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Research Article

# The Analytic Solution of Initial Periodic Boundary Value Problem Including Sequential Time Fractional Diffusion Equation

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**Abstract.** In this study, the separation of variables method is applied to form the analytic solution of periodic boundary value problem including time fractional differential equation with periodic boundary conditions in one dimension. The solution is obtained in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem.

**Keywords.** Caputo fractional derivative; Space-fractional diffusion equation; Mittag-Leffler function; Periodic-boundary-value problems; Spectral method

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## 1. Introduction

As PDEs of fractional order is an important research topic in modelling for the numerous processes and systems in various scientific research areas such as mathematical biology, engineering, physics chemistry etc., the interest of this topic is increasing enormously. Since the fractional derivative is non-local, the model with fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non linear processes. That is why attracts increasing number of researchers. The derivatives in the sense of Caputo is one of the most common one since modelling of physical processes with fractional differential equations including Caputo derivative is much more better than other models. In literature increasing number of studies can be found supporting this conclusion [1], [2], [5], [6], [4], [3],[7], [8], [10],[11], [12], [13], [14]. Moreover, the Caputo derivative of constant is zero which is not hold by many fractional derivatives. The solutions of fractional PDEs and ODEs are determined in terms of Mittag-Leffler function.

## 2. Preliminary Results

In this part, we recall fundamental definition and well known results about fractional derivative in Caputo sense.

**Definition 1.** The $q$th order fractional derivative of $u(t)$ in Caputo sense is defined as

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds, \quad t \in [t_0, t_0+T], \tag{1}$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$, $n-1 < q < n$. Note that Caputo fractional derivative is equal to integer order derivative when the order of the derivative is integer.

**Definition 2.** If $0 < q < 1$, the $q$th order Caputo fractional derivative is defined as

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds, \quad t \in [t_0, t_0+T]. \tag{2}$$

The Mittag-Leffler function with two-parameters is defined as

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0 \tag{3}$$

including constant $\lambda$. Especially, for $t_0 = 0$, $\alpha = \beta = q$ we have

$$E_{\alpha,\beta}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, \quad q > 0. \tag{4}$$

Mittag-Leffler function coincides with exponential function, i.e., $E_{1,1}(\lambda t) = e^{\lambda t}$ for $q = 1$ (for details see [9], [15]).

We determined the solution of following time fractional differential equation with periodic boundary conditions and initial condition in this study:

$$D_t^\alpha u(x,t) = \gamma^2 u_{xx}(x,t), \quad 0 < \alpha < 1, -l \le x \le l, 0 \le t \le T, \gamma \in R, \tag{5}$$

$$\begin{cases} u(-l,t) = u(l,t), & t > 0 \\ u_x(-l,t) = u_x(l,t), & t > 0 \end{cases} \quad 0 \le t \le T, \tag{6}$$

$$u(x,0) = f(x), \quad -l \le x \le l. \tag{7}$$

## 3. Main Results

By means of separation of variables method, the solution of problem (5)-(7) is constructed in analytical form. Thus a solution of problem (5)-(7) have the following form:

$$u(x,t;\alpha) = X(x)T(t;\alpha), \tag{8}$$

where $0 \le x \le l$, $0 \le t \le T$.

Plugging (8) into (5) and arranging it, we have

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2. \tag{9}$$

Equation (9) produces a fractional differential equation with respect to time and a ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of eq. (9). Hence with boundary conditions (6), we have the following problem:

$$X''(x) + \lambda^2 X(x) = 0, \tag{10}$$

$$\begin{cases} X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases} \tag{11}$$

The solution of eigenvalue problem (10)-(11) is accomplished by making use of the exponential function of the following form:

$$X(x) = e^{rx}. \tag{12}$$

Hence the characteristic equation is computed in the following form:

$$r^2 + \lambda^2 = 0 \tag{13}$$

**Case 1.** If $\lambda = 0$, the eq. (13) have two coincident roots $r_1 = r_2$, leading to the general solution of the eigenvalue problem (10)-(11) having the following form:

$$X(x) = k_1 x + k_2, \tag{14}$$

$$X'(x) = k_1. \tag{15}$$

The first boundary condition yields

$$X(-l) = -k_1 l + k_2 = k_1 l + k_2 = X(l) \Rightarrow k_1 = 0. \tag{16}$$

This result leads to

$$X(x) = k_2. \tag{17}$$

Similarly, second boundary condition leads to

$$X'(-l) = 0 = X'(l). \tag{18}$$

Hence we obtain the solution as follows:

$$X_0(x) = k_2. \tag{19}$$

**Case 2.** If $\lambda > 0$, the eq. (13) have two distinct real roots $r_1$, $r_2$leading to the general solution of the eigenvalue problem (13)-(14) having the following form:

$$X(x) = k_1 e^{r_1 x} + k_2 e^{r_2 x}. \tag{20}$$

The first boundary condition yields

$$X(-l) = k_1 e^{-r_1 l} + k_2 e^{-r_2 l} = k_1 e^{r_1 l} + k_2 e^{r_2 l} = X(l), \tag{21}$$

$$k_1(e^{-r_1 l} - e^{r_1 l}) + k_2(e^{-r_2 l} - e^{r_2 l}) = 0. \tag{22}$$

Since $e^{-r_1 l} - e^{r_1 l}$ and $e^{-r_2 l} - e^{r_2 l}$ are linearly independent the equation (22) is satisfied if and only if $k_1 = 0 = k_2$ which implies that $X(x) = 0$ which means that we don't have any solution for $\lambda > 0$.

**Case 3.** If $\lambda < 0$, the characteristic equation have two complex conjugate roots leading to the general solution of the eigenvalue problem (10)-(11) having the following form:

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x). \tag{23}$$

The first boundary condition yields

$$X(-l) = c_1 \cos(\lambda l) - c_2 \sin(\lambda l) = c_1 \cos(\lambda l) + c_2 \sin(\lambda l) = X(l) \tag{24}$$

which implies that

$$\Rightarrow 2c_2 \sin(\lambda l) = 0 \Rightarrow c_2 = 0. \tag{25}$$

Hence the solution becomes

$$X(x) = c_1 \cos(\lambda x), \tag{26}$$

$$X'(x) = -c_1 \lambda \sin(\lambda x). \tag{27}$$

Similarly, last boundary condition leads to

$$X'(-l) = c_1 \lambda \sin(\lambda l) + c_2 \lambda \cos(\lambda l) = -c_1 \lambda \sin(\lambda l) + c_2 \lambda \cos(\lambda l) = X'(l) \tag{28}$$

$$\Rightarrow \quad 2c_1 \lambda \sin(\lambda l) = 0 \tag{29}$$

which implies that

$$\sin(\lambda l) = 0. \tag{30}$$

Let $\lambda_n l = n\pi$. Hence the eigenvalues can be represented as follows:

$$\lambda_n = \frac{n\pi}{l}, \quad \lambda_1 < \lambda_2 < \lambda_3 < \dots. \tag{31}$$

The representation of the solution is obtained as follows:

$$X_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3, \dots. \tag{32}$$

The second equation in (9) for every eigenvalue $\lambda_n$ is determined as follows:

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = -\gamma^2 \lambda_n^2 \tag{33}$$

which yields the following solution

$$T_n(t;\alpha) = E_{\alpha,1}\left(-\gamma^2 \left(\frac{n\pi}{l}\right)^2 t^\alpha\right), \quad n = 0, 1, 2, 3, \dots. \tag{34}$$

The solution for every eigenvalue $\lambda_n$ is constructed as follows:

$$u_n(x,t;\alpha) = X_n(x) T_n(t;\alpha) = \cos\left(\frac{n\pi x}{l}\right) E_{\alpha,1}\left(-\gamma^2 \left(\frac{n\pi}{l}\right)^2 t^\alpha\right), \quad n = 0, 1, 2, 3, \dots. \tag{35}$$

Hence the general solution becomes

$$u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2}{l^2} t^\alpha\right). \tag{36}$$

Note that boundary conditions and fractional differential equation are satisfied by this solution.

The coefficients in (36) are obtained by making use of initial condition (7):

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right). \tag{37}$$

Through the inner product in $L_2[-l,l]$, we obtain the coefficients $A_n$, for $n = 0,1,2,3,\dots$ as follows:

$$A_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx, \tag{38}$$

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right). \tag{39}$$

## 4. Illustrative Example

In this part, we first take the following periodic initial boundary value problem:

$$u_t(x,t) = u_{xx}(x,t), \quad -1 \le x \le 1, 0 \le t \le T$$

$$\begin{cases} u(-1,t) = u(1,t), & t > 0 \\ u_x(-1t,t) = u_x(1,t), & t > 0 \end{cases}$$

$$u(x,0) = \cos(\pi x), \quad -1 \le x \le 1 \tag{40}$$

which has the solution in the following form:

$$u(x,t) = \cos(\pi x)\, e^{-\pi^2 t} \tag{41}$$

Now, we take the problem below named as heat-like problem into consideration

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t), \quad 0 < \alpha < 1, -1 \le x \le 1, 0 \le t \le T \tag{42}$$

$$\begin{cases} u(-1,t) = u(1,t), & t > 0 \\ u_x(-1,t) = u_x(1,t), & t > 0 \end{cases} \tag{43}$$

$$u(x,0) = \cos(\pi x), \quad -1 \le x \le 1 \tag{44}$$

Applying separation of the variables to (42) leads to the equation

$$\frac{D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = \frac{X''(x)}{X(x)} = -\lambda^2. \tag{45}$$

Equation (45) produces a fractional differential equation with respect to time and a ordinary differential equation with respect to space. The first fractional differential equation is obtained by taking the equation on the right hand side of eq. (45). Hence with boundary conditions (43), we have the following problem:

$$X'(x) + \lambda^2 X(x) = 0, \tag{46}$$

$$\begin{cases} X(-1) = X(1) \\ X'(-1) = X'(1) \end{cases} \tag{47}$$

Hence the eigenvalue problem (46)-(47) yields the following solution:

$$X_n(x) = \cos(n\pi x), \quad n = 0,1,2,3,\dots. \tag{48}$$

The second equation (45) for every $\lambda_n = n\pi$ leads to the following fractional differential equation:

$$\frac{D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = -\lambda_n^2 \tag{49}$$

which yields the following solution

$$T_n(t;\alpha) = E_{\alpha,1}(-(n\pi)^2 t^{\alpha}), \quad n = 0,1,2,3,\dots. \tag{50}$$

The solution is established for every $\lambda_n$ as follows:

$$u_n(x,t;\alpha) = E_{\alpha,1}(-(n\pi)^2 t^{\alpha})\cos(n\pi x), \quad n = 0,1,2,3,\dots \tag{51}$$

and hence we have the following sum:

$$u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n E_{\alpha,1}(-(n\pi)^2 t^{\alpha})\cos(n\pi x). \tag{52}$$

Plugging $t = 0$ in to the general solution (52), we have

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x). \tag{53}$$

We obtain the coefficients $A_n$ for $n = 0,1,2,3,\dots$ as follows:

$$A_0 = \frac{1}{2}\int_{-1}^{1} \cos(\pi x)dx = \left(\frac{1}{2\pi}\sin(\pi x)\right)_{x=-1}^{x=1} = 0, \tag{54}$$

$$A_n = \int_{-1}^{1} \cos(\pi x)\cos(n\pi x)dx. \tag{55}$$

For $n \neq 1$, $A_n = 0$ and for $n = 1$, we get

$$A_1 = \int_{-1}^{1} \cos^2(\pi x)dx = \int_{-1}^{1}\left(\frac{1}{2} + \frac{\cos(2\pi x)}{2}\right)dx = \left(\frac{x}{2} + \frac{\sin(2\pi x)}{4\pi}\right)\Bigg|_{x=-1}^{x=1} = 1. \tag{56}$$

Thus

$$u(x,t;\alpha) = \cos(\pi x)E_{\alpha,1}(-\pi^2 t^{\alpha}). \tag{57}$$

It is important to note that plugging $\alpha = 1$ in to the solution (57) gives the solution (41) which confirm the accuracy of the method we apply.

## 5. Conclusion

In this research, the analytic solution of initial periodic boundary value problem with periodic boundary conditions, arising in varies applications, is constructed. By making use of separation of variables the solution is formed in the form of a Fourier series in terms of Mittag-Leffler function and exponential function.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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