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The Analytic Reformulation of the Hadwiger Conjecture

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Abstract. 0. The *Hadwiger conjecture* [recall that the famous four-color problem is a special case of the Hadwiger conjecture] states that every graph G satisfies $\chi(G) \leq \eta(G)$ [where $\chi(G)$ is the *chromatic number of* G, and $\eta(G)$ is the *hadwiger number* of G (i.e. the maximum of p such that G is *contractible* to the complete graph K_p)]. In this paper, via an original speech and simple results, we rigorously simplify the understanding of the Hadwiger conjecture. It will appear that to solve the famous Hadwiger conjecture is equivalent to solve an analytic conjecture stated on a very small class of graphs.

1. The *Hadwiger conjecture* (see [3] or [4] or [6] or [7] or [8] or [9] or [10]) and the *Berge problem* (see [2] or [4] or [5] or [6] or [8]) are well known. Recall in a graph $G = [V(G), E(G), \chi(G), \omega(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number, $\omega(G)$ is the clique number and $\bar{G}$ is the complementary graph of G. The *Hadwiger conjecture* [recall that the famous four-color problem is a special case of the Hadwiger conjecture] states that every graph G satisfies $\chi(G) \leq \eta(G)$ [where $\eta(G)$ is the *hadwiger number* of G (i.e. the maximum of p such that G is *contractible* to the complete graph K_p)]. We say that a graph B is *berge* if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. Indeed, the *Berge problem* (see [4] or [5] or [6]) consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B [we recall (see [**0**]), that the *Berge problem* was solved in a paper of 146 pages long by Chudnovsky, Robertson, Seymour and Thomas]. In [4], it is presented an original investigation around the *Berge problem* and the *Hadwiger conjecture*, and, via two simple Theorems, it is shown that the *Berge problem* and the *Hadwiger conjecture* are curiously resembling, so resembling that they seem identical [indeed, they can be restated in ways that resemble each other (see [4])]. *Now, in this paper, via only original speech and results, we rigorously simplify the understanding of the Hadwiger conjecture. Moreover, it will appear that to solve the Hadwiger conjecture is equivalent to solve an analytic conjecture stated on a very small class of graphs*.

Introduction

The Hadwiger conjecture states that every graph G is $\eta(G)$ colorable [i.e. we can color all vertices of G with $\eta(G)$ colors such that two adjacent vertices do

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not receive the same color. The integer $\eta(G)$ is the *hadwiger number* of G and is the maximum of p such that G is *contractible* to the complete graph K_p]. In this paper, we introduce some definitions that are not standard in the literature of Graph Theory, and, using these non-standard definitions mixing with original results, we rigorously simplify the understanding of the Hadwiger conjecture. Moreover, it will appear that to solve the Hadwiger conjecture is equivalent to solve an analytic conjecture stated on a very small class of graphs [for more information around the four-color problem and the Hadwiger conjecture, see [3] and [10], or see [9] and [10]]. All results of this paper are original, and therefore, are not related to investigations that have been done on the Hadwiger conjecture in the past. That being so, this paper is divided into two sections. In section 1 (Prologue), we recall standard definitions known in Graph Theory; we also recall (see [4]) the graph parameter denoted by τ [the graph parameter τ (see [4]) is called the *hadwiger index*], and, via the parameter τ, we recall (see [4]) *the surgical reformulation of the Hadwiger conjecture*. This surgical reformulation is simple and determining for the simplification of the Hadwiger conjecture. After the stating of the surgical reformulation of the Hadwiger conjecture, we define *relative subgraphs* and *uniform graphs* [*relative subgraphs* and *uniform graphs* are crucial for the proof of the result which rigorously simplifies the understanding of the Hadwiger conjecture, and the resulting analytic conjecture] and we give some properties concerning these graphs. These properties are elementary and play an important role for the simplification of the Hadwiger conjecture. In section 2 we introduce another graph parameter denoted by a [the graph parameter a is called the *hadwiger caliber* and is related to the *hadwiger index* τ], and using the parameter a, we give *the strong reformulation of the Hadwiger conjecture*. This strong reformulation is determinant and crucial for the simplification and the understanding of the *Hadwiger conjecture*. Moreover, using this strong reformulation of the *Hadwiger conjecture*, it will immediately follow that to prove the Hadwiger conjecture is equivalent to prove an analytic conjecture stated on a very small class of graphs. Here, every graph is finite, is simple and undirected. We start

1. **Prologue**

Standard definitions, some non-standard definitions, the surgical reformulation of the Hadwiger conjecture, uniform graphs, relative subgraphs and natural isomorphisms.

In a graph $G = [V(G), E(G)]$, $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. $\bar{G}$ is the *complementary* graph of G [recall that $\bar{G}$ is the *complementary* graph of G if $V(G) = V(\bar{G})$ and two vertices are adjacent in G if and only if they are not adjacent in $\bar{G}$]. A graph F is a *subgraph* of a graph G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that F is an *induced subgraph* of G by Y, if F is a subgraph of G such that $V(F) = Y$, $Y \subseteq V(G)$, and two vertices of F are adjacent in F if and only if they are adjacent in G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G *induced* by $V(G) \setminus X$. A *clique* of a graph G is a subgraph of G that is *complete*; it is necessarily an induced subgraph [recall that a graph K is *complete* if every pair of vertices of K is an edge of K]; $\omega(G)$ is the size of a largest clique of G, and $\omega(G)$ is called the *clique number* of G. A *stable set* of a graph G is a set of vertices of G that induces a subgraph with no edges; $\alpha(G)$ is the size of a largest stable set of G, and $\alpha(G)$ is called the *stability number* of G [observe that the *stability* number of G is clearly the *clique number* of $\bar{G}$]. The *chromatic number* of a graph G [denoted by $\chi(G)$], is the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color. The *hadwiger number* of a graph G [denoted by $\eta(G)$], is the maximum of p such that G is *contractible* to the complete graph K_p. [[Recall that, if e is an edge of G incident to x and y, we can obtain a new graph from G by removing the edge e and identifying x and y so that the resulting vertex is incident to all those edges (other than e) originally incident to x or to y. This is called *contracting* the edge e. If a graph F can be obtained from G by a succession of such edge-contractions, then, G is *contractible* to F. The maximum of p such that G is contractible to the complete graph K_p is the *hadwiger number* of G, and is denoted by $\eta(G)$]]. Clearly we have:

Assertion 1.0. *Let G be a graph and F be a subgraph of G. Then $\omega(G) \leq \chi(G)$ and $\eta(F) \leq \eta(G)$.*

Definitions 0. We recall (see [4]) that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$. *trpl*(F) denotes the set of all true pals of F, and Ω denotes the set of all *complete multipartite graphs* [[i.e. the set of all graphs Q which are *complete $\omega(Q)$-partite* [recall that a graph Q is a *complete $\omega(Q)$-partite* graph, if there exists a partition $\Xi(Q) = \{Y_1, \ldots, Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable sets, such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q]]]. It is immediate that $\chi(Q) = \omega(Q)$ for all $Q \in \Omega$. It is also immediate that, for every $Q \in \Omega$, the partition $\Xi(Q) = \{Y_1, \ldots, Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable sets is canonical, and every $Y \in \Xi(Q)$ is a *maximal stable set* of Q.

So $G \in$ *trpl*(F) **means** G is a true pal of F, and $Q \in \Omega$ **means** Q is a complete $\omega(Q)$-partite graph [We **recall** (see above) that Ω is the set of all *complete multipartite graphs*. For example, if G is a multipartite graph with $\omega(G) = 1$, then $G \in \Omega$; if G is a multipartite graph with $\omega(G) = 2$, then $G \in \Omega$;if G is a multipartite graph with $\omega(G) = 3$, then $G \in \Omega$;...etc]. Now, using the previous definitions, then the following Assertion becomes immediate.

Assertion 1.1. *Let F be a graph. Then, there exists a graph $P \in \Omega$ such that P is a true pal of F [i.e. there exists $P \in \Omega$ such that $P \in$ trpl(F)].*

Proof. [Indeed, let F be a graph and let $\Xi(F) = \{Y_1, \dots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(F)$; (ii) $\Xi(Q) = \{Y_1, \dots, Y_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$). Now put $Q = P$; Assertion 1.1 follows.] □

Using Assertion 1.1, let us define.

