete equation. This equation can be effectively solved by tridiagonal method. This method performs convergence analysis and the method produces second-order consistent convergence. The numerical experiments are used to illustrate the method. The absolute error has been proposed to compare with other methods in the literature to prove the rationality of the method.

Keywords. Non-uniform grid; Finite difference method; Singularly perturbed boundary value problem; Boundary layer

MSC. 65L10; 65L11; 65L12

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1. Introduction

Differential equations with a highest derivative multiplied by a small parameter play an important role in various fields of science and engineering. Such as nuclear engineering, control theory, elasticity, fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, reaction diffusion processes, geophysics, etc. Various authors have proposed different numerical methods to solve the *Singular Perturbation Problem* (SPP) [7, 12, 14].

Kadalbajoo *et al*. [5] considered the numerical solution of the SPP using spline functions.

It knows well that most of the methods fail on an hour basis with respect to the grid size used for the discretization of the SPP. As usually, general numerical methods cannot give a best approximation to those equations. Awoke and Reddy [1] proposed a fitting method for solving these equations. Habib and El-Zahar [4] have used an algorithm for solving the mechanistic SPP. Kadalbajoo and Sharma [8] briefly investigated numerical methods for solving SPP. Kadalbajoo and Kapil [9] have chosen a numerical method based on finite-difference singularly perturbed boundary value problems for delay differential equations. Mohammadi [10] proposed a uniformly converged uniform grid difference scheme using adaptive cubic splines to solve SPP. Kadalbajoo *et al*. [5] gives a second-order method, which becomes a special case of the method given in [10]. Kadalbajoo *et al*. [6] derived a finite-difference method for the uniform convergence of a fitting grid for this problem. Natesan and Ramanujam [11] proposed an enhanced method for SPPs that arise in chemical reactor theory. Rao and Chakravarty [13] used a finite difference method to deal with a singularly perturbed differential difference equation with layers and oscillation behavior. Surla *et al*. [15] proposed a quadratic spline discrete minimum principle for the SPP. Roos *et al*. [14] discuss numerical methods for solving SPP. Uniform numerical methods for the initial and boundary layer problems are given by Doolan *et al*. [3]. Bigge and Bohl [2] proposed the deformation of the bifurcation diagram due to the discretization.

2. Numerical Approach

Consider a model singular perturbed one parameter equation of the form:

$$L[\theta(u)] \equiv \varepsilon\theta''(u) + a(u)\theta'(u) + b(u)\theta(u) = f(u), \quad 0 \le u \le 1 \tag{2.1}$$

with boundary conditions

$$\theta(0) = \gamma_0 \text{ and } \theta(1) = \gamma_1 \tag{2.2}$$

where ε $(0 < \varepsilon \ll 1)$ is a very small positive perturbation parameter and the functions $a(u)$, $b(u)$ and $f(u)$ are sufficiently smooth functions in the given domain satisfying the conditions $a(u) \ge \widetilde{a} > 0$, $b(u) \le -\rho < 0$ where ρ is a positive constant.

Let $[0,1]$ be divided into N subintervals with variable mesh size $h_i = u_i - u_{i-1}$ for $i = 1$ to N and $h_{i+1} = \tau_i h_i$. To start the computational implementation, we have to determine the value of h_1. Denote $R = u_N - u_0$. Then

$$\begin{aligned} R &= (u_N - u_{N-1}) + (u_{N-1} - u_{N-2}) + \ldots + (u_1 - u_0) \\ &= h_N + h_{N-1} + \ldots + h_1 \\ &= (\tau_1 + \tau_1\tau_2 + \ldots + \tau_1\tau_2\tau_3 \ldots \tau_{N-1})h_1 . \end{aligned}$$

Then $h_1 = \frac{R}{(\tau_1+\tau_1\tau_2+\ldots+\tau_1\tau_2\tau_3\ldots\tau_{N-1})}$ determines the value of the starting step length with which we can compute the subsequent step lengths h_2, h_3, etc. In singular perturbation problems, if the layer is at the left- end boundary $u = 0$, then a large collection of nodal points nearer to this point are required. Likewise, when the layer is at the right- end boundary then a large cluster of nodal points at this boundary is needed. The following process achieves this distribution of nodal points.Choose $\tau_i = \tau =$ constant for $i = 1,2,\ldots,N$. Then the step length h_1 reduces to $h_1 = \frac{R(1-\tau)}{(1-\tau^N)}$.

If the layer is at left- end point, then we take $\tau > 1$. It assurances that more number of nodal points exist near to the left end boundary. If the layer exists at the right- end, then choose $\tau < 1$ which ensures a collection of large number of nodal points near the right end boundary. When the layer is at the both end points of the interval then take $\tau > 1$ in the left half, $\tau < 1$ in the right half of the respective intervals. Then we have a symmetric mesh with a greater number of nodal points at both ends of the domain.

Consider a non uniform higher order finite difference approximation of first and second derivatives:

$$\theta_i' = \widetilde{\theta}_i' - \frac{\tau_i h_i^2}{6}\theta_i''' + \nu_{i1}, \tag{2.3}$$

$$\theta_i'' = \widetilde{\theta}_i'' + \frac{(1-\tau_i)h_i}{3}\theta_i''' + \nu_{i2}, \tag{2.4}$$

where $\widetilde{\theta}_i' = \frac{\theta_{i+1}-\tau_i^2\theta_{i-1}+(\tau_i^2-1)\theta_i}{\tau_i(1+\tau_i)h_i}$ and $\widetilde{\theta}_i'' = \frac{2[\theta_{i+1}+\tau_i\theta_{i-1}-(1+\tau_i)\theta_i]}{\tau_i(1+\tau_i)h_i^2}$ with truncation errors $\nu_{i1} = \frac{\tau_i(\tau_i-1)}{24}h_i^3\theta_i^{(iv)}$ and $\nu_{i2} = \frac{(\tau_i^2-\tau_i+1)}{12}h_i^2\theta_i^{(iv)}$.

Computing θ_i''' using eq. (2.1), replacing it in eq. (2.3) and eq. (2.4), we get

$$\theta_i' = \widetilde{\theta}_i' - B_i\Big(\frac{f_i^\iota - a_i\theta_i^{\iota\iota} - k_3(i)\theta_i^\iota - b_i^\iota\theta_i}{\varepsilon}\Big), \tag{2.5}$$

$$\theta_i'' = \widetilde{\theta}_i'' + A_i\Big(\frac{f_i^\iota - a_i\theta_i^{\iota\iota} - k_3(i)\theta_i' - b_i^\iota\theta_i}{\varepsilon}\Big), \tag{2.6}$$

where $A_i = \dfrac{(1-\tau_i)h_i}{3}$, $B_i = \dfrac{h_i\tau_i^2}{6}$, $k_1(i) = \tau(1+\tau)h_i^2$, $k_2(i) = \tau(1+\tau)h_i$, $k_3(i) = (a_i' + b_i)$.

Now inserting eqs. (2.5) and (2.6) in eq. (2.1), we get the following tridiagonal relation

$$U_i\theta_{i-1} + V_i\theta_i + W_i\theta_{i+1} = Z_i \tag{2.7}$$

where $U_i = \big(2L_i\tau - \tau^2 M_i h_{i-1}\big)$, $V_i = \big(M_i(\tau^2-1)h_i - 2L_i(1+\tau) + O_i\tau(1+\tau)h_i^2\big)$, $W_i = (2L_i + M_i h_{i-1})$, $L_i = \varepsilon - a_iA_i + \frac{a_i^2B_i}{\varepsilon}$, $M_i = a_i - A_ik_3(i) + \frac{a_iB_ik_3(i)}{\varepsilon}$, $O_i = \frac{a_iB_ib_i^\iota}{\varepsilon} + b_i - A_ib_i^\iota$, $Z_i = f_i + \Big(\frac{a_iB_i}{\varepsilon} - A_i\Big)f_i^\iota$.

3. Convergence Analysis

Truncation error in the proposed scheme is

$$T_i(h_i) = \frac{\tau(1+\tau)}{3}\left[\frac{(\tau-1)^2}{3}a_i\theta_i''' + \frac{(\tau^2-\tau+1)}{4}\left(f_i'' - a_i\theta_i''' - (2a_i' + b_i')\theta_i'' - (a_i'' + 2b_i')\theta_i' - b_i''\theta_i\right)\right]$$
$$\cdot h_i^4 + O(h_i^5). \tag{3.1}$$

Consider the tridiagonal system eq. (3.1) in matrix form

$$P\theta = D, \tag{3.2}$$

where $P = [p_{ij}]$ for $1 \le i, j \le (N-1)$ is a tri-diagonal matrix with $p_{ii+1} = W_i$, $p_{ii} = V_i$, $p_{ii-1} = U_i$ and $D = (v_i)$ is a column vector with $v_i = Z_i$.

We also have

$$P\overline{\theta} - T_i(h_i) = D, \tag{3.3}$$

where $\overline{\theta} = (\overline{\theta}_0, \overline{\theta}_1, \dots, \overline{\theta}_N)^t$ represents original solution and $T_i(h_i) = (T_0(h_0), T_1(h_1), \dots, T_N(h_N))^t$ is the local truncation error.

From eq. (3.2) and eq. (3.3), it is clear that

$$P(\overline{\theta} - \theta) = T_i(h_i) \tag{3.4}$$

so that the error equation is

$$PE = T_i(h_i), \tag{3.5}$$

where $E = \overline{Y} - Y = (e_0, e_1, \dots, e_N)^t$. Let S_i be the sum of elements of the ith row of matrix P, then we have

$$\begin{aligned} S_i &= \sum_{j=1}^{N-1} m_{ij} = -2\tau\varepsilon + \left(a_i\tau + \frac{2a_i(1-\tau)}{3}\right)h_i + O(h_i^2) && \text{for } i = 1 \\ &= \sum_{j=1}^{N-1} m_{ij} = \frac{(\tau+1)b_i}{\tau}h_i^2 + \left(\frac{(\tau+1)(\tau-1)b_i'}{3\tau^2}\right)h_i^3 + O(h_i^4) = \beta_i h_i^2 + O(h_i^3) && \text{for } i = 2,3,\dots,N-2 \\ &= \sum_{j=1}^{N-1} m_{ij} = -2\varepsilon + \left(\frac{(1-\tau)2a_i}{3} - a_i\right)\frac{h_i}{\tau} + O(h_i^2) && \text{for } i = N-1 \end{aligned}$$

Since $0 < \varepsilon \ll 1$, P^{-1} exists and it has non-negative elements. So that from eq. (3.5), it has

$$E = P^{-1}T(h) \tag{3.6}$$

and

$$\|E\| \le \|P^{-1}\| \cdot \|T(h)\|. \tag{3.7}$$

Let $\overline{m}_{ki}$ be the (ki)th element of P^{-1}. Since $\overline{m}_{ki} \ge 0$, from the matrix theory, we have

$$\sum_{i=1}^{N-1} m_{ki}S_i = 1, \quad k = 1,2,\dots,N-1. \tag{3.8}$$

Therefore,

$$\sum_{i=1}^{N-1} \overline{m}_{ki} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{\beta_i} \le \frac{1}{|\beta_i|}. \tag{3.9}$$

We define $\|P^{-1}\| = \max\limits_{1\le k\le N-1} \sum\limits_{i=1}^{N-1} |\overline{m}_{ki}|$ and $\|T(h)\| = \max\limits_{1\le i\le N-1} |T_i(h)|$.

