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Research Article

A New Approach to Multivalued Certain Contraction Mappings

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Abstract. In the submit study, we establish the notion of generalization of partial Hausdorff metric space. Also, we state an extension of the concept of f-weak compatibility of Pathak [12] on metric space in generalization of partial metric space. We introduced some common fixed point theorems for multivalued mappings satisfying generalized weak contraction conditions on a complete G_p metric spaces. Also, a example is given to illustrate the main theorem. Further, our theorems generalize several formerly obtained fixed point results.

Keywords. Fixed point; f-weakly compatible mappings; G_p-metric space; Multivalued mappings

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1. Introduction

Banach [5], introduced fixed point result, besides known as the Banach contraction principle. The Banach contraction theorem and its many extensions have been generalized by latest growing concept of weakly contractive mappings. Alber and Guerre-Delabriere [1] was defined the concept of φ-weak contraction mappings. In 2001 Rhoades [13] proved the φ-weak contraction single-valued mappings. Later Dutta and Choudhury [8] extended this concept to (φ,ψ)-weak contraction mappings and proved fixed point theorem for (φ,ψ)-weak contraction mappings. Many authors proved fixed point for (φ,ψ)-weak contractive mappings (see for example [6,7]). Lately, Zand and Nezhad [14] established a new generalized metric space G_p based on the two over metric spaces as both a generalization of the partial metric space and a G metric spaces. Some of these works may be considered in [4, 9]. Lately, Nadler [11], established the notion of multivalued contractive mapping and demonstrated well recognized Banach contraction principle. Several authors followed Nadler's opinion and offer their supports in that sense, see for example [2, 3]. Aydi *et al*. [3] introduced the Banach species fixed point results for set valued mapping in partial metric spaces.

Now, we mention briefly some fundamental definitions.

Definition 1 ([14])**.** Let X be a nonempty set and let $G_p : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(GP1) $0 \le G_p(\mathcal{A},\mathcal{A},\mathcal{A}) \le G_p(\mathcal{A},\mathcal{A},w) \le G_p(\mathcal{A},w,v)$, for all $\mathcal{A},w,v \in X$;

(GP2) $G_p(\mathcal{A},w,v) = G_p(\mathcal{A},v,w) = G_p(w,v,\mathcal{A})\dots$;

(GP3) $G_p(\mathcal{A},w,v) \le G(\mathcal{A},k,k) + G_p(k,w,v) - G_p(k,k,k)$, for any $k,\mathcal{A},w,v \in X$;

(GP4) $\mathcal{A} = w = v$ if $G_p(\mathcal{A},w,v) = G_p(\mathcal{A},\mathcal{A},\mathcal{A}) = G_p(w,w,w) = G_p(v,v,v)$;

Then the pair (X,G_p) is called a G_p metric space.

Proposition 1 ([14])**.** *Every G_p-metric space (X,G_p) describes a metric space (X,d_{G_p}) as follows:*

$$d_{G_p}(\mathcal{A},w) = G_p(\mathcal{A},w,w) + G_p(w,\mathcal{A},\mathcal{A}) - G_p(\mathcal{A},\mathcal{A},\mathcal{A}) - G_p(w,w,w) \tag{1.1}$$

for all $\mathcal{A},w \in X$.

Definition 2. The two classes of following mappings are defined

$$\Psi = \{\psi : [0,\infty) \to [0,\infty) |\ \psi \text{ is continuous, nondecreasing, and } \psi(t) = 0 \Leftrightarrow t = 0\} \text{ and } \psi^{-1}(0) = 0$$

and

$$\Phi = \{\phi : [0,\infty) \to [0,\infty) | \phi \text{ is lower semi continuous and } \phi(t) = 0 \Leftrightarrow t = 0\} \text{ and } \varphi^{-1}(0) = 0.$$

2. Multivalued Mappings in G_p Metric Spaces

The following expressions and Lemma's will be significant in the main results. (X,G_p), G_p-metric space can get its respectively for the metric case in [11], generalized metric space in [10] and partial metric space in [3].

Let X be a G_p metric space. We would state $CB^{G_p}(X)$ the family of all nonempty closed bounded subsets of X. Let $H_{G_p}(\cdot,\cdot,\cdot)$ be the Hausdorff G_p distance on $CB^{G_p}(X)$,

$$H_{G_p}(K,L,M) = \max\left\{\sup_{\mathcal{A}\in K} G_p(\mathcal{A},L,M), \sup_{\mathcal{A}\in L} G_p(\mathcal{A},M,K), \sup_{\mathcal{A}\in M} G_p(\mathcal{A},K,L)\right\},$$

where

$$G_p(\mathcal{A},L,M) = d_{G_p}(\mathcal{A},L) + d_{G_p}(L,M) + d_{G_p}(\mathcal{A},M),$$

$$d_{G_p}(\mathcal{A},L) = \inf\{d_{G_p}(\mathcal{A},w), w \in L\},$$

$$d_{G_p}(K,L) = \inf\{d_{G_p}(k,l), k \in K, l \in L\}.$$

Remind that $G_p(\mathcal{A},w,M) = \inf\{G_p(\mathcal{A},w,v), v \in M\}$. A mapping $E : X \to 2^X$ is named a multivalued mapping. A point $\mathcal{A} \in X$ is called a fixed point of E if $\mathcal{A} \in E\mathcal{A}$. Express by $P(X)$ the family of all nonempty sub-sets of X, $CB^{G_p}(X)$ the family of all nonempty, closed and bounded sub-sets of X and $K^{G_p}(X)$ the family of all nonempty compact subsets of X.