Definitions 1. We say that a graph P is a *parent* of a graph F, if $P \in \Omega \cap trpl(F)$. In other words, a graph P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F [note that such a P clearly exists, via Assertion 1.1]. *parent*(F) denotes the set of all parents of F.

So $P \in parent(F)$ **means** P is a parent of F. Using the definition of a parent and the definition of a true pal, then the following two Assertions are immediate.

Assertion 1.2. *Let F be a graph and let $P \in parent(F)$; then $\chi(F) = \chi(P) = \omega(P)$.*

Assertion 1.3. *Let G be a graph. Then, there exists a graph S such that G is a true pal of S and $\eta(S)$ is minimum for this property.*

Definition 2. Let G be a graph and put $\mathscr{A}(G) = \{H; G \in trpl(H)\}$; [clearly $\mathscr{A}(G)$ is the set of all graphs H, such that G is a true pal of H]. Then the *hadwiger index* of G (see [4]) is denoted by $\tau(G)$, where $\tau(G) = \min_{F \in \mathscr{A}(G)} \eta(F)$.

Using Assertion 1.3, then it becomes immediate to see that for every graph G, $\tau(G)$ exists and is well defined. Clearly we have:

Proposition 1.4. *Let $G \in \Omega$; if $\omega(G) \leq 1$ or if G is a complete graph, then $\eta(G) = \omega(G) = \tau(G) = \chi(G)$.*

Proposition 1.5 ([4]). *Let F be a graph and let $G \in trpl(F)$. Then $\tau(G) \leq \tau(F)$.*

Corollary 1.6 ([4]). *Let F be a graph and let $P \in parent(F)$; then $\tau(P) \leq \tau(F)$.*

Now, the following is the surgical reformulation of the Hadwiger conjecture.

Theorem 1.7 ([4]) (The surgical reformulation of the Hadwiger conjecture). *The following are equivalent.*

(1) *The Hadwiger conjecture holds (i.e. For every graph H, we have $\chi(H) \leq \eta(H)$).*
(2) *For every graph F, we have $\chi(F) \leq \tau(F)$.*
(3) *For every $G \in \Omega$, we have $\omega(G) = \tau(G)$.*

In section 2, *the strong reformulation of Hadwiger* which is crucial for the rigorous simplification of the Hadwiger conjecture, will be implicitly based on the surgical reformulation of the Hadwiger conjecture given by Theorem 1.7. Now, we are going to define *uniform graphs* and *relative subgraphs* [properties related to *uniform graphs* and *relative subgraphs* are elementary, and curiously, are also crucial for the proof of the result which rigorously simplifies the Hadwiger conjecture and the resulting analytic conjecture]. Before, let us define.

Definitions 3. An *optimal coloration* of a graph G is a partition $\Xi(G) = \{Y_1, \dots, Y_{\chi(G)}\}$ of $V(G)$ into $\chi(G)$ stable sets; $\ominus(G)$ denotes the set of all optimal colorations of G.

So, $\Xi(G) \in \ominus(G)$ **means** $\Xi(G)$ is an optimal coloration of G.

Definition 4. Let G be a graph and let $\Xi(G) \in \ominus(G)$. We say that $\Xi(G)$ is the *canonical coloration* of G, if and only if $\ominus(G) = \{\Xi(G)\}$ [observe that such a canonical coloration does not always exists].

Using the definition of $\ominus(G)$, then the following Assertion is immediate.

Assertion 1.8. *Let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$. Then $\ominus(G) = \{\Xi(G)\}$ [i.e. $\Xi(G)$ is the canonical coloration of G, via* Definition 4].

So, let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$; then Assertion 1.8 *clearly says that $\Xi(G)$ is the canonical coloration of G [i.e. we have no choice, since $\ominus(G) = \{\Xi(G)\}$].*

Definitions 5. For a set X, $card(X)$ is the cardinality of X. Now let $G \in \Omega$, and let $\Xi(G) \in \ominus(G)$ [*note $\Xi(G)$ is the canonical coloration of G, via* Assertion 1.8 *and* Definition 4]; then we define $T(G)$, $t(G)$, $N_2(G)$ and $n(G)$ as follows: $T(G) = \{x \in V(G); \exists\ Y \in \Xi(G)$, and $Y = \{x\}\}$, $t(G) = card(T(G))$,
$N_2(G) = \{Y \in \Xi(G); card(Y) \geq 2\}$, and $n(G) = card(N_2(G))$.

It is immediate that these definitions make sense, since $G \in \Omega$, and so $\Xi(G)$ is the canonical coloration of G, via Assertion 1.8. Using Definitions 5, we clearly have:

Assertion 1.9. *Let $G \in \Omega$ and let $T(G)$. Now put $G' = G \setminus T(G)$ [note G' is the induced subgraph of G by $V(G) \setminus T(G)$]. Then we have the following three properties.*
(1.9.0) *$G' \in \Omega$.*
(1.9.1) *$\omega(G') = n(G)$.*
(1.9.2) *$\omega(G) = n(G) + t(G) = \omega(G') + t(G)$.*

Now we define *uniform* graphs and *relative* subgraphs.

Definition 6 (Fundamental 0). Let G be a graph and let $\Xi(G) \in \ominus(G)$ [see Definitions 3]; we say that G is *uniform*, if $G \in \Omega$ and for all $Y \in \Xi(G)$, we have $card(Y) = \alpha(G)$.

This definition makes sense, since $G \in \Omega$ and so $\Xi(G)$ is canonical (by using Assertion 1.8). Using the definition of a uniform graph, then the following three Assertions are immediate.

Assertion 1.10. *If G is a complete graph or if $V(G) = \emptyset$ or if $\omega(G) \leq 1$, then G is uniform.*

Assertion 1.11. *Let G be uniform. We have the following three properties.*

(1.11.0) *If G is a complete graph or if $\alpha(G) \leq 1$ or if $n(G) = 0$, then $\omega(G) = t(G)$.*
(1.11.1) *If G is not a complete graph or if $\alpha(G) > 1$ or if $n(G) \neq 0$, then $\omega(G) = n(G)$.*
(1.11.2) *If $t(G) \neq 0$, then G is a complete graph.*

Assertion 1.12. *Let F be a graph (F is not necessarily in Ω); then there exist a uniform graph P such that P is a parent of F [see* Definitions 1 *for the meaning of parent].*

Proof. [**if** $\omega(F) \leq 1$, clearly F is uniform (use Assertion 1.10); now put $P = F$, clearly P is a uniform graph which is a parent of F. Now, **if** $\omega(F) \geq 2$, let $\Xi(F) = \{Y_1, \ldots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $\Xi(Q) = \{Z_1, \ldots, Z_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that, $x \in Z_j \in \Xi(Q)$, $y \in Z_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q; (ii) For every $j = 1, 2, \ldots, \chi(F)$ and for every $Z_j \in \Xi(Q) = \{Z_1, \ldots, Z_{\chi(F)}\}$, $card(Z_j) = \alpha(F)$. Clearly $Q \in \Omega$, $card(V(Q)) = \chi(F)\alpha(F)$, Q is uniform, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly “isomorphic” to a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is “isomorphic” to a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$) and Q is uniform. Using the previous and the definition of a parent, then we immediately deduce that Q is a uniform graph which is a parent of F. Now put $Q = P$; Assertion 1.12 follows.] □

Definition 7 (Fundamental 1). Let G and F be uniform. Let $\ominus(G) = \{\Xi(G)\}$ and $\ominus(F) = \{\Xi(F)\}$. We say that F is a *relative subgraph* of G, **if** $\Xi(F) \subseteq \Xi(G)$.

It is immediate that this definition makes sense, since in particular $(G, F) \in \Omega \times \Omega$ [because G and F are uniform], and so $\Xi(G)$ and $\Xi(F)$ are canonical (i.e. $\ominus(G) = \{\Xi(G)\}$ and $\ominus(F) = \{\Xi(F)\}$). *It is also immediate that relative subgraphs are defined for uniform graphs, and only for uniform graphs.* Using the definition of a relative subgraph, then the following three assertions are immediate.

Assertion 1.13. *Let R and P be uniform such that $\omega(P) \geq 1$ and $\omega(R) \geq 1$. If R is a relative subgraph of P, then $\alpha(R) = \alpha(P)$ and $\omega(R) \leq \omega(P)$.*

Assertion 1.14. *Let P be uniform and let R be a relative subgraph of P. Then we have the following.*

(1.14.0) $\alpha(R) = 0$ *or* $\alpha(R) = \alpha(P)$ *[the law of all or nothing].*
(1.14.1) **If** $\alpha(R) \geq 1$ *or if* $\omega(R) \geq 1$*, then* $\alpha(R) = \alpha(P)$ *and* $\omega(R) \leq \omega(P)$*.*
(1.14.2) R *is uniform.*

Assertion 1.15. *Let* G *be uniform and let* R *be a relative subgraph of* G*. Now let* $\Xi(G)$ *be the canonical coloration of* G*, and let* $\Xi(R)$ *be the canonical coloration of* R*. Then we have the following two properties.*
(1.15.0) $\omega(R) = \omega(G) \Leftrightarrow R = G \Leftrightarrow \Xi(R) = \Xi(G) \Leftrightarrow V(R) = V(G)$*.*
(1.15.1) $\omega(R) < \omega(G) \Leftrightarrow$ *there exists* $Y \in \Xi(G)$ *such that* R *is a relative subgraph of* $G \setminus Y$*.*

We will see in section 2 that uniform graphs play a major role in the proof of Theorem which rigorously simplifies the Hadwiger conjecture and the resulting analytic conjecture. That being so, uniform graphs have also nice properties related to *isomorphisms* [recall that two graphs are *isomorphic* if there exists a one to one correspondence between their vertex set that preserves adjacency]. Using the definition of a uniform graph, the definition of a relative subgraph and the definition of isomorphism of graphs, then the following two assertions are immediate.

Assertion 1.16 (Natural Isomorphism 1). *Let* P *and* Q *be uniform. Then the following are equivalent.*
(1) P *is isomorphic to* Q*.*
(2) $\omega(P) = \omega(Q)$ *and* $\alpha(P) = \alpha(Q)$*.*

Assertion 1.17 (Natural Isomorphism 2. *Let* R *and* P *be uniform such that* $\omega(P) \geq 1$ *and* $\omega(R) \geq 1$*. Then the following are equivalent.*
(1) $\omega(P) \geq \omega(R)$ *and* $\alpha(P) = \alpha(R)$*.*
(2) R *is isomorphic to a relative subgraph of* P*.*

Some of the previous elementary assertions will help us to give the strong reformulation of the Hadwiger conjecture and to deduce the analytic conjecture which is equivalent to the Hadwiger conjecture.

2. The strong reformulation of the Hadwiger conjecture and the statement of the analytic conjecture which is explicitly equivalent to the Hadwiger conjecture

Here, we use again definitions that are not standard; in particular we introduce another graph parameter denoted by a [the graph parameter a is called the *hadwiger caliber* and is related to the hadwiger index τ, via uniform graphs introduced in PROLOGUE], and using the parameter a, we give the strong reformulation of the Hadwiger conjecture. This strong reformulation immediately implies that, to solve the Hadwiger conjecture is equivalent to solve an analytic conjecture stated on uniform graphs [see Definition 6 for the meaning of uniform graphs]. Before, let us define.

Definition 8 (Fundamental 2). We say that a graph G is *hadwigerian*, **if** G is uniform and **if** $\omega(G) = \tau(G)$ [see Definition 2 for the meaning of $\tau(G)$].

Using Proposition 1.4, then it becomes immediate to see that every complete graph is hadwigerian. So the set of all complete graphs is an obvious example of hadwigerian graphs.

Definitions 9 (Fundamental 3). Let G be uniform. We say that a graph F is a *hadwigerian subgraph* of G, if F is *hadwigerian* and is a relative subgraph of G. We say that F is a *maximal hadwigerian* subgraph of G [recall that G is uniform], **if** F is a *hadwigerian subgraph* of G and $\omega(F)$ is **maximum** for this property [it is immediate that such a F exists and is well defined].

Now we define the *hadwiger caliber*.

Definition 10 (Fundamental 4). Let G be uniform, and let F be a *maximal hadwigerian* subgraph of G [see Definitions 9], then the *hadwiger caliber* of G is denoted by $a(G)$, where $a(G) = \omega(F)$.

It is immediate that $a(G)$ exists and is well defined. *It is also immediate that the hadwiger caliber [i.e. the graph parameter a] is defined for uniform graphs and only for uniform graphs*. From the definition of a uniform graph, the definition of a relative subgraph and the definition of the hadwiger caliber, then the following two Assertions are simple.