Using eq. (3.1), eq. (3.6) and eq. (3.9), we get

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{ki}T_i(h_i), \quad j = 1,2,3,\dots,N-1$$

which implies

$$e_j \le \frac{kh_i^4}{|\beta_i|h_i^2}, \quad \text{for } j = 1,2,\dots,N-1 \tag{3.10}$$

where $k = \frac{\tau(1+\tau)}{3}\left[\frac{(\tau-1)^2}{3}a_i\theta_i''' + \frac{(\tau^2-\tau+1)}{4}\left(f_i'' - a_i\theta_i''' - (2a_i' + b_i')\theta_i'' - (a_i'' + 2b_i')\theta_i' - b_i''\theta_i\right)\right]$ is a constant independent of h and $\beta_i = \frac{(\tau+1)b_i}{\tau}$.

Therefore, using eq. (3.10), we have

$$\|E\| = O\left(h_i^2\right)$$

i.e., the method is second order uniformly convergent on the non-uniform mesh.

4. Numerical Experiments

To demonstrate the suggested method, it is implemented on three test examples. The maximum error in the solution of the problem is computed and tabulated with comparison.

Example 1. $\varepsilon\theta''(u)+\theta'(u)=0$, with $\theta(0)=1$, $\theta(1)=e^{\frac{-1}{\varepsilon}}$.

The exact solution is $\theta(u)=e^{-u/\varepsilon}$.

Table 1. Comparison of the maximum absolute error of Example 1

$\varepsilon\downarrow$ \ $N\rightarrow$	64	128	256	512	1024
	Results by the proposed scheme				
2^{-6}	3.20(−2)	7.70(−3)	1.90(−3)	4.93(−4)	1.4509(−4)
2^{-7}	1.142(−1)	3.17(−2)	7.60(−3)	1.90(−3)	4.6970(−4)
2^{-8}	2.75(−1)	1.136(−1)	3.13(−2)	7.40(−3)	1.80(−3)
2^{-9}	4.881(−1)	2.741(−1)	1.125(−1)	3.05(−2)	7.00(−3)
	Results in [10]				
2^{-6}	3.45(−2)	7.87(−3)	1.92(−3)	4.79(−4)	1.19(−4)
2^{-7}	1.35(−1)	3.45(−2)	7.87(−3)	1.92(−3)	4.79(−4)
2^{-8}	3.510(−1)	1.35(−1)	3.45(−2)	7.87(−3)	1.92(−3)
2^{-9}	6.00(−1)	3.51(−1)	1.35(−1)	3.45(−2)	7.87(−3)

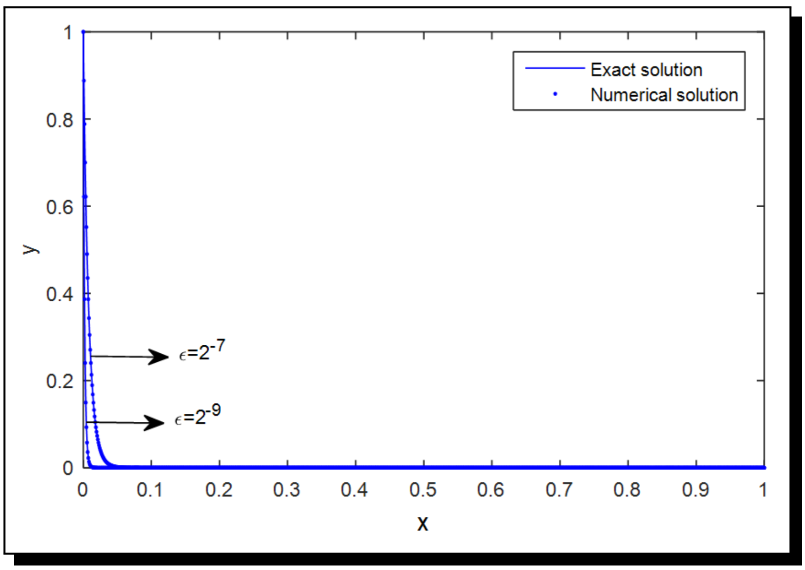

Figure 1. Graphical representation of the solution in Example 1

Example 2. $-\varepsilon\theta''(u)+\theta'(u)=e^u$, $\theta(0)=0$, $\theta(1)=0$.

The exact solution is $\theta(u)=\dfrac{1}{\varepsilon-1}\left[1-\dfrac{e-1}{e^{\frac{1}{\varepsilon}}-1}+\dfrac{(e-1)e^{\frac{u}{\varepsilon}}}{e^{\frac{1}{\varepsilon}}-1}-e^u\right]$.

Table 2. Comparison of the maximum absolute error of Example 2

$\varepsilon\downarrow$ / $N\rightarrow$	64	128	256	512	1024
Results by the proposed scheme					
2^{-1}	5.0796(−6)	2.0215(−6)	5.0708(−7)	1.2865(−7)	3.5256(−8)
2^{-4}	3.1(−3)	7.5966(−4)	1.8963(−4)	4.7332(−5)	1.1771(−5)
2^{-8}	5.986(−1)	2.31(−1)	5.91(−2)	1.35(−2)	3.30(−3)
Results in [10]					
2^{-1}	5.34(−5)	1.33(−5)	3.34(−6)	8.35(−7)	2.08(−7)
2^{-4}	3.53(−3)	8.79(−4)	2.19(−4)	5.48(−5)	1.37(−5)
2^{-8}	6.06(−1)	2.33(−1)	5.95(−2)	1.35(−2)	3.32(−3)

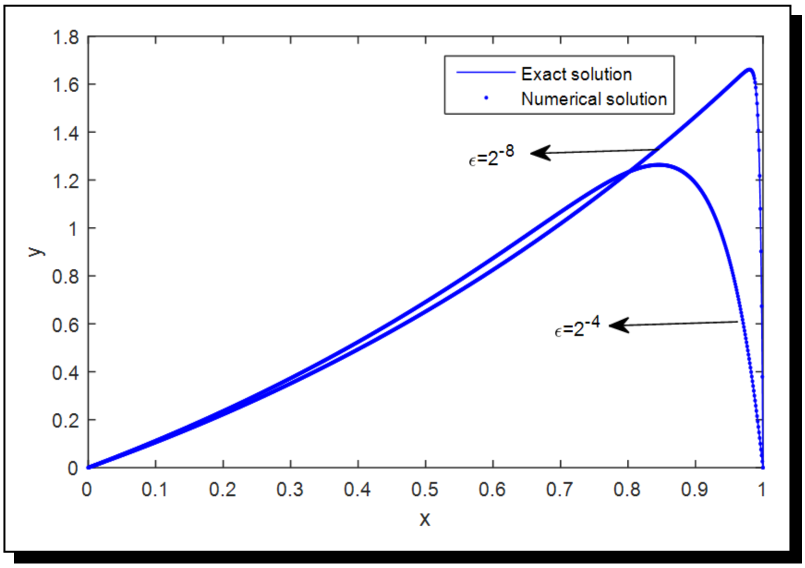

Figure 2. Graphical representation of the solution in Example 2

Example 3. $\varepsilon\theta''-\dfrac{1}{u+1}\theta'-\dfrac{1}{u+2}\theta=f(u)$ with $\theta(0)=1+2^{\frac{-1}{\varepsilon}}$,

$\theta(1)=e+2f(u)=e^u\left(\varepsilon-\frac{1}{u+1}-\dfrac{1}{u+2}\right)-2^{\frac{-1}{\varepsilon}}\dfrac{(u+1)^{1+\frac{1}{\varepsilon}}}{u+2}$.

The exact solution is $\theta(u)=e^u+2^{\frac{-1}{\varepsilon}}(u+1)^{1+\frac{1}{\varepsilon}}$.

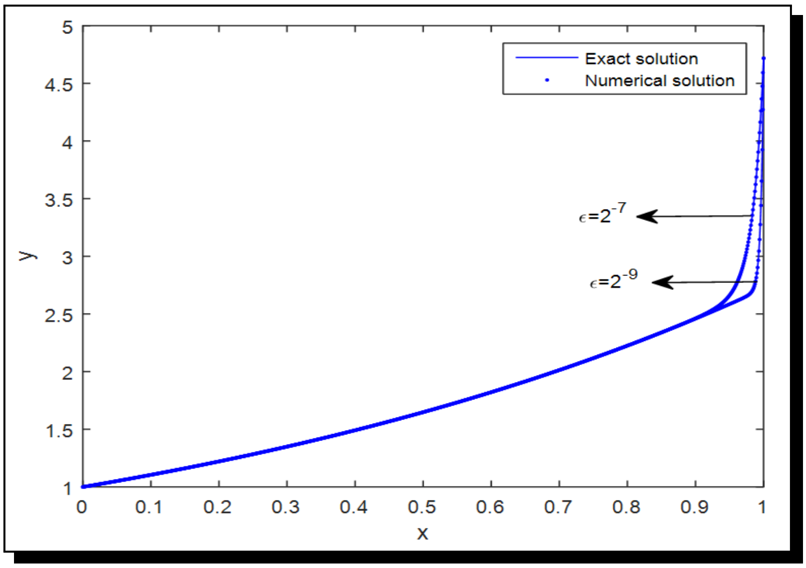

Figure 3. Graphical representation of the solution in Example 3

Table 3. Comparison of the maximum absolute error of Example 3

$\varepsilon \downarrow$ \ $N \rightarrow$	64	128	256	512	1024
Results by the proposed scheme					
2^{-4}	8.1296(−4)	2.0378(−4)	5.1420(−5)	1.3359(−5)	3.9075(−6)
2^{-5}	3.5308(−3)	8.8004(−4)	2.2076(−4)	5.6146(−5)	1.5032(−5)
2^{-6}	1.4909(−2)	3.6816(−3)	9.1909(−4)	2.3137(−4)	5.9725(−5)
2^{-7}	6.3188(−2)	1.5220(−2)	3.7648(−3)	9.4190(−4)	2.3885(−4)
2^{-8}	2.2808(−1)	6.3711(−2)	1.5385(−2)	3.8116(−3)	9.5725(−4)
2^{-9}	5.5208(−1)	2.2869(−1)	6.3987(−2)	1.5476(−2)	3.8424(−3)
Results in [10]					
2^{-4}	1.00(−3)	2.50(−4)	6.27(−5)	1.56(−5)	3.91(−6)
2^{-5}	3.90(−3)	9.71(−4)	2.41(−4)	6.06(−5)	1.50(−5)
2^{-6}	1.57(−2)	3.86(−3)	9.62(−4)	2.39(−4)	6.00(−5)
2^{-7}	6.91(−2)	1.56(−2)	3.85(−3)	9.59(−4)	2.39(−4)
2^{-8}	2.69(−1)	6.90(−2)	1.57(−2)	3.85(−3)	9.58(−4)
2^{-9}	6.97(−1)	2.70(−1)	6.90(−2)	1.57(−2)	3.85(−3)

5. Discussions and Conclusion

For the solution of the SPP, a finite difference method on non-uniform grid is proposed. The discretized equation is obtained by inserting a high-order finite-difference approximation to the first-order and second-order derivatives of the problem using a geometric grid. To illustrate the proposed solution, it was verified in three examples. The numerical results are compared with the results of the method [7] to prove the proposed solution. We noticed that the proposed method gives better results. From the graphical representation of the solutions, we observed that the numerical solution is very accurate. The convergence of the suggested method is established. This method is very simple and requires little computational effort to produce accurate results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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