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Existence of Weak Solutions for A Class of Nonuniformly Nonlinear Elliptic Equations of p-Laplacian Type

G.A. Afrouzi, Z. Naghizadeh, F. Mirzadeh, and M. Amirian

Abstract. The goal of this paper is to study the existence of non-trivial weak solutions for the nonuniformly nonlinear elliptic equation in an unbounded domain. The solution will be obtained in a subspace of the Sobolev space and the proofs rely essentially on a variation of the mountain pass theorem.

1. **Introduction**

Let Ω be an unbounded domain in R^N $(N \geq 3)$ with smooth boundary $\partial\Omega$. We study the existence of non-trivial weak solutions of the following Dirichlet problem.

$$\begin{cases} -\operatorname{div}(h(x)|\nabla u|^{p-2}\nabla u) + q(x) \mid u \mid^{p-2} u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{1.1}$$

where the functions h and q satisfy the hypotheses $h(x) \in L^1_{loc}(\Omega)$, $h(x) \geq 1$, a.e. $x \in \Omega$ and $q(x) \in C(\Omega)$, there exists $q_0 > 0$ such that

$$q(x) \geq q_0 > 0, \quad \text{a.e. } x \in \Omega,\ q(x) \to +\infty \text{ as } |x| \to \infty. \tag{1.2}$$

We remark that in the case when $h(x) = 1$ in Ω and $p = 2$, problem (1.1) has been studied in the article [5] and if Ω is a bounded domain it has been studied in the article [6]. We reduce problem (1.1) to a uniform one by using an appropriate weighted Sobolev space. Then applying a variation of the mountain pass theorem in the article [3], [4] we prove that problem (1.1) admits a non-trivial weak solution in a subspace of the Sobolev space $W_0^{1,p}(\Omega)$.

In order to state our main theorem, let us introduce the following hypotheses:

(F_1): $f(x,z) \in C^1(\Omega \times R, R)$, $f(x,0) = 0$, a.e. $x \in \Omega$.

(F_2): There exists a function $\tau(x) \geq 0$, a.e. $x \in \Omega$, $\tau(x) \in L^{p_0}(\Omega) \cap L^\infty(\Omega)$, where $\alpha \in \left(1, \frac{N+p}{N-p}\right)$, $p_0 = \frac{pN}{pN-(p+1)(N-p)}$ such that

$$|f'_z(x,z)| \leq \tau(x)|z|^{\alpha-1} \quad \text{a.e.} \quad x \in \Omega,\ \forall\, z \in R.$$

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(F_3): There exists a constant $\mu > p$ such that

$$0 < \mu F(x,w) \leq z \cdot f(x,z)$$

for all $x \in \Omega$, $z \in R \setminus \{0\}$, where $F(x,z) = \int_0^z f(x,s)ds$.

For example let $f(x,u) = u^3$, $\mu = 4$, and $p = 3$. It can be shown that the function f satisfy condition (F_3). We define the norm of $u \in W_0^{1,p}(\Omega)$ by

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p + |u|^p dx \right)^{\frac{1}{p}}$$

and consider the following subspace

$$E = \left\{ u \in W_0^{1,p}((\Omega) : \int_\Omega (|\nabla u|^p + q(x)|u|^p) dx)^{\frac{1}{p}} < +\infty \right\}$$

then E is a Banach space with the norm

$$\|u\|_E^p = \int_\Omega (|\nabla u|^p + q(x)|u|^p) dx, \quad u \in W_0^{1,p}(\Omega).$$

Furthermore, we have

$$\|u\|_E \geq m_0^{\frac{1}{2}} \|u\|_{W_0^{1,p}(\Omega)} \quad \forall\, u \in E$$

where $m_0 = \min(1, q_0)$ and the continuous embeddings

$$E \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad p \leq q \leq p^* = \frac{pN}{N-p}$$

hold true(see [1] or [5]).

Moreover, the embedding $E \hookrightarrow L^p(\Omega)$ is compact (see [2]).