Assertion 2.0. *Let G be uniform. Now let $a(G)$ be the hadwiger caliber of G, and let $\tau(G)$ be the hadwiger index of G. We have the following five properties.*

(2.0.0) $\omega(G) \geq a(G)$.

(2.0.1) *G is hadwigerian* $\Leftrightarrow \tau(G) = \omega(G) = a(G)$.

(2.0.2) *G is not hawigerian* $\Leftrightarrow \omega(G) > a(G) \Leftrightarrow \omega(G) \neq \tau(G) \Leftrightarrow \omega(G) \neq a(G)$.

(2.0.3) *If $\omega(G) \leq 1$ or if $\alpha(G) \leq 1$ or if G is a complete graph or if $a(G) = 0$, then G is hadwigerian.*

(2.0.4) *If $\omega(G) \geq 1$, then $a(G) \geq 1$.*

Property (2.0.3) *of* Assertion 2.0 *gives classical examples of hadwigerian graphs*.

Assertion 2.1. *Let G be uniform and let R be a relative subgraph of G. Now let $a(G)$ be the hadwiger caliber of G, and let $a(R)$ be the hadwiger caliber of R. Then we have the following two simple properties.*

(2.1.0) $a(R) \leq a(G)$.

(2.1.1) *If G is hadwigerian, then R is also hadwigerian [in other words, if $\omega(G) = a(G)$, then $\omega(R) = a(R)$ (see property* (2.0.1) *of* Assertion 2.0)].

Definition 11 (Fundamental 5). Let G be uniform, and let $a(G)$ be the *hadwiger caliber* of G [see definition 10]; then we define f_G as follows. $f_G = 48a(G)^{a(G)} + 1000$. It is immediate that f_G exists and is well defined.

Now the following is the strong reformulation of the Hadwiger conjecture; this strong reformulation of the Hadwiger conjecture is based on the surgical reformulation given by Theorem 1.7.

Theorem 2.2 (The strong reformulation of the Hadwiger conjecture). *The following are equivalent.*

(i) *For every uniform graph U, we have $\omega(U) = a(U)$.*
(ii) *Every uniform graph Q is hadwigerian.*
(iii) *For every uniform graph G, we have $f_G^4 \geq \omega(G)$.*

Using Assertion 1.16 or Assertion 1.17, then the following proposition becomes easy to prove.

Proposition 2.3. *Let Q be uniform and let R be a relative subgraph of Q. Now let $a(Q)$ be the hadwiger caliber of Q, and let $a(R)$ be the hadwiger caliber of R.* **If** *$\omega(R) > a(R)$ [i.e.* **if** *R is not hadwigerian], then $a(Q) = a(R)$.*

Proof. Otherwise [we reason by reduction to absurd], observing that $Q \in \Omega$, and since R is a relative subgraph of Q, then property (2.1.0) of Assertion 2.1 implies that $a(Q) > a(R)$. Now, let R' be a relative subgraph of R such that $\omega(R') = a(R)+1$ [*It is immediate that such a R' exists, since $\omega(R) > a(R)$*]; clearly R' is a relative subgraph of the uniform graph Q (because R' is a relative subgraph of R and R is a relative subgraph of the uniform graph Q) and $\omega(R') \geq 1$. So R' is a relative subgraph of the uniform graph Q and $\omega(R') \geq 1$; properties (1.14.1) and (1.14.2) of Assertion 1.14 imply that $\alpha(R') = \alpha(Q)$ and R' is uniform. Now, let F be a maximal hadwigerian subgraph of Q; observing that $\omega(F) = a(Q)$ (because F is a maximal hadwigerian subgraph of Q), clearly $\omega(F) \geq \omega(R')$ (since we have seen above that $a(Q) > a(R)$ and $\omega(R') = a(R) + 1$). In particular F is a relative subgraph of Q (because F is a maximal hadwigerian subgraph of Q), now, observing that $\omega(F) \geq 1$ (since we have seen above that $\omega(F) \geq \omega(R')$ and $\omega(R') \geq 1$), clearly F is a relative subgraph of the uniform graph Q and $\omega(F) \geq 1$; properties (1.14.1) and (1.14.2) of Assertion 1.14 imply that $\alpha(F) = \alpha(Q)$ and F is uniform. Now, consider R' and F, by using the previous, we easily deduce that R' and F are uniform such that $\alpha(F) = \alpha(R') = \alpha(Q)$ and $\omega(F) \geq \omega(R') \geq 1$; clearly R' and F are uniform, and $\alpha(F) = \alpha(R')$ and $\omega(F) \geq \omega(R') \geq 1$, then, by using Assertion 1.17 (Natural Isomorphism (2)), it follows that R' is isomorphic to a relative subgraph of F. Since in particular F is hadwigerian (because F is a maximal hadwigerian subgraph of Q), clearly R' is isomorphic to a relate if subgraph of F and F is hadwigerian; then by applying property (2.1.1) of Assertion 2.1, it follows that R' is also hadwigerian. Therefore $a(R') = \omega(R')$, and clearly $a(R') = a(R) + 1$ (since we he seen above that $\omega(R') = a(R) + 1$). Now, recalling that R' is a relative subgraph of R, then property (2.1.0) of Assertion 2.1 implies that $a(R') \leq a(R)$. This last inequality is impossible, since $a(R') = a(R)+1$. Proposition 2.3 follows. □

