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Research Article

Common Fixed Points for Hybrid Pair of Maps with CLR-Property in Convex Metric Space

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Abstract. In present work, we prove common fixed point theorem and best proximity point theorem for two pairs of hybrid mappings in convex metric space satisfying $(\psi-\phi)$-contractive conditions under *common limit range property* with respect to q. We prove both theorems for two pairs of hybrid mappings which can be utilized to derive common fixed point and best proximity point theorem including any number of finite mappings. We also present an example to support our main result.

Keywords. Convex metric space; Common limit range property; Common fixed point; Best proximity point; Compatible maps; q-affine; pq-affine

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1. Introduction

Aamri and Moutawakil [1] two prominent mathematicians brought an idea of E.A. property for a single pair of selfmaps in 2002. Undoubtedly, it was an innovative contribution on their part in the field of fixed point theory. Further, this concept of *E.A. property* was generalized by Liu *et al*. [14] for two pair of selfmaps. They came with a new notion of *Common Property* (*E.A.*) in setup of metric space.

With the passage of time and changing methods Rathee and Kumar [13] redefined *E.A.* property with a setup of convex metric space for two selfmaps. Rathee *et al*. [18] have given their contribution by bringing some new changes. They have explained *E.A. Property* for four self maps in convex metric space. In addition to this concept common limit range property was introduced by Sintunavarat and Kumam [23, 24] and Imdad *et al*. [10] generalized this idea for four self maps in metric space and in this paper we have tried to explain the concept of common limit range property in convex metric space for two hybrid pairs in which one map is single valued map and other is multivalued map.

Before going to the main work, we recall some known definitions and results which is required in the sequel.

Definition 1 ([1]). Let A and S be two mappings from a metric space (X,d) into itself. Then the mappings are said to satisfy the property (*E.A.*) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Liu *et al*. [14], an innovative mind in 2005 defined the idea of common property (*E.A.*) for hybrid pair of mappings which also satisfy the (*E.A.*) *Property*.

Definition 2. Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) are said to satisfy the common property *E.A.* if two sequence $\{x_n\}$ and $\{y_n\}$ in X exist such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

Sintunavarat and Kumam [23], in 2011, coined a new idea "*Common Limit Range Property*". Recently, this term has been modified with some new change by Imdad *et al*. [10] by introducing common limit range property to two pairs of self mappings.

Definition 3. A pair (A,S) of self mappings of a metric space (X,d) is said to satisfy common limit range property with respect to S denoted by CLR_S, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

where $t \in S(X)$.

Thus one can conclude that a pair (A,S) justifying the *E.A.* property along with the closedness of subspace finds that CLR_S property more useful with respect to mapping S.

Definition 4. Two pairs (A,S) and (B,T) of self mapping of a metric space (X,d) are said to satisfy common limit range property with respect to mappings S and T, denoted by CLR_{ST} if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

where $t \in S(X) \cap T(X)$.

Definition 5. Let (X,d) be a metric space and $f : X \to CB(X)$ and $T : X \to X$ then the pair $\{f,T\}$ is said to be compatible if and only if $Tfx \in CB(X)$ for each $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \to M \in CB(X)$ and $Tx_n \to t \in M$.

Definition 6. Let (X,d) be a metric space. Two mappings $f : X \to X$ and $T : X \to CB(X)$ are said to satisfy common limit range property of f with respect to T if there exists a sequence $\{x_n\}$ in X and $A \in CB(X)$ such that

$$\lim_{n\to\infty} f(x_n) = f(u) \in A = \lim_{n\to\infty} Tx_n$$

for some $u \in X$.

Remark 7. If $f(X)$ is closed, then a noncompatible hybrid pair (f,T) satisfies the CLR_f with respect to T.

Definition 8 ([13])**.** Let (X,d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is called a convex structure on X if, for all $x, y \in X$ and $\lambda \in [0,1]$, we have

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y), \tag{1.1}$$

for all $u \in X$.

A metric space (X,d) endowed with a convex structure is called convex metric space.

Definition 9. A subset M of a convex metric space (X,d) is called a convex set if $W(x,y,\lambda)$ for all $x, y \in M$ and $\lambda \in [0,1]$. The set M is said to be q-starshaped if there exists $q \in M$ such that $W(x,q,\lambda) \in M$ for all $x \in M$ and $\lambda \in [0,1]$.

Definition 10 ([13])**.** A convex metric space (X,d) is said to satisfy the property I, if for all $x, y, z \in X$ and $\lambda \in [0,1]$,

$$d(W(x,z,\lambda),W(y,z,\lambda)) \le \lambda d(x,y).$$

Definition 11 ([13])**.** Let (X,d) be convex metric space and M be a subset of X. A mapping $I : M \to X$ is said to be

(1) affine, if M is convex and $I(W(x,y,\lambda)) = W(Ix,Iy,\lambda)$ for all $x, y \in M$ and $\lambda \in [0,1]$.

(2) q-affine, if M is q-starshaped and $I(W(x,q,\lambda)) = W(Ix,q,\lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.

Definition 12. If $A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\} \neq \phi$, then the pair (A,B) is said to have P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$:

$$d(x_1,y_1) = d(A,B) \ \text{ and } \ d(x_2,y_2) = d(A,B) \Longrightarrow d(x_1,x_2) = d(y_1,y_2).$$

Definition 13 ([25])**.** Let (X,d) be a metric space. we denote by $CB(X)$ the set of all nonempty closed and bounded subsets of X. The Hausdorff distance $H : CB(X) \times CB(X) \to [0,\infty)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right\},$$

where $d(x,A) = \inf_{y\in A} d(x,y)$.

In 1997, Alber and Guerre-Delabriere [2] introduced the following notion:

Consider the following set of real functions $\Phi = \{\phi : [0,\infty) \to [0,\infty) : \phi$ is lower semi-continuous and $\phi(\{0\}) = \{0\}\}$.

Let us consider the following set of real functions: $\Psi = \{[0,\infty) \to [0,\infty) : \psi$ is continuous non-decreasing and $\psi(\{0\}) = \{0\}\}$.

2. Main Results

Now we state and prove our main results for four mappings justifying Common Limit Range Property in Convex metric space. Firstly, we define CLR-property with respect to q.

Definition 14. Let (X,d) be a convex metric space. Two hybrid pair (f,S) and (g,T) such that $f,g : X \to CB(X)$ and $S,T : X \to X$ are said to satisfy common limit range property (CLR_{ST}) with respect to q if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such th at

$$\lim_{n\to\infty} S_\lambda x_n = S(u) \in C = \lim_{n\to\infty} f x_n \quad\text{and}\quad \lim_{n\to\infty} T_\lambda x_n = T(v) \in D = \lim_{n\to\infty} g y_n$$

for some $u,v \in X$ and $C,D \in CB(X)$ and $Su = Tv$.

Let us pose the following example for $(CLR)_{ST}$-property for hybrid pair of maps:

Example 15. Let $X = R$ endowed with usual metric and let $M = \left[-1, \frac{2}{3}\right]$. Define $f,g : M \to CB(M)$ and $S,T : M \to M$ by:

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } -1 \le x \le \frac{1}{3} \\ \left[\frac{5}{3} - 4x, \frac{1}{3}\right] & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \end{cases} \quad\text{and}\quad S(x) = \begin{cases} \frac{1}{3} & \text{if } -1 \le x \le \frac{1}{3} \\ \frac{x}{2} + \frac{1}{6} & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{3} & \text{if } -1 \le x \le \frac{1}{3} \\ \left[1 - 2x, \frac{1}{3}\right] & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \end{cases} \quad\text{and}\quad T(x) = \begin{cases} \frac{1}{3} & \text{if } -1 \le x \le \frac{1}{3} \\ \frac{x}{4} + \frac{1}{4} & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \end{cases}$$

Then (X,d) is a convex metric space with the convex structure $W(x,y,\lambda) = (\lambda)x + (1-\lambda)y$.

We have to check the following:

(i) f and g is q-affine with $q = \frac{1}{3}$.

(ii) The pair (f,S) and (g,T) satisfying (CLR_{ST})-property with respect to $q = \frac{1}{3}$.

Proof. **(i)** If $x \in \left[-1, \frac{1}{3}\right]$, then $W\left(x, \frac{1}{3}, \lambda\right) = (\lambda)x + (1-\lambda)\frac{1}{3} \in \left[-1, \frac{1}{3}\right]$.

That implies $f\left(W\left(x, \frac{1}{3}, \lambda\right)\right) = W\left(fx, \frac{1}{3}, \lambda\right)$.

Again, if $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then $W\left(x, \frac{1}{3}, \lambda\right) = (\lambda)x + (1-\lambda)\frac{1}{3} \in \left[\frac{1}{3}, \frac{2}{3}\right]$, so we get

$$\begin{aligned} f\left(W\left(x, \frac{1}{3}, \lambda\right)\right) &= \left[\frac{5}{3} - 4\left(W\left(x, \frac{1}{3}, \lambda\right)\right), \frac{1}{3}\right] \\ &= \left[\frac{5}{3} - 4\lambda x - 4(1-\lambda)\frac{1}{3}, \frac{1}{3}\right] \\ &= \left[\frac{1}{3} - 4\lambda x + \frac{4}{3}\lambda, \frac{1}{3}\right] \\ &= \left[\frac{1}{3} + 4\lambda\left(\frac{1}{3} - x\right), \frac{1}{3}\right] \end{aligned}$$

and

$$W\left(fx,\frac{1}{3},\lambda\right)=\bigcup_{a\in f(x)}W\left(a,\frac{1}{3},\lambda\right)\left[\lambda\left(\frac{5}{3}-4x\right)+(1-\lambda)\frac{1}{3},\lambda\left(\frac{1}{3}\right)+(1-\lambda)\frac{1}{3}\right]$$
$$=\left[\frac{1}{3}+\frac{4}{3}\lambda-4\lambda x,\frac{1}{3}\right]$$
$$=\left[\frac{1}{3}+4\lambda\left(\frac{1}{3}-x\right),\frac{1}{3}\right].$$

Thus, $f\left(W\left(x,\frac{1}{3},\lambda\right)\right)=W\left(fx,\frac{1}{3},\lambda\right)$ for all $x\in M$ and hence f is q-affine with $q=\frac{1}{3}$.

Now, we shall prove that g is q-affine with $q=\frac{1}{3}$.

For this, if $x\in\left[-1,\frac{1}{3}\right]$, then $g\left(W\left(x,\frac{1}{3},\lambda\right)\right)=W\left(gx,\frac{1}{3},\lambda\right)$, and

if $x\in\left[\frac{1}{3},\frac{2}{3}\right]$, then $W\left(x,\frac{1}{3},\lambda\right)=(\lambda)x+(1-\lambda)\frac{1}{3}\in\left[\frac{1}{3},\frac{2}{3}\right]$.

Therefore, we have

$$g\left(W\left(x,\frac{1}{3},\lambda\right)\right)=\left[1-2\left(W\left(x,\frac{1}{3},\lambda\right)\right),\frac{1}{3}\right]$$
$$=\left[1-2\left(\lambda x-(1-\lambda)\frac{1}{3}\right),\frac{1}{3}\right]$$
$$=\left[\frac{1}{3}+2\lambda\left(\frac{1}{3}-x\right),\frac{1}{3}\right]$$

and

$$W\left(gx,\frac{1}{3},\lambda\right)=\bigcup_{b\in g(x)}W\left(b,\frac{1}{3},\lambda\right)$$
$$=\left[\lambda-2\lambda x+\frac{1}{3}-\frac{1}{3}\lambda,\lambda\frac{1}{3}+(1-\lambda)\frac{1}{3}\right]$$
$$=\left[\frac{1}{3}+2\lambda\left(\frac{1}{3}-x\right),\frac{1}{3}\right].$$

So, $g\left(W\left(x,\frac{1}{3},\lambda\right)\right)=W\left(gx,\frac{1}{3},\lambda\right)$ for each $x\in M$. This implies that g is q-affine with $q=\frac{1}{3}$.

(ii) Clearly $f\left(\frac{1}{3}\right)=g\left(\frac{1}{3}\right)=\left\{\frac{1}{3}\right\}$.

Consider $x_n=\frac{1}{3}-\frac{1}{n+2}$, $n\geq 1$ and $y_n=\frac{1}{3}-\frac{1}{3n}$, $n\geq 1$,

then for each n, x_n and $y_n\in\left[0,\frac{1}{3}\right]$ and for each $\lambda\in[0,1]$, we have

$$\limsup_{n\to\infty}S_\lambda x_n=W\left(\frac{1}{3},\frac{1}{3},\lambda\right)=\frac{1}{3}\in\left\{\frac{1}{3}\right\}=\lim_{n\to\infty}fx_n$$

and

$$\limsup_{n\to\infty}T_\lambda y_n=W\left(\frac{1}{3},\frac{1}{3},\lambda\right)=\frac{1}{3}\in\left\{\frac{1}{3}\right\}=\lim_{n\to\infty}gy_n$$
$$S\left(\frac{1}{3}\right)=T\left(\frac{1}{3}\right).$$

This implies that the pair (A,S) and (B,T) satisfying (CLR_{ST}) with respect to $q=\frac{1}{3}$ □

Definition 16. Let (X,d) be a convex metric space and A and B be two nonempty subsets of X. A mapping $I : A \to B$ is called pq-affine if A is p-starshaped set and B is q-starshaped set and $I(W(x,p,\lambda)) = W(Ix,q,\lambda)$.

Definition 17 ([18])**.** Let (X,d) be a convex metric space abd A and B be two nonempty subsets of X such that B is q-starshaped set. A pair (f,S) of two nonself maps from A to B to be proximally commuting if for some $\lambda \in [0,1]$ whenever

$$d(x,(Su,q,\lambda)) = d(y,fu) = d(A,B) \Longrightarrow W(Sy,q,\lambda) = fx.$$

Theorem 18. *Let (X,d) be a convex metric space and M be a starshaped subset of a convex metric space with Property I. Let $f,g : M \to CB(M)$ and $S,T : M \to M$ such that the hybrid pairs (f,S) and (g,T) satisfies (CLR_{ST})-property with respect to q and the mappings f, g, S and T are compatible maps. Also, assume that f,g are q-affine, M is compact and*

$$\psi(H(fx,gy)) \le \psi(m(x,y)) - \phi(m(x,y)), \tag{2.1}$$

where

$$m(x,y) = \max\left\{dist([Sx,q],[Ty,q]), \frac{d(fx,[Sx,q])d(gy,[Ty,q])}{1+d([Sx,q],[Ty,q])}, \frac{d([Sx,q],gy)d([Ty,q],fx)}{1+d([Sx,q],[Ty,q])}\right\}$$

then $M \cap F(f) \cap F(g) \cap F(S) \cap F(T) \ne \phi$.

Proof. For each $n \in \mathbb{N}$, we define $T_n : M \to M$ and $S_n : M \to M$ by $T_n(y) = W(Ty,q,\lambda_n)$ and $S_n(x) = W(Sx,q,\lambda_n)$ for all $x,y \in M$ where λ_n is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Now, we have to prove that for each $n \in \mathbb{N}$, the hybrid pair (f,S) and (g,T) are OWC. Since the hybrid pairs (f,S) and (g,T) satisfies CLR_{ST}-property with respect to q therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} S_\lambda x_n = S(u) \in C = \lim_{n\to\infty} fx_n \quad \text{and} \quad \lim_{n\to\infty} T_\lambda y_n = T(v) \in D = \lim_{n\to\infty} gy_n$$

for $u,v \in X$ and $C,D \in CB(X)$. Since M is compact and every compact set is sequentially compact. As a consequence M is sequentially compact so every sequence has a convergent sub sequence say $\{x_m\}$ of $\{x_n\}$ and $\{y_m\}$ of $\{y_n\}$ such that for $u,v \in M$

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} y_n = v.$$

Now, since f, g, S and T are sharing common limit range property with respect to q then for two sequences $\{x_m\}$ and $\{y_m\}$ in M, we have

$$\lim_{m\to\infty} S_\lambda x_m = Su \in C = \lim_{m\to\infty} fx_m \quad \text{and} \quad \lim_{m\to\infty} T_\lambda y_m = Tv \in D = \lim_{m\to\infty} gy_m. \tag{2.2}$$

Also, $S(u) = T(v) = t$ (say). Now, we claim that $S(u) \in f(u)$ and $T(v) \in g(v)$. That is $t \in f(u)$ and $t \in g(v)$.

For this consider

$$\lim_{m\to\infty} S_n x_m = \lim_{m\to\infty} W(Sx_m,q,\lambda_n) = \lim_{m\to\infty} S_{\lambda_n}(x_m) = S(u).$$

By follow this process, we observe

$$\lim_{m\to\infty} S_n x_m = S(u) \in C = \lim_{m\to\infty} fx_m \quad \text{and} \quad \lim_{m\to\infty} T_n y_m = T(v) \in D = \lim_{m\to\infty} gy_m. \tag{2.3}$$

Taking into account eq. (2.1) with $x = u$ and $y = y_m$, we have

$$\psi(H(fu, gy_m)) \le \psi(m(u, y_m)) - \phi(m(u, y_m)), \tag{2.4}$$

where

$$m(u, y_m) = \max\left\{d([Su,q],[Ty_m,q]), \frac{d(fu,[Su,q])d(gy_m,[Ty_m,q])}{1+d([Su,q],[Ty_m,q])}, \frac{d([Su,q],gy_m), d([Ty_m,q],fu)}{1+d([Su,q],[Ty_m,q])}\right\}.$$

Taking limit $m \to \infty$, we find

$$\begin{aligned}\lim_{m\to\infty}\psi(H(fu,gy_m)) &\le \lim_{m\to\infty}[\psi(m(u,y_m)) - \phi(m(u,y_m))] \\ &= \lim_{m\to\infty}\psi(m(u,y_m)) - \lim_{m\to\infty}\phi(m(u,y_m)) \\ \psi(H(fu,D)) &\le \psi\left(\lim_{m\to\infty} m(u,y_m)\right) - \phi\left(\lim_{m\to\infty} m(u,y_m)\right).\end{aligned} \tag{2.5}$$

Consider $m(u, y_m) = \left\{d(S_nu, T_ny_m), \dfrac{d(fu,S_nu)d(gy_m,T_ny_m)}{1+d(S_nu,T_ny_m)}, \dfrac{d(S_nu,gy_md(T_ny_m,fu))}{1+d(S_nu,T_ny_m)}\right\}$.

Taking limit $m \to \infty$ and using eq. (2.2) and (2.3), we have

$$\lim_{n\to\infty} m(u,y_m) = \max\left\{d(Su,Tv), \frac{d(fu,Su)d(Tv,D)}{1+d(Su,Tv)}, \frac{d(Su,D)d(Tv,fu)}{1+d(Su,Tv)}\right\}.$$

This implies $\lim_{m\to\infty} m(u, y_m) = 0$.

Thus eq. (2.5) implies that

$$\begin{aligned} &\psi(H(fu,D)) \le \psi(0) - \phi(0) \\ \Longrightarrow\quad &\psi(H(fu,D)) = 0 \\ \Longrightarrow\quad &(H(fu,D)) = 0.\end{aligned}$$

Since $Tv \in D$. It follows from the definition of Hausdorff metric space that

$$\begin{aligned} &d(fu,Tv) \le H(fu,D) = 0 \\ &d(fu,Tv) = 0 \\ &Tv \in f(u) \\ \Longrightarrow\quad &t \in f(u)\end{aligned}$$

Similarly, we can prove that $t \in g(v)$.

Thus $S(u) \in f(u)$ and $T(v) \in g(v)$.

This implies that u is a coincidence point of f and S and v is a coincidence point of g and T. Now, we shall prove that f and S commute at u. For this we use the fact that f is q-affine and Property I.

$$\begin{aligned} H(S_nfx_m, fS_nx_m) &= H(W(Sfx_m,q,\lambda_n), f(W(Sx_m,q,\lambda_n))) \\ &= H\left(W(Sfx_m,q,\lambda_n), \bigcup_{y_m\in fsx_m} W(y_m,q,\lambda_n)\right) \\ &\le \lambda_n H(Sfx_m, fSx_m).\end{aligned}$$

This implies

$$H(S_n f x_m, f S_n x_m) \le \lambda_n H(S f x_m, f S x_m).$$

Similarly,

$$H(T_n g y_m, g T_n y_m) \le \lambda_n H(T g y_m, g T y_m).$$

Since mappings f, g, S and T satisfies common limit range property with respect to q so mappings also satisfy *E.A.* property with respect to q and the mappings f, g, S and T are compatible. Therefore taking limit $m,n \to \infty$, we have

$$\lim_{m,n\to\infty} H(S_n f x_m, f S_n x_m) \le 0$$

$$\lim_{m,n\to\infty} H(S_n f x_m, f S_n x_m) = 0$$

$$\implies \quad Sfu = fSu$$

Similarly, $Tgv = gTv$.

The pair (f,S) and (g,T) have OWC. Now, we are left with $Su \in f(u)$ and $T(v) \in g(v)$ and $fSu = Sfu$ and $gTv = Tgv$.

Consider

$$St = SSu \in Sfu = f(Su) = f(t)$$

$$Tt = TTv = \in Tgv = gTv = gt$$

$$St \in f(t) \ \text{ and } \ T(t) \in g(t) \tag{2.6}$$

Now, we shall prove that $t \in M \cap F(f) \cap F(g) \cap F(S) \cap F(T)$.

Put $x = t$ and $y = v$ in inequality (2.1)

$$\psi(H(ft, gv)) \le \psi(m(t,v)) - \phi(m(t,v)), \tag{2.7}$$

where

$$m(t,v) = \max\left\{ d([St,q],[Tv,q]), \frac{d(ft,[St,q])d(gv,[Tv,q])}{1+d([St,q],[Tv,q])}, \frac{d([St,q],gv)d([Tv,q],ft)}{1+d([St,q],[Tv,q])} \right\}$$

$$= \max\left\{ d(St,t), 0, \frac{d([St,q],Tv)d([Tv,q],St)}{1+d([St,q],[Tv,q])} \right\}. \tag{2.8}$$

Now consider

$$\frac{d([St,q],Tv)d([Tv,q],St)}{1+d([St,q],[Tv,q])} d([St,q],t) \le 1 + d([St,q],t)$$

$$\frac{d([St,q],t)}{1+d([St,q],t)} \le 1.$$

$$\frac{d([St,q],t)d([Tv,q],St)}{1+d([St,q],t)} \le d([Tv,q],St) = d([t,q],St) \le d(t,St)$$

So $\dfrac{d([St,q],t)d([Tv,q],St)}{1+d([St,q],t)} \le d(t,St)$

From eq. (2.8),

$$m(t,v) = d(St,t). \tag{2.9}$$

Also,

$$H(ft,gv)=\max\left\{\sup_{a\in ft} d(a,gv),\sup_{t\in gv} d(ft,t)\right\}\geq d(St,t)$$

$$\implies\quad d(St,t)\leq H(ft,gv)$$

$$\implies\quad \psi(d(St,t))\leq \psi(H(ft,gv)) \tag{2.10}$$

Using eqs. (2.7), (2.8), (2.9) and (2.10), we have

$$\implies\quad \psi(d(St,t))\leq \psi(d(St,t))-\phi(d(St,t))$$

$$-\phi(d(St,t))\geq 0$$

$$\phi(d(St,t))\leq 0$$

$$\implies\quad \phi(d(St,t))=0.$$

This implies $d(St,t)=0$. Thus $St=t$.

Hence $t=St\in f(t)$ and similarly $t=Tt\in g(t)$.

Hence $t\in M\cap F(f)\cap F(g)\cap F(S)\cap F(T)$. □

Theorem 19. *Let (A,B) be a pair of nonempty closed subsets of a convex metric space (X,d). Suppose that A is a p-starshaped and B is q-starshaped with property I. Also suppose that A_0 is closed. Let S and T be continuous non self maps from A to B and $f,g:A\to CB(B)$ satisfying the conditions:*

(i) *Two pairs (f,S) and (g,T) have CLR-property with respect to q and commute proximally.*

(ii) $f(A_0)\subseteq T_n(A_0)$, $g(A_0)\subseteq S_nA_0$, $S_nA_0\subseteq B_0$, $T_nA_0\subseteq B_0$.

(iii) *The pair (A,B) has P-property.*

(iv) *f, g, S and T satisfying the condition*

$$\psi(H(fx,gy))\leq \psi(m(x,y))-\phi(m(x,y)) \tag{2.11}$$

for all $x,y\in X$, where

$$m(x,y)=\max\left\{d([Sx,q],[Ty,q]),\frac{d(fx,[Sx,q])d(gy,[Ty,q])}{1+d([Sx,q][Ty,q])},\frac{d([Sx,q],gy)d([Ty,q],fx)}{1+d([Sx,q],[Ty,q])}\right\}.$$

(v) *S and T are pq-affine.*

Proof. Now for fix x_0 in A_0, since $f(A_0)\subseteq T_n(A_0)$ then there exists an element x_1 in A_0 such that $T_n(x_1)\in f(x_0)$. Similarly, a point $X_2\in A_0$ can be chosen such that $S_n(x_2)\in g(x_1)$, continuing this process we obtain a sequence $\{x_n\}\in A_0$ such that

$$T_n(x_{2n+1})\in f(x_{2n})\ \text{ and }\ S_n(x_{2n+2})\in g(x_{2n+1}). \tag{2.12}$$

Since $S_n(A_0)\subseteq B_0$ and $T_nA_0\subseteq B_0$, there exists $\{u_n\}\in A_0$ such that

$$d(u_{2n},S_nx_{2n})=d(A,B)\ \text{ and }\ d(u_{2n+1},T_nx_{2n+1})=d(A,B). \tag{2.13}$$

As the pair (A,B) has P-property, then by using eq. (2.13)

$$d(u_{2n},u_{2n+1})=d(S_nx_{2n},T_nx_{2n+1}). \tag{2.14}$$

Now $d(S_nx_{2n},T_nx_{2n+1})\leq H(fx_{2n},gx_{2n-1})$. This implies $d(u_{2n},u_{2n+1})\leq H(fx_{2n},gx_{2n-1})$.

$$\psi(d(u_{2n},u_{2n+1}))\leq \psi(H(fx_{2n},gx_{2n-1}))$$

$$\leq \psi(m(x_{2n},x_{2n-1}))-\phi(m(x_{2n},x_{2n-1})), \tag{2.15}$$

where

$$\begin{aligned} m(x_{2n},x_{2n-1}) &= \max\left\{d([Sx_{2n},q],[Tx_{2n-1},q]), \frac{d(fx_{2n},[Sx_{2n},q]d(gx_{2n-1},[Tx_{2n-1},q]))}{1+d([Sx_{2n},q],[Tx_{2n-1},q])},\right. \\ &\qquad \left.\frac{d([Sx_{2n},q],gx_{2n-1})d([Tx_{2n-1},q],fx_{2n})}{1+d([Sx_{2n},q],[Tx_{2n-1},q])}\right\} \\ &= \max\left\{d(u_{2n},u_{2n-1}), \frac{d(fx_{2n},[Sx_{2n},q])d(gx_{2n-1},[Tx_{2n-1},q])}{1+d([Sx_{2n},q][Tx_{2n-1},q]),0}\right\}. \end{aligned}$$

Now consider

$$\frac{d(fx_{2n},[Sx_{2n},q])d(gx_{2n-1},[Tx_{2n-1},q])}{1+d([Sx_{2n},q][Tx_{2n-1},q])} \leq \frac{d(T_nx_{2n+1},S_nx_{2n})d(S_nx_{2n},T_nx_{2n-1})}{1+d(S_nx_{2n},T_nx_{2n-1})}. \tag{2.16}$$

For this

$$\begin{aligned} & d(S_nx_{2n},T_nx_{2n-1}) \leq 1+d(S_nx_{2n},T_nx_{2n-1}) \\ \implies & \frac{d(S_nx_{2n},T_nx_{2n-1})}{1+d(S_nx_{2n},T_nx_{2n-1})} \leq 1 \end{aligned}$$

Thus eq. (2.16) implies that

$$\frac{d(fx_{2n},Sx_{2n})d(gx_{2n-1},[Tx_{2n-1},q])}{1+d([Sx_{2n},q],[Tx_{2n-1},q])} \leq d(T_nx_{2n+1},S_nx_{2n}) = d(u_{2n+1},u_{2n}).$$

Thus eq. (2.15) becomes

$$m(x_{2n},x_{2n-1}) = \max\{d(u_{2n},u_{2n-1}),d(u_{2n+1},u_{2n})\}.$$

Now if $m(x_{2n},x_{2n-1}) = d(u_{2n+1},u_{2n})$ then eq. (2.14)

$$\psi(d(u_{2n},u_{2n+1})) \leq \psi(d(u_{2n+1},u_{2n}))-\phi(d(u_{2n},u_{2n+1})).$$

This implies $d(u_{2n},u_{2n+1}) = 0$.

This implies $\langle u_n\rangle$ is a Cauchy sequence and when $m(x_{2n},x_{2n-1}) = d(u_{2n},u_{2n-1})$ then eq. (2.14) becomes

$$\begin{aligned} &\psi(d(u_{2n},u_{2n+1})) \leq \psi(d(u_{2n},u_{2n-1}))-\phi(d(u_{2n},u_{2n-1})) \\ &\psi(d(u_{2n},u_{2n+1})) < \psi(d(u_{2n},u_{2n-1})) \\ &\quad d(u_{2n},u_{2n+1}) < d(u_{2n},u_{2n-1}). \end{aligned}$$

Thus in both cases

$$d(u_{2n},u_{2n+1}) \leq d(u_{2n},u_{2n-1}).$$

Also, it can be written as for $\lambda_n \in (0,1)$,

$$\begin{aligned} d(u_{2n},u_{2n+1}) &\leq \lambda_n d(u_{2n-1},u_{2n}) \leq d(u_{2n},u_{2n-1}) \\ d(u_n,u_{n+1}) &\leq \lambda_n d(u_{n-1},u_n) \\ &\leq \lambda_n(\lambda_n d(u_{n-2},u_{n-1})) \\ &\vdots \\ &\leq (\lambda_n)^n d(u_0,u_1) \end{aligned} \tag{2.17}$$

$$d(u_n,u_{n+1}) \leq (\lambda_n)^n d(u_0,u_1) \tag{2.18}$$

Let $m, n \in \mathbb{N}$ and $m < n$, we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_n) \\ &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + d(u_{m+2}, u_n) \\ &\vdots \\ &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \ldots + d(u_{n-1}, u_n). \end{aligned}$$

Using eq. (2.18)

$$\begin{aligned} d(u_m, u_n) &\leq (\lambda_n)^m d(u_0, u_1) + (\lambda_n)^{m+1} + \ldots + (\lambda_n)^{n-1} d(u_0, u_1) \\ &\leq (\lambda_n)^m [d(u_0, u_1) + \lambda_n d(u_0, u_1) + \ldots + (\lambda_n)^{n-m-1} d(u_0, u_1)] \\ &= (\lambda_n)^m d(u_0, u_1)[1 + \lambda_n + (\lambda_n)^2 + \ldots + (\lambda_n)^{n-m-1}] \\ &= (\lambda_n)^m d(u_0, u_1) \frac{1}{1 - \lambda - n} \to 0 \end{aligned}$$