Lemma 1. *Let (X,G_p) generalization of partial metric space, $K,L \in CB^{G_p}(X)$ and $h > 1$. For any $k \in K$, there exists $l = l(k) \in L$ so much so that*

$$G_p(k,l,l) \le hH_{G_p}(K,L,L).$$

Lemma 2. *Let (X,G_p) generalization of partial metric space, $K,L \in CB^{G_p}(X)$ and $k \in K$. Therefore, for $\xi > 0$, there exists a point $l \in L$ so much so that*

$$G_p(k,l,l) \le H_{G_p}(K,L,L) + \xi.$$

Lemma 3. *Let (X,G_p) generalization of partial metric space, $K,L \in CB^{G_p}(X)$. Therefore, For any $k \in K$,*

$$G_p(k,L,L) \le H_{G_p}(K,L,L).$$

Our results is related to mappings $E : X \to K^{G_p}(X)$. Then, we shall use the following Lemma:

Lemma 4. *Let (X,G_p) generalization of partial metric space and a compact subsets of X. Then, for $\mathcal{A} \in X$, there exists $k \in K$ such that*

$$G_p(\mathcal{A},k,k) = G_p(\mathcal{A},K,K).$$

3. Main Results

In this section, firstly, we present an extension of the concept of f-weak compatibility of Pathak [12] on metric space in generalization of partial metric space. We obtain some common fixed point theorems for multi-valued mappings using generalized weak contraction and f-weak compatible of multivalued mappings on G_p metric space.

Definition 3. Let (X,G_p) be G_p metric space. The mapping $f : X \to X$ and $E : X \to CB^{G_p}(X)$ are f-weak compatible if and only if $fE\mathcal{A} \in CB^{G_p}(X)$ for all $\mathcal{A} \in X$ and the following limits exists and supplying

(i) $\lim_{n\to\infty} H_{G_p}(fE\mathcal{A}_n, Ef\mathcal{A}_n, Ef\mathcal{A}_n) \le \lim_{n\to\infty} H_{G_p}(Ef\mathcal{A}_n, E\mathcal{A}_n, E\mathcal{A}_n)$,

(ii) $\lim_{n\to\infty} G_p(fE\mathcal{A}_n, f\mathcal{A}_n, f\mathcal{A}_n) \le \lim_{n\to\infty} H_{G_p}(Ef\mathcal{A}_n, E\mathcal{A}_n, E\mathcal{A}_n)$,

whenever $\{\mathcal{A}_n\}$ is a sequence X such that $E\mathcal{A}_n \to N \in CB^{G_p}(X)$ and $f\mathcal{A}_n \to h \in N$.

Lemma 5. *If a sequence $\{\mathcal{A}_n\}$ in X is not Cauchy, then there exist $\xi > 0$ and two subsequences $\{\mathcal{A}_{m_k}\}$ and $\{\mathcal{A}_{n_k}\}$ of $\{\mathcal{A}_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$*

$$G_p(\mathcal{A}_{n_k}, \mathcal{A}_{m_k}, \mathcal{A}_{m_k}) \ge \xi \tag{3.1}$$

and

$$G_p(\mathcal{A}_{n_k}, \mathcal{A}_{m_k-1}, \mathcal{A}_{m_k-1}) < \xi \tag{3.2}$$

Moreover, suppose that $\lim_{n\to\infty} G_p(a_n, a_{n+1}, a_{n+1}) = 0$. *Then we have:*

(i) $\lim_{n\to\infty} G_p(a_{n_k}, a_{m_k}, a_{m_k}) = \xi$,

(ii) $\lim_{n\to\infty} G_p(a_{n_k-1}, a_{m_k-1}, a_{m_k-1}) = \xi$,

(iii) $\lim_{n\to\infty} G_p(a_{n_k-1}, a_{m_k}, a_{m_k}) = \xi$,

(iv) $\lim_{n\to\infty} G_p(a_{n_k}, a_{m_k-1}, a_{m_k-1}) = \xi$.

Also, let a function $f : X \to [0,\infty)$, here X is a metric space, is named lower semi continuous if, for all $a \in X$ and $a_n \in X$ with $\lim_{n\to\infty} a_n = a$, we have $f a \le \lim_{n\to\infty} \inf f a_n$.

Theorem 1. *Let* (X,G_p) *be a complete* G_p *metric space,* $f : X \to X$ *and* $E : X \to CB^{G_p}(X)$ *be* f*-weak compatible continuous mappings such that* $E(X) \subseteq f(X)$ *and*

$$H_{G_p}(E a, Ew, Ev) \le \alpha G_p(f a, f w, f v) \tag{3.3}$$

is satisfied for all $a, w, v \in X$, $\alpha \in (0,1)$. *Therefore there exists a point* $h \in X$ *such that* $fh \in Eh$.

Proof. Let a_0 be an arbitrary point of X and choose $a_1 \in X$ such that $f a_1 \in E a_0$. This is probable from $E a_0 \subseteq f(X)$. If $\alpha = 0$, we have

$$G_p(f a_1, E a_1, E a_1) \le H_{G_p}(E a_0, E a_1, E a_1) = 0$$

$f a_1 \in E a_1$ since $E a_1$ is closed. We given that that $0 < \alpha < 1$ and let $s = \frac{1}{\sqrt{\alpha}}$. Now using description of H_{G_p}, there exists a point $y_1 \in E a_1$ such that

$$G_p(f a_1, y_1, y_1) \le s H_{G_p}(E a_0, E a_1, E a_1).$$

From this $E a_1 \subseteq f(X)$, let $a_2 \in X$ be such that $y_1 = f a_2$. Generally, hands chosen $a_n \in X$ we choose $a_{n+1} \in X$, so that $w_n = v_n = f a_{n+1} \in E a_n$ and

$$G_p(f a_n, y_n, y_n) = G_p(f a_n, f a_{n+1}, f a_{n+1}) \le s H_{G_p}(E a_{n-1}, E a_n, E a_n) \tag{3.4}$$

for each $n \ge 1$. Now using (3.3) we obtain

$$\begin{aligned} G_p(f a_n, f a_{n+1}, f a_{n+1}) &\le s H_{G_p}(E a_{n-1}, E a_n, E a_n) \\ &\le \sqrt{\alpha} G_p(f a_{n-1}, f a_n, f a_n) \end{aligned}$$

for each $n \in \mathbb{N}$. If we continue in similar, we obtain

$$\begin{aligned} G_p(f a_n, f a_{n+m}, f a_{n+m}) &\le G_p(f a_n, f a_{n+1}, f a_{n+1}) \\ &\quad + G_p(f a_{n+1}, f a_{n+2}, f a_{n+2}) + \cdots + G_p(f a_{n+m-1}, f a_{n+m}, f a_{n+m}) \\ &\le (\sqrt{\alpha^{n-1}} + \sqrt{\alpha^n} + \cdots + \sqrt{\alpha^{n+m-2}}) G_p(f a_1, f a_2, f a_2) \\ &= \sqrt{\alpha^{n-1}} (1 + \sqrt{\alpha} + \sqrt{\alpha^2} + \cdots + \sqrt{\alpha^{m-1}}) G_p(f a_1, f a_2, f a_2) \\ &\le \frac{\sqrt{\alpha^{n-1}}}{1 - \sqrt{\alpha}} G_p(f a_1, f a_2, f a_2), \end{aligned}$$

where we get the limit for $m,n \to \infty$, this show that $G_p(f a_n, f a_{m+n}, f a_{m+n}) \to 0$. As $\sqrt{\alpha} < 1$ and X is complete, this demonstrate that $\{f a_n\}$ is a Cauchy sequence converging to some point $h \in X$. Then

$$\lim_{n\to\infty} G_p(f a_n, f a_m, f a_m) = \lim_{n\to\infty} G_p(f a_n, h, h) = G_p(h,h,h) = 0. \tag{3.5}$$

Further, we attain

$$H_{G_p}(E\mathcal{A}_{n-1},E\mathcal{A}_n,E\mathcal{A}_n) \le \alpha G_p(f\mathcal{A}_{n-1},f\mathcal{A}_n,f\mathcal{A}_n).$$

This refer that $\{E\mathcal{A}_n\}$ is a Cauchy sequence in the complete G_p metric space $(CB^{G_p}(X),H_{G_p})$. Thus, let $E\mathcal{A}_n \to N \in CB^{G_p}(X)$. Now, we get

$$\begin{aligned} G_p(h,N,N) &\le G_p(h,f\mathcal{A}_n,f\mathcal{A}_n)+G_p(f\mathcal{A}_n,N,N)-G_p(f\mathcal{A}_n,f\mathcal{A}_n,f\mathcal{A}_n) \\ &\le G_p(h,f\mathcal{A}_n,f\mathcal{A}_n)+G_p(E\mathcal{A}_{n-1},N,N) \to 0. \quad (\text{as } n\to\infty) \end{aligned}$$

Since N is closed, $h \in N$ and the f-weak compatibility and the continuity of f and E refers that

$$H_{G_p}(fN,Eh,Eh) \le H_{G_p}(Eh,N,N),$$
$$G_p(fh,h,h) \le H_{G_p}(Eh,N,N).$$

Now, we have

$$\begin{aligned} G_p(fh,Eh,Eh) &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+G_p(ff\mathcal{A}_{n+1},Eh,Eh)-G_p(ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1}) \\ &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+H_{G_p}(fE\mathcal{A}_n,Eh,Eh) \\ &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+H_{G_p}(fE\mathcal{A}_n,Ef\mathcal{A}_n,Ef\mathcal{A}_n)+H_{G_p}(Ef\mathcal{A}_n,Eh,Eh) \\ &\quad -\inf_{u\in Ef\mathcal{A}_n}\{G_p(u,u,u)\}. \end{aligned}$$

Letting $n\to\infty$ in the upper inequality, we obtain

$$G_p(fh,Eh,Eh) \le H_{G_p}(Eh,N,N).$$

Now, using (3.3), we have

$$\begin{aligned} H_{G_p}(Eh,E\mathcal{A}_n,E\mathcal{A}_n) &\le \alpha G_p(fh,f\mathcal{A}_n,f\mathcal{A}_n) \\ &= \alpha G_p(fh,h,h) \qquad (\text{as } n\to\infty) \\ &\le \alpha H_{G_p}(Eh,N,N). \end{aligned}$$

Then

$$H_{G_p}(Eh,N,N)) \le \alpha H_{G_p}(Eh,N,N).$$

This is only with $H_{G_p}(Eh,N,N)=0$. Therefore, $G_p(fh,Eh,Eh)=0$, $fh \in Eh$ since E closed. □

Corollary 1. *Let (X,G_p) be a complete G_p metric space. Given that $E: X \to CB^{G_p}(X)$ supplied the chase condition*

$$H_{G_p}(E\mathcal{A},Ew,Ev) \le \alpha G_p(\mathcal{A},w,v) \tag{3.6}$$

for all $\mathcal{A},w,v \in X$, $\alpha \in (0,1)$. Therefore, E has a fixed point. That is, $h \in Eh$.

Theorem 2. *Let (X,G_p) be a complete G_p metric space, $f: X \to X$ and $E: X \to K^{G_p}(X)$ be f-weak compatible continuous mappings such that $E(X) \subseteq f(X)$ and*

$$\psi(H_{G_p}(E\mathcal{A},Ew,Ev)) \le \psi(G_p(f\mathcal{A},fw,fv))-\phi(G_p(f\mathcal{A},fw,fv)) \tag{3.7}$$

is supplied for all $\mathcal{A},w,v \in X$, $\psi \in \Psi$ and $\phi \in \Phi$. Therefore, there exists a point $h \in X$ such that $fh \in Eh$.

Proof. Let $\mathscr{A}_0$ be an arbitrary point of X and choose $\mathscr{A}_1 \in X$ such that $f\mathscr{A}_1 \in E\mathscr{A}_0$. This is maybe since $E\mathscr{A}_0 \subseteq f(X)$. In general, having chosen $\mathscr{A}_n \in X$ we choose $\mathscr{A}_{n+1} \in X$, so that $w_n = v_n = f\mathscr{A}_{n+1} \in E\mathscr{A}_n$. Given that there exists $n \in \mathbb{N}$ for which $y_n = y_{n+1}$. We obtain $f\mathscr{A}_n \in E\mathscr{A}_n$. Now assume that $n \in \mathbb{N}$ for which $y_n \neq y_{n+1}$. Since $E\mathscr{A}_n$ is compact, we give that $y_n \in E\mathscr{A}_n$ such that $G_p(f\mathscr{A}_n, y_n, y_n) = G_p(f\mathscr{A}_n, E\mathscr{A}_n, E\mathscr{A}_n)$. This implies that

$$\psi(G_p(f\mathscr{A}_n, y_n, y_n)) = \psi(G_p(f\mathscr{A}_n, E\mathscr{A}_n, E\mathscr{A}_n)) \le \psi(H_{G_p}(E\mathscr{A}_{n-1}, E\mathscr{A}_n, E\mathscr{A}_n)) \tag{3.8}$$

for each $n \ge 1$. Now, using (3.7), we have

$$\psi(G_p(f\mathscr{A}_n, f\mathscr{A}_{n+1}, f\mathscr{A}_{n+1})) \le \psi(G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n)) - \phi(G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n)) \tag{3.9}$$

for each $n \in \mathbb{N}$. Since ψ is nondecreasing, therefore we possess

$$G_p(f\mathscr{A}_n, f\mathscr{A}_{n+1}, f\mathscr{A}_{n+1}) \le G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n).$$

Thus, the sequence $\{G_p(f\mathscr{A}_n, f\mathscr{A}_{n+1}, f\mathscr{A}_{n+1})\}$ is monotone nonincreasing and bounded below and hence there exists $j^* \ge 0$ such that

$$\lim_{n\to\infty} G_p(f\mathscr{A}_n, f\mathscr{A}_{n+1}, f\mathscr{A}_{n+1}) = j^*.$$

Suppose that $j^* > 0$. Letting $n \to \infty$ (3.9), by the continuity of ψ and the lower semi continuity of ϕ it follows that

$$\begin{aligned}\psi(j^*) &\le \psi(j^*) - \lim_{n\to\infty}\inf \phi(G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n)) \\ &\le \psi(j^*) - \phi(j^*).\end{aligned}$$

Since $j^* > 0$, $\phi(j^*) > 0$. Hence

$$\psi(j^*) \le \psi(j^*) - \phi(j^*) < \psi(j^*),$$

a contraction. Then

$$\lim_{n\to\infty} G_p(f\mathscr{A}_n, f\mathscr{A}_{n+1}, f\mathscr{A}_{n+1}) = 0. \tag{3.10}$$

We indicate that the sequence $\{f\mathscr{A}_n\}$ is Cauchy. If $\{f\mathscr{A}_n\}$ is not Cauchy, then by Lemma 5 there exist $\xi > 0$ and subsequences $\{f\mathscr{A}_{m_k}\}$ and $\{f\mathscr{A}_{n_k}\}$ such that (3.1) and (3.2) hold. From (3.7), we have

$$\begin{aligned}\psi(G_p(f\mathscr{A}_{n_k+1}, E\mathscr{A}_{m_k}, E\mathscr{A}_{m_k})) &\le \psi((H_{G_p}(E\mathscr{A}_{n_k}, E\mathscr{A}_{m_k}, E\mathscr{A}_{m_k})) \\ &\le \psi(G_p(f\mathscr{A}_{n_k}, f\mathscr{A}_{m_k}, f\mathscr{A}_{m_k})) - \phi(G_p(f\mathscr{A}_{n_k}, f\mathscr{A}_{m_k}, f\mathscr{A}_{m_k})).\end{aligned} \tag{3.11}$$

Letting $k \to \infty$ in (3.11) and applying Lemma 5, the continuity of ψ, the lower semi continuity of ϕ and (3.10), we have

$$\psi(\xi) \le \psi(\xi) - \phi(\xi),$$

that is a contraction because $\phi(\xi) > 0$. Hence the sequence $\{f\mathscr{A}_n\}$ is Cauchy. Therefore X is complete, these show that $\{f\mathscr{A}_n\}$ is a Cauchy sequence converging to some point $h \in X$. Then

$$\lim_{n\to\infty} G_p(f\mathscr{A}_n, f\mathscr{A}_m, f\mathscr{A}_m) = \lim_{n\to\infty} G_p(f\mathscr{A}_n, h, h) = G_p(h, h, h) = 0. \tag{3.12}$$

We obtain

$$\psi(H_{G_p}(E\mathscr{A}_{n-1}, E\mathscr{A}_n, E\mathscr{A}_n)) \le \psi(G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n)) - \phi(G_p(f\mathscr{A}_{n-1}, f\mathscr{A}_n, f\mathscr{A}_n)).$$

This refer that $\{E\mathcal{A}_n\}$ is a Cauchy sequence in the complete G_p metric space $(K^{G_p}(X),H_{G_p})$. Therefore, let $E\mathcal{A}_n \to N \in K^{G_p}(X)$. Now, we get

$$\begin{aligned}G_p(h,N,N) &\le G_p(h,f\mathcal{A}_n,f\mathcal{A}_n)+G_p(f\mathcal{A}_n,N,N)-G_p(f\mathcal{A}_n,f\mathcal{A}_n,f\mathcal{A}_n)\\ &\le G_p(h,f\mathcal{A}_n,f\mathcal{A}_n)+G_p(E\mathcal{A}_{n-1},N,N)\to 0 \qquad (\text{as } n\to\infty).\end{aligned}$$

Since N is closed, $h \in N$ and the f-weak compatibility and the continuity of f and E implies that

$$H_{G_p}(fN,Eh,Eh) \le H_{G_p}(Eh,N,N), \tag{3.13}$$

$$G_p(fh,h,h) \le H_{G_p}(Eh,N,N). \tag{3.14}$$

Now, we have

$$\begin{aligned}&G_p(fh,Eh,Eh)\\ &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+G_p(ff\mathcal{A}_{n+1},Eh,Eh)-G_p(ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})\\ &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+H_{G_p}(fE\mathcal{A}_n,Eh,Eh)\\ &\le G_p(fh,ff\mathcal{A}_{n+1},ff\mathcal{A}_{n+1})+H_{G_p}(fE\mathcal{A}_n,Ef\mathcal{A}_n,Ef\mathcal{A}_n)+H_{G_p}(Ef\mathcal{A}_n,Eh,Eh)\\ &\quad -\inf_{u\in Ef\mathcal{A}_n}\{G_p(u,u,u)\}.\end{aligned}$$

Letting $n\to\infty$ in the upper inequality, we get

$$G_p(fh,Eh,Eh) \le H_{G_p}(Eh,N,N).$$

Now, using (3.7), we have

$$\begin{aligned}\psi(H_{G_p}(Eh,E\mathcal{A}_n,E\mathcal{A}_n)) &\le \psi(G_p(fh,f\mathcal{A}_n,f\mathcal{A}_n))-\phi(G_p(fh,f\mathcal{A}_n,f\mathcal{A}_n))\\ &= \psi(G_p(fh,h,h))-\phi(G_p(fh,h,h)). \quad (\text{as } n\to\infty)\end{aligned}$$

Therefore, using (3.14), we get

$$\psi(H_{G_p}(Eh,N,N)) \le \psi(H_{G_p}(Eh,N,N))-\phi(H_{G_p}(Eh,N,N)).$$

Therefore $H_{G_p}(Eh,N,N)=0$. Then $G_p(fh,Eh,Eh)=0$, $fh\in Eh$ since E closed. □

Corollary 2. *Let (X,G_p) be a complete G_p metric space. Given that $E: X\to K^{G_p}(X)$ satisfied the following condition*

$$\psi(H_{G_p}(E\mathcal{A},Ew,Ev)) \le \psi(G_p(\mathcal{A},w,v))-\phi(G_p(\mathcal{A},w,v)) \tag{3.15}$$

for all $\mathcal{A},w,v\in X$, $\psi\in\Psi$ and $\phi\in\Phi$. Therefore, there exist a point $h\in X$ such that $h\in Eh$.

Theorem 3. *Let (X,G_p) be a complete G_p metric space, $f: X\to X$ and $E: X\to K^{G_p}(X)$ be f-weak compatible continuous mappings such that $E(X)\subseteq f(X)$ and*

$$\psi(H_{G_p}(E\mathcal{A},Ew,Ev)) \le \alpha(G_p(f\mathcal{A},fw,fv))-\beta(G_p(f\mathcal{A},fw,fv)) \tag{3.16}$$

(i) $\alpha:[0,\infty)\to[0,\infty)$ *is continuous, nondecreasing and* $\alpha(\chi)=0 \Leftrightarrow \chi=0$

(ii) $\beta:[0,\infty)\to[0,\infty)$ *is lower semi continuous and* $\beta(\chi)=0\Leftrightarrow\chi=0$

(iii) $\psi(\chi)-\alpha(\chi)+\beta(\chi)>0$*, for all* $\chi>0$

is satisfied for all $\mathcal{A},w,v\in X$, $\psi\in\Psi$. Therefore, there exists a point $h\in X$ such that $fh\in Eh$.

Proof. Let $\mathcal{A}_0$ be an arbitrary point of X and choose $\mathcal{A}_1 \in X$ such that $f\mathcal{A}_1 \in E\mathcal{A}_0$. This is hands since $E\mathcal{A}_0 \subseteq f(X)$. In general, having chosen $\mathcal{A}_n \in X$ we choose $\mathcal{A}_{n+1} \in X$, so that $w_n = v_n = f\mathcal{A}_{n+1} \in E\mathcal{A}_n$. Given that there exists $n \in \mathbb{N}$ for which $y_n = y_{n+1}$. We attain $f\mathcal{A}_n \in E\mathcal{A}_n$. Now, suppose that $n \in \mathbb{N}$ for which $y_n \neq y_{n+1}$. Since $E\mathcal{A}_n$ is compact, we attain that $y_n \in E\mathcal{A}_n$ such that $G_p(f\mathcal{A}_n, y_n, y_n) = G_p(f\mathcal{A}_n, E\mathcal{A}_n, E\mathcal{A}_n)$. This implies that

$$\psi(G_p(f\mathcal{A}_n, y_n, y_n)) = \psi(G_p(f\mathcal{A}_n, E\mathcal{A}_n, E\mathcal{A}_n)) \leq \psi(H_{G_p}(E\mathcal{A}_{n-1}, E\mathcal{A}_n, E\mathcal{A}_n)) \tag{3.17}$$

for each $n \geq 1$. Now, using (3.16), we have

$$\psi(G_p(f\mathcal{A}_n, f\mathcal{A}_{n+1}, f\mathcal{A}_{n+1})) \leq \alpha(G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n)) - \beta(G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n)) \tag{3.18}$$

for each $n \in \mathbb{N}$. Since ψ and α is nondecreasing, therefore we have

$$G_p(f\mathcal{A}_n, f\mathcal{A}_{n+1}, f\mathcal{A}_{n+1}) \leq G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n).$$

Thus, the sequence $\{G_p(f\mathcal{A}_n, f\mathcal{A}_{n+1}, f\mathcal{A}_{n+1})\}$ is monotone nonincreasing and bounded below and so there exists $j^* \geq 0$ such that

$$\lim_{n\to\infty} G_p(f\mathcal{A}_n, f\mathcal{A}_{n+1}, f\mathcal{A}_{n+1}) = j^*.$$

Suppose that $j^* > 0$. Letting $n \to \infty$ in (3.18), by the continuity of ψ and α and the lower semi continuity of β it follows that

$$\begin{aligned}\psi(j^*) &\leq \alpha(j^*) - \lim_{n\to\infty} \inf \beta(G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n)) \\ &\leq \alpha(j^*) - \beta(j^*).\end{aligned}$$

Hence

$$\psi(j^*) \leq \alpha(j^*) - \beta(j^*),$$

but

$$\psi(j^*) - \alpha(j^*) + \beta(j^*) > 0$$

a contraction. Hence

$$\lim_{n\to\infty} G_p(f\mathcal{A}_n, f\mathcal{A}_{n+1}, f\mathcal{A}_{n+1}) = 0. \tag{3.19}$$

Now, we demonstrate that the sequence $\{f\mathcal{A}_n\}$ is Cauchy. If $\{f\mathcal{A}_n\}$ is not Cauchy, thus using Lemma 5 there exist $\xi > 0$ and subsequences $\{f\mathcal{A}_{m_k}\}$ and $\{f\mathcal{A}_{n_k}\}$ such that (3.1) and (3.2) hold. From (3.16), we have

$$\begin{aligned}\psi(G_p(f\mathcal{A}_{n_k+1}, E\mathcal{A}_{m_k}, E\mathcal{A}_{m_k})) &\leq \psi((H_{G_p}(E\mathcal{A}_{n_k}, E\mathcal{A}_{m_k}, E\mathcal{A}_{m_k})) \\ &\leq \alpha(G_p(f\mathcal{A}_{n_k}, f\mathcal{A}_{m_k}, f\mathcal{A}_{m_k})) - \beta(G_p(f\mathcal{A}_{n_k}, f\mathcal{A}_{m_k}, f\mathcal{A}_{m_k})).\end{aligned} \tag{3.20}$$

Letting $k \to \infty$ in (3.20) and applying Lemma 5, the continuity of ψ and α the lower semi continuity of β and (3.19), we have

$$\psi(\xi) \leq \alpha(\xi) - \beta(\xi),$$

which is a contraction because

$$\psi(\xi) - \alpha(\xi) + \beta(\xi) > 0.$$

Hence the sequence $\{f\mathcal{A}_n\}$ is Cauchy. Therefore X is complete, these give that $\{f\mathcal{A}_n\}$ is a Cauchy sequence converging to some point $h \in X$.

Then

$$\lim_{n\to\infty} G_p(f\mathcal{A}_n, f\mathcal{A}_m, f\mathcal{A}_m) = \lim_{n\to\infty} G_p(f\mathcal{A}_n, h, h) = G_p(h,h,h) = 0. \tag{3.21}$$

We obtain

$$\psi(H_{G_p}(E\mathcal{A}_{n-1}, E\mathcal{A}_n, E\mathcal{A}_n)) \le \alpha(G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n)) - \beta(G_p(f\mathcal{A}_{n-1}, f\mathcal{A}_n, f\mathcal{A}_n)).$$

This refer that $\{E\mathcal{A}_n\}$ is a Cauchy sequence in the complete G_p metric space $(K^{G_p}(X), H_{G_p})$. Therefore, let $E\mathcal{A}_n \to N \in K^{G_p}(X)$. Now, we hold

$$\begin{aligned} G_p(h,N,N) &\le G_p(h, f\mathcal{A}_n, f\mathcal{A}_n) + G_p(f\mathcal{A}_n, N, N) - G_p(f\mathcal{A}_n, f\mathcal{A}_n, f\mathcal{A}_n) \\ &\le G_p(h, f\mathcal{A}_n, f\mathcal{A}_n) + G_p(E\mathcal{A}_{n-1}, N, N) \to 0 \qquad (\text{as } n\to\infty). \end{aligned}$$

Since N is closed, $h \in N$ and the f-weak compatibility and the continuity of f and E implies that

$$H_{G_p}(fN, Eh, Eh) \le H_{G_p}(Eh, N, N), \tag{3.22}$$

$$G_p(fh, h, h) \le H_{G_p}(Eh, N, N). \tag{3.23}$$

Now we have

$$\begin{aligned} G_p(fh, Eh, Eh) \le\ & G_p(fh, ff\mathcal{A}_{n+1}, ff\mathcal{A}_{n+1}) + G_p(ff\mathcal{A}_{n+1}, Eh, Eh) \\ & - G_p(ff\mathcal{A}_{n+1}, ff\mathcal{A}_{n+1}, ff\mathcal{A}_{n+1}) \\ \le\ & G_p(fh, ff\mathcal{A}_{n+1}, ff\mathcal{A}_{n+1}) + H_{G_p}(fE\mathcal{A}_n, Eh, Eh) \\ \le\ & G_p(fh, ff\mathcal{A}_{n+1}, ff\mathcal{A}_{n+1}) + H_{G_p}(fE\mathcal{A}_n, Ef\mathcal{A}_n, Ef\mathcal{A}_n) \\ & + H_{G_p}(Ef\mathcal{A}_n, Eh, Eh) - \inf_{u\in Ef\mathcal{A}_n}\{G_p(u,u,u)\}. \end{aligned}$$

Letting $n \to \infty$ in the upper inequality, we attain

$$G_p(fh, Eh, Eh) \le H_{G_p}(Eh, N, N).$$

Now, using (3.16), we possess

$$\begin{aligned} \psi(H_{G_p}(Eh, E\mathcal{A}_n, E\mathcal{A}_n)) &\le \alpha(G_p(fh, f\mathcal{A}_n, f\mathcal{A}_n)) - \beta(G_p(fh, f\mathcal{A}_n, f\mathcal{A}_n)) \\ &= \alpha(G_p(fh,h,h)) - \beta(G_p(fh,h,h)). \quad (\text{as } n\to\infty) \end{aligned}$$

Then, using (3.23), we possess

$$\psi(H_{G_p}(Eh,N,N)) \le \alpha(H_{G_p}(Eh,N,N)) - \beta(H_{G_p}(Eh,N,N)).$$

Therefore, $H_{G_p}(Eh,N,N) = 0$. Then $G_p(fh, Eh, Eh) = 0$, $fh \in Eh$ since E closed. □

Corollary 3. *Let (X, G_p) be a complete G_p metric space. Given that $E : X \to K^{G_p}(X)$ supplied the chase condition*

$$\psi(H_{G_p}(E\mathcal{A}, Ew, Ev)) \le \alpha(G_p(\mathcal{A}, w, v)) - \beta(G_p(\mathcal{A}, w, v)) \tag{3.24}$$

(i) $\alpha : [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing and* $\alpha(\chi) = 0 \Leftrightarrow \chi = 0$

(ii) $\beta : [0,\infty) \to [0,\infty)$ *is lower semi continuous and* $\beta(\chi) = 0 \Leftrightarrow \chi = 0$

(iii) $\psi(\chi) - \alpha(\chi) + \beta(\chi) > 0$, *for all* $\chi > 0$

for all $\mathcal{A}, w, v \in X$, $\psi \in \Psi$. Therefore E has a fixed point. That is, $h \in Eh$.

Example 1. Let $X=[0,1]$ and define

$$G_p(\mathscr{A},w,v)=\max\{|\mathscr{A}-w|,|w-v|,|\mathscr{A}-v|\},$$

for all $\mathscr{A},w,v\in X$. Thus (X,G_p) is a complete G_p metric space. Also, defined $E:X\to K^{G_p}(X)$ a compact mapping, where

$$E(\mathscr{A})=\left[0,\frac{\mathscr{A}}{8}\right]$$

and $f:X\to X$, where $f\mathscr{A}=\mathscr{A}$ for all $\mathscr{A}\in X$, $E(X)\subseteq f(X)$. Let $\psi(s)=s$ and $\phi(s)=\frac{s}{2}$. Then, from Theorem 2 we obtain

$$\psi(H_{G_p}(E\mathscr{A},Ew,Ev))\le\psi(G_p(f\mathscr{A},fw,fv))-\phi(G_p(f\mathscr{A},fw,fv)).\tag{3.25}$$

By (1.1), we have

$$d_{G_p}(\mathscr{A},w)=2|\mathscr{A}-w|\tag{3.26}$$

for all $\mathscr{A},w\in X$.