***Proof of Theorem* 2.2.** It is immediate that (i)⇒(ii) and it is also immediate that (ii)⇒(iii).
(iii)⇒(i)]. Assume otherwise [we reason by reduction to absurd]. Let U be uniform such that $\omega(U) \neq a(U)$, clearly (by using property (2.0.0) of Assertion 2.0), we have

$$\omega(U) > a(U) \qquad (\mathbf{2.2.0});$$

fix once and for all U [U is fixed once and for all; so U does not move anymore]; then we have these simple remarks. □

Remark 2.2.1. U is uniform such that $\omega(U) > 1$ and $\alpha(U) > 1$ and $a(U) \geq 1$.

Otherwise, clearly U is uniform such that $\omega(U) \leq 1$ or $\alpha(U) \leq 1$ or $a(U) = 0$, and using property (2.0.3) of Assertion 2.0, we easily deduce that U is hadwigerian; therefore $\omega(U) = a(U)$ and this contradicts (2.2.0).

Remark 2.2.2. Look at the uniform graph U and consider f_U^4 [see Definition 11 for the meaning of f_U]. Then there exists a uniform graph U' such that $\omega(U') = 3000[\omega(U) + 1000 f_U^4]$ and $\alpha(U') = \alpha(U)$.

Indeed, observing [by using Remark 2.2.1] that U is uniform such that $\alpha(U) > 1$, then it becomes immediate to deduce that there exists a uniform graph U' such that $\omega(U') = 3000[\omega(U) + 1000 f_U^4]$ and $\alpha(U') = \alpha(U)$.

Remark 2.2.3. Look at the uniform graph U and let the uniform graph U' defined in Remark 2.2.2. Then U is isomorphic to a relative subgraph of U'.

Indeed, noticing [by using Remark 2.2.1] that $\omega(U) > 1$, then, using the previous inequality and the definition of U', we immediately deduce that $\omega(U') > \omega(U) > 1$ and $\alpha(U') = \alpha(U)$; clearly U and U' are uniform such that $\alpha(U) = \alpha(U')$ and $\omega(U') > \omega(U) > 1$; now using the previous and Assertion 1.17, we easily deduce that U is isomorphic to a relative subgraph of U'. Remark 2.2.3 follows.

Remark 2.2.4. Look at the uniform graph U and let the uniform graph U' defined in Remark 2.2.2. Then $a(U') = a(U)$.

Indeed, observing [by (2.2.0)] that $\omega(U) > a(U)$ and remarking [by Remark 2.2.3] that U is isomorphic to a relative subgraph of U', then, using Proposition 2.3, we immediately deduce that $a(U') = a(U)$. Remark 2.2.4 follows.

Remark 2.2.5. Let the uniform graph U' defined in Remark 2.2.2. Then $\omega(U') \leq f_{U'}^4$.

Indeed, observing that U' is uniform, then U' satisfies property (iii) of Theorem 2.2 and therefore $\omega(U') \leq f_{U'}^4$. Remark 2.2.5 follows.

Remark 2.2.6. Look at the uniform graph U and let the uniform graph U' defined in Remark 2.2.2. Then $\omega(U') \leq f_U^4$.

Indeed observing [by Remark 2.2.4] that $a(U) = a(U')$, then the previous equality immediately implies that $f_U = f_{U'}$ and therefore

$$f_U^4 = f_{U'}^4 \,. \tag{2.2.6.0}$$

Now using equality (2.2.6.0) and Remark 2.2.5, we easily deduce that $\omega(U') \leq f_U^4$. Remark 2.2.6 follows.

These simple remarks made, let U' be the uniform graph introduced in Remark 2.2.2, then [by Remark 2.2.6] $\omega(U') \leq f_U^4$; observing [by the definition of U'] that $\omega(U') = 3000[\omega(U) + 1000 f_U^4]$, then the previous inequality immediately becomes $3000[\omega(U) + 1000 f_U^4] \leq f_U^4$; and this is clearly impossible, since $\omega(U) > 1$ [by using Remark 2.2.1] and $f_U^4 > 1$ [by the definition of f_U]. So, assuming that property (iii) of Theorem 2.2 is true and property (i) of Theorem 2.2 is false give rises to a serious contradiction. So (iii)$\Rightarrow$(i)]. Theorem 2.2 clearly follows.

Corollary 2.4. *If for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then for every graph H, we have $\chi(H) \leq \tau(H)$.*

Proof. Indeed, let H be a graph and let Q be a uniform graph such that Q is a parent of H [it is immediate that such a Q exists, by using Assertion 1.12]; then using Corollary 1.6, we immediately deduce that

$$\tau(Q) \leq \tau(H)\,. \tag{2.4.0}$$

Now, observing [by the hypotheses] that for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then, using Theorem 2.2, we immediately deduce that

$$\text{every uniform graph } U \text{ is Hadwigerian.} \tag{2.4.1}$$

Using property (2.0.1) of Assertion 2.0, then we immediately deduce that (2.4.1) implies that

$$\text{for every uniform graph } U, \text{ we have } \omega(U) = a(U) = \tau(U)\,. \tag{2.4.2}$$

Recalling that Q is uniform, then, using (2.4.2), we immediately deduce that inequality (2.4.0) is of the form

$$\omega(Q) \leq \tau(H)\,. \tag{2.4.3}$$

Now, observing that $\omega(Q) = \chi(H)$ [note that Q is a parent of H and use Assertion 1.2], then, inequality (2.4.3) immediately becomes $\chi(H) \leq \tau(H)$. □

Corollary 2.5. *If for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then the Hadwiger conjecture follows.*

Proof. Use Corollary 2.4 and Theorem 1.7. □

Corollary 2.5 *clearly says that, if for every uniform U, we have $f_U^4 \geq \omega(U)$, then, the Hadwiger conjecture follows*. Using Corollary 2.5 and Theorem 2.2, then the following Corollary becomes immediate.

Corollary 2.6. *The following are equivalent.*

(i) *For every uniform graph U, we have $\omega(U) = a(U)$.*
(ii) *Every uniform graph Q is hadwigerian.*
(iii) *For every uniform graph G, we have $f_G^4 \geq \omega(G)$.*
(iv) *For every uniform graph U, we have $f_U^4 \geq \omega(U)$*
(v) *The Hadwiger conjecture holds.*

From Corollary 2.6, it immediately follows that to prove the Hadwiger conjecture is equivalent to prove the following analytic conjecture stated on uniform graphs.

Conjecture 0. *For every uniform graph U, we have $f_U^4 \geq \omega(U)$, where $f_U = 48a(U)^{a(U)} + 1000$, and where $a(U)$ is the hadwiger caliber of U.*

It is easy to show that Conjecture 0 is equivalent to the following:

Conjecture 1. *There exists a fixed positive integer t such that for every uniform graph U, we have $\omega(U) \leq a(U) + t$, where $a(U)$ is the hadwiger caliber of U.*

So, to give a proof of the Hadwiger conjecture is equivalent to show that: *for every uniform graph U, we have $f_U^4 \geq \omega(U)$* or *there exists a fixed positive integer t such that for every uniform graph U, we have $\omega(U) \leq a(U) + t$.*

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