1.1

We now introduce the space

$$H = \left\{ u \in E : \left(\int_\Omega (h(x)|\nabla u|^p + q(x)|u|^p) dx \right)^{\frac{1}{p}} < \infty \right\}$$

endowed with the norm

$$\|u\|_H^p = \int_\Omega (h(x)|\nabla u|^p + q(x)|u|^p) dx.$$

As [4] it can be easily shown that H is a Banach space with the above norm.

Remark 1.1. (i) Since $h(x) \geq 1$, a.e.$x \in \Omega$ we have

$$\|u\|_H \geq \|u\|_E, \quad \forall\, u \in H$$

(ii) $\forall\, v \in C_0^\infty(\Omega)$, $\int_\Omega (h(x)|\nabla v|^2 + q(x)|v|^2) dx < +\infty$.

Hence $C_0^\infty(\Omega) \subset H$.

Definition 1.2. We say that a function $u \in H$ is a weak solution of (1.1) if

$$\int_\Omega (h(x)|\nabla u|^{p-2} \nabla u \nabla v + q(x) \mid u \mid^{p-2} uv) dx - \int_\Omega f(x,u) v dx = 0$$

2. Main result

2.1

Our main result is stated as follows:

Theorem 2.1. *Assuming* (1.2) *and* (F_1)-(F_3) *are satisfied, then problem* (1.1) *has at least one nontrivial weak solution in* H.

It is clear that equation (1.1) has a variational structure. Let $J : H \to R$ defined by

$$J(u) = \frac{1}{p}\int_\Omega (h(x)|\nabla u|^p + q(x) \mid u \mid^p)dx - \int_\Omega F(x,u)dx$$
$$= T(u) - p(u) \quad \text{for} \quad u \in H \tag{2.1}$$

where

$$T(u) = \frac{1}{p}\int_\Omega (h(x)|\nabla u|^p + q(x) \mid u \mid^p)dx \tag{2.2}$$

$$P(u) = \int_\Omega F(x,u)dx. \tag{2.3}$$

Definition 2.2. Let J be a functional from a Banach space Y into R. We say that J is weakly continuously differentiable on Y if and only if the following conditions are satisfied:

(i) J is continuous on Y.

(ii) For any $u \in Y$, there exists a linear map $J'(u)$ from Y into R such that

$$\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \langle J'(u), v\rangle \quad \forall\ v \in Y.$$

(iii) For any $v \in Y$, the map $u \to \langle J'(u), v\rangle$ is continuous on Y.

Proposition 2.3. *Under the assumptions of* Theorem 2.1*, the functional* $J(u)$ *is weakly continuously differentiable on* H *and*

$$\langle J'(u), v\rangle = \int_\Omega (h(x)|\nabla u|^{p-2}\nabla u \nabla v + q(x)|u|^{p-2}uv)dx - \int_\Omega f(x,u)v dx$$

for all $u, v \in H$.

Proof. Following exactly the same procedures as in the proof of Proposition 2.2 in article [4]. □

Proposition 2.4 ([4]). *Suppose that* $\{u_m\}$ *is a sequence weakly converging to* u *in* E. *Then we have:*

(i) $\lim_{m\to+\infty} P(u_m) = P(u)$

(ii) $T(u) \le \liminf_{m\to+\infty} T(u_m)$

Proposition 2.5 ([4]). *The functional $J(u)$, $u \in H$ given by* (2.1) *satisfies the Palais-Smale condition.*

Proof. Let $\{u_m\} \subset H$ be a Palais-Smale sequence, i.e.

$$\lim_{m\to+\infty} J(u_m) = c, \quad \lim_{m\to+\infty} \|J'(u_m)\|_{H^*} = 0.$$

First we should prove that $\{u_m\}$is bounded in H.We suppose by contradiction that $\{u_m\}$ is not bounded in H. Then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\|u_{m_j}\|_H \to \infty \quad \text{as} \quad j \to \infty.$$