when $m \to \infty$.
This implies $d(u_m, u_n) \to 0$ as $m \to \infty$.
Hence $\{u_n\}$ is a Cauchy sequence. Since $\{u_n\} \subset A_0$ and A_0 is a closed subset of the convex metric space (X, d), we can find $u \in A_0$ such that $\lim\limits_{n \to \infty} u_n = u$. Since (f, S) and (g, T) have common limit range property with respect to q so there exists a sequence $\{u_m\}$ in A such that

$$\lim_{n \to \infty} S_\lambda u_m = S(p) \in C = \lim_{m \to \infty} f u_m \quad \text{and} \quad \lim_{m \to \infty} T_\lambda u_m = T(r) \in D = \lim_{m \to \infty} g y_{u_m}.$$

Considering

$$\lim_{m \to \infty} = \lim_{m \to \infty} W(T u_m, q, \lambda_n) = \lim_{m \to \infty} T_{\lambda_n} u_m = T(r).$$

Thus, we have

$$\lim_{m \to \infty} T_n u_m = T(r) \in D = \lim_{m \to \infty} g(u_m) \tag{2.19}$$

and similarly

$$\lim_{m \to \infty} S_n u_m = S(p) \in C = \lim_{m \to \infty} f(u_m). \tag{2.20}$$

From eqs. (2.19) and (2.20), we obtain

$$S_n u \in f(u) \quad \text{and} \quad T_n u \in g(u).$$

As limit $n \to \infty$

$$S u \in f(u) \quad \text{and} \quad T(u) \in g(u).$$

Since $S_n(A_0) \subseteq B_0$, there exists $x \in A_0$ such that

$$d(A, B) = d(x, S_n u) \geq d(x, f u), \tag{2.21}$$

where $x \in A$ and $f(u) \subseteq B$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$

$$\implies \quad d(A, B) \leq d(a, B) \text{ for } a \in A.$$

This implies $d(A, B) \leq d(x, f(u))$ and $d(A, B) = d(x, f u)$.
Hence $d(A, B) = d(x, f u) = d(x, S_n u)$.
Similarly, $d(A, B) = d(x, g u) = d(x, T_n u)$

$$d(x, f u) = d(x, S_n u) = d(x, g u) = d(x, T_n u) = d(A, B). \tag{2.22}$$

Taking limit $n \to \infty$

$$d(x, fu) = d(x, Su) = d(x, gu) = d(x, Tu) = d(A, B). \tag{2.23}$$

As (f, S) and (g, T) are proximally commuting so $Sx \in fx$ and $Tx \in gx$. Since $S_n A_0 \subseteq B_0$, there exists $z \in A_0$ such that

$$d(z, S_n x) = d(z, fx) = d(z, gx) = d(z, T_n x) = d(A, B). \tag{2.24}$$

Taking limit $n \to \infty$

$$d(z, Sx) = (z, fx) = d(z, gx) = d(z, Tx) = d(A, B). \tag{2.25}$$

Because the pair (A, B) has p-property so by using eq. (2.24) and (2.25)

$$d(x, z) = d(Su, Tx) \le H(fu, gx)$$

$$\psi(d(x, z)) \le \psi(H(fu, gx))$$

$$\le \psi(m(u, x)) - \phi(m(u, x)), \tag{2.26}$$

where

$$m(u, x) = \max\left\{d([Su, q], [Tx, q]), \frac{d(fu, [Su, q])d(gx, [Tx, q])}{1 + d([Su, q], [Tx, q])}, \frac{d([Su, q], gx)d([Tx, q]), fu}{1 + d([Su, q][Tx, q])}\right\}$$

$$= d(x, z).$$

Thus from eq. (2.26)

$$\psi(d(x, z)) \le \psi(d(x, z)) - \phi(d(x, z))$$

$$d(x, z) = 0$$

$$\implies \quad x = z.$$

Hence

$$d(A, B) = d(x, fx) = d(x, gx) = d(x, Sx) = d(x, Tx). \tag{2.27}$$

Suppose that y is another best proximity point of the mapping f, g, S and T such that

$$d(A, B) = d(y, fy) = d(y, gy) = d(y, Sy) = d(y, Ty). \tag{2.28}$$

Then by using P-property and using (2.27) and (2.28) $x = y$.
Hence the result. □

Conclusion

In this note, we defined common limit range property in the context of convex metric space for two pairs of hybrid mappings in which one mapping is single valued and other is multivalued. Due to this, we have been able to obtained a set of common fixed point and best proximity point The concept plays an important role in solving many kind of physical science problems which can be recast in terms of common fixed point problems.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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