To see (3.25), let $\mathscr{A},w,v\in X$. If $\mathscr{A}=w=v=0$, then

$$H_{G_p}(E\mathscr{A},Ew,Ev)=0\le\psi(G_p(f\mathscr{A},fw,fv))-\phi(G_p(f\mathscr{A},fw,fv)).$$

Thus, with out loss of generality, we suppose that $\mathscr{A}\le w\le v$. Therefore

$$\begin{aligned}H_{G_p}(E\mathscr{A},Ew,Ev)&=H_{G_p},\left(\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{w}{8}\right],\left[0,\frac{v}{8}\right]\right)\\&=\max\left\{\begin{array}{l}\sup\limits_{0\le a\le\frac{\mathscr{A}}{8}}G_p\left(a,\left[0,\frac{w}{8}\right],\left[0,\frac{v}{8}\right]\right),\\\sup\limits_{0\le b\le\frac{w}{8}}G_p\left(b,\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{v}{8}\right]\right),\\\sup\limits_{0\le c\le\frac{v}{8}}G_p\left(c,\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{w}{8}\right]\right)\end{array}\right\}.\end{aligned}$$

Since $\mathscr{A}\le w\le v$, we obtain

$$\left[0,\frac{\mathscr{A}}{8}\right]\subseteq\left[0,\frac{w}{8}\right]\subseteq\left[0,\frac{v}{8}\right]$$

which implies that

$$d_{G_p}\left(\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{w}{8}\right]\right)=d_{G_p}\left(\left[0,\frac{w}{8}\right],\left[0,\frac{v}{8}\right]\right)=d_{G_p}\left(\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{v}{8}\right]\right)=0.$$

For each $0\le a\le\frac{\mathscr{A}}{8}$, we possess

$$G_p\left(a,\left[0,\frac{w}{8}\right],\left[0,\frac{v}{8}\right]\right)=d_{G_p}\left(a,\left[0,\frac{w}{8}\right]\right)+d_{G_p}\left(\left[0,\frac{w}{8}\right],\left[0,\frac{v}{8}\right]\right)+d_{G_p}\left(a,\left[0,\frac{v}{8}\right]\right)=0.$$

Moreover, for each $0\le b\le\frac{w}{8}$, we obtain

$$\begin{aligned}G_p\left(b,\left[0,\frac{v}{8}\right],\left[0,\frac{\mathscr{A}}{8}\right]\right)&=d_{G_p}\left(b,\left[0,\frac{\mathscr{A}}{8}\right]\right)+d_{G_p}\left(\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{v}{8}\right]\right)+d_{G_p}\left(b,\left[0,\frac{v}{8}\right]\right)\\&=\begin{cases}0, & \text{if } b\le\frac{\mathscr{A}}{8}\\2b-\frac{\mathscr{A}}{4}, & \text{if } b>\frac{\mathscr{A}}{8}\end{cases}\end{aligned}$$

this show that

$$\sup_{0\le b\le\frac{w}{8}}G_p\left(b,\left[0,\frac{\mathscr{A}}{8}\right],\left[0,\frac{v}{8}\right]\right)=\frac{w}{4}-\frac{\mathscr{A}}{4}.$$

Also, for each $0 \le c \le \frac{v}{8}$, we obtain

$$G_p\left(c,\left[0,\frac{\mathcal{A}}{8}\right],\left[0,\frac{w}{8}\right]\right) = d_{G_p}\left(c,\left[0,\frac{\mathcal{A}}{8}\right]\right) + d_{G_p}\left(\left[0,\frac{\mathcal{A}}{8}\right],\left[0,\frac{w}{8}\right]\right) + d_{G_p}\left(c,\left[0,\frac{w}{8}\right]\right)$$

$$= \begin{cases} 0, & \text{if } c < \frac{\mathcal{A}}{8} \\ 2c - \frac{\mathcal{A}}{4}, & \text{if } \frac{\mathcal{A}}{8} \le c \le \frac{w}{8} \\ 4c - \frac{\mathcal{A}}{4} - \frac{w}{4}, & \text{if } c > \frac{w}{8} \end{cases}$$

which implies that

$$\sup_{0 \le c \le \frac{v}{8}} G_p\left(c,\left[0,\frac{\mathcal{A}}{8}\right],\left[0,\frac{w}{8}\right]\right) = \frac{v}{2} - \frac{\mathcal{A}}{4} - \frac{w}{4}.$$

We conclude that

$$H_{G_p}(E\mathcal{A},Ew,Ev) = \frac{v}{2} - \frac{\mathcal{A}}{4} - \frac{w}{4}.$$

Then by (3.25)

$$H_{G_p}(E\mathcal{A},Ew,Ev) \le \frac{1}{2}\max\{|\mathcal{A}-w|,|w-v|,|\mathcal{A}-v|\}.$$

Thus, this is satisfying the condition of Theorem 2.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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