Observe further that

$$\begin{aligned} J(u_{m_j}) - \frac{1}{\mu}\langle J'(u_{m_j}), u_{m_j}\rangle &= T(u_{m_j}) - \frac{1}{\mu}\langle T'(u_{m_j}), u_{m_j}\rangle + \frac{1}{\mu}\langle P'(u_{m_j}), u_{m_j}\rangle - P(u_{m_j}) \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_{m_j}\|_H^p \end{aligned}$$

yields

$$\begin{aligned} J(u_{m_j}) &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_{m_j}\|_H^p + \frac{1}{\mu}\langle J'(u_{m_j}), u_{m_j}\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_{m_j}\|_H^p - \frac{1}{\mu}\|J'(u_{m_j})\|_{H^*}\|u_{m_j}\|_H \\ &\geq \|u_{m_j}\|_H\left(\gamma_0\|u_{m_j}\|_H^{p-1} - \frac{1}{\mu}\|J'(u_{m_j})\|_{H^*}\right), \end{aligned} \tag{2.4}$$

where $\gamma_0 = \left(\frac{1}{p} - \frac{1}{\mu}\right) > 0$.

From (2.4) letting $j \to \infty$ since $\|u_{m_j}\|_H \to \infty$ and $\|J'(u_{m_j})\|_{H^*} \to 0$ we deduce $J(u_{m_j}) \to \infty$ which yields a contradiction. Hence, $\{\|u_m\|_H\}$ is bounded.

Since $\|u_m\|_E \leq \|u_m\|_H$, $\{u_m\}$is also bounded in E. Therefore, there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ converging weakly to u in E. By Proposition 2.4 we have

$$T(u) \leq \liminf_{k\to\infty} T(u_{m_k}) = \lim_{k\to\infty}[P(u_{m_k}) + J(u_{m_k})] = P(u) + c < \infty.$$

Therefore $u \in H$.

Furthermore, since the embedding $E \hookrightarrow L^{p^*}(\Omega)$ is continuous, $\{u_{m_k}\}$ is weakly convergent to u in $L^{p^*}(\Omega)$. Then it is clear that the sequence $\{|u_{m_k}|^{p-1}u_{m_k}\}$ converges weakly to $|u|^{p-1}u$ in $L^{\frac{p^*}{p}}(\Omega)$ by

$$\langle k(u), w\rangle = \int \tau(x)uw\,dx, \quad w \in L^{\frac{p^*}{p}}(\Omega).$$

We remark $k(u)$ is linear and continuous provided that $\tau(x) \in L^{p_0}(\Omega)$, $u \in L^{p^*}(\Omega)$, $w \in L^{\frac{p^*}{p}}(\Omega)$ and $\frac{1}{p_0} + \frac{1}{p^*} + \frac{p}{p^*} = 1$.

By (F_1) and (F_2) we obtain

$$\lim_{k\to\infty}\int_\Omega f(x,u_{m_k})(u_{m_k}-u)dx=0$$

i.e.

$$\lim_{k\to\infty}\langle P'(u_{m_k}),u_{m_k}-u\rangle=0. \tag{2.5}$$

It follows from (2.5) that

$$\lim_{k\to\infty}\langle T'(u_{m_k}),u_{m_k}-u\rangle=\lim_{k\to\infty}\langle J'(u_{m_k}),u_{m_k}-u\rangle+\lim_{k\to\infty}\langle P'(u_{m_k}),u_{m_k}-u\rangle=0.$$

Moreover, since T is convex the following inequality holds true

$$T(u)-T(u_{m_k})\geq\langle T'(u_{m_k}),u-u_{m_k}\rangle.$$

Letting $k\to\infty$ we obtain that

$$T(u)-\lim_{k\to\infty}T(u_{m_k})=\lim_{k\to\infty}[T(u)-T(u_{m_k})]\geq\lim_{k\to\infty}\langle T'(u_{m_k}),u-u_{m_k}\rangle=0.$$

Thus

$$T(u)\geq\lim_{k\to\infty}T(u_{m_k}). \tag{2.6}$$

On the other hand, by (ii) of Proposition 2.4 we have

$$T(u)\leq\liminf_{k\to\infty}T(u_{m_k}). \tag{2.7}$$

Combining (2.6) and (2.7) we get $\lim_{k\to\infty}T(u_{m_k})=T(u)$. Now, we shall prove that $u_{m_k}\to u$ strongly in H. i.e.

$$\lim_{k\to\infty}\|u_{m_k}-u\|_H=0.$$

Indeed, we suppose by contradiction that $\{u_{m_k}\}$ does not converge strongly to $u\in H$. Then there exist a constant $\epsilon_0>0$ and a subsequence $\{u_{m_{k_j}}\}$ of $\{u_{m_k}\}$ such that

$$\|u_{m_{k_j}}-u\|_H\geq\epsilon_0,\quad\forall\ j=1,2,\dots.$$

Recalling the equality

$$\left|\frac{\alpha+\beta}{2}\right|^2+\left|\frac{\alpha-\beta}{2}\right|^2=\frac{1}{2}(\alpha^2+\beta^2),\quad\forall\ \alpha,\beta\in R$$

we deduce that for any $j=1,2,...$

$$\frac{1}{2}T(u_{m_{k_j}})+\frac{1}{2}T(u)-T\left(\frac{u_{m_{k_j}}+u}{2}\right)=k_1\|u_{m_{k_j}}-u\|_H^p\geq k_1\epsilon_0^p\quad k_1>0. \tag{2.8}$$

Again instead of the remark that since $\left\{\frac{u_{m_{k_j}}+u}{2}\right\}$ converges weakly to u in E, by (ii) of Proposition 2.4 we have

$$T(u)\leq\liminf_{j\to\infty}T\left(\frac{u_{m_{k_j}}+u}{2}\right).$$

Then from (2.8), letting $j \to \infty$ we obtain

$$T(u) - \liminf_{j\to\infty} T\left(\frac{u_{m_{k_j}} + u}{2}\right) \geq k_1 \epsilon_0^p$$

hence $0 \geq k_1 \epsilon_0^p$, which is a contradiction. There fore $\{u_{m_k}\}$ converges strongly to $u$in H. Thus, the functional J satisfiesthe Palais-Smale condition on H. The proof of Proposition 2.5 is complete.

We remark that the critical points of the functional J correspond to the weak solutions of problem (1.1). To apply the Mountain pass theorem we shall prove the following proposition which shows that the functional J has the Mountain pass geometry.

Proposition 2.6. (i) *There exist $\alpha > 0$ and $r > 0$ such that $J(u) \geq \alpha > 0$, for all $u \in H$ and $\| u \|_H = r$.*

(ii) *There exists $u_0 \in H$ such that $\| u_0 \|_H > r$ and $J(u_0) < 0$.*

Proof. From (F_2), (F_3) there exist a constant $c_1 > 0$such that

$$F(x,z) < c_1 |z|^{\alpha+1} \quad \forall\ z \in R \text{ a.e. } x \in \Omega. \tag{2.9}$$

Remark that $p < \alpha + 1 < p^*$, we have

$$\lim_{|z|\to 0} \frac{F(x,z)}{|z|^p} = 0, \tag{2.10}$$

$$\lim_{|z|\to \infty} \frac{F(x,z)}{|z|^{p^*}} = 0. \tag{2.11}$$

Then for a constant $\epsilon > 0$ there exist two positive constants δ_1 and δ_2 $(\delta_1 < \delta_2)$ such that

$$F(x,z) < \epsilon |z|^p \text{ for all } z \text{ with } |z| < \delta_1,$$
$$F(x,z) < \epsilon |z|^{p^*} \text{ for all } z \text{ with } |z| > \delta_2.$$

On the other hand, from (2.9) there exists a constant $c_2 > 0$ such that

$$F(x,z) < c_2 \text{ for all } z \text{ with } |z| \in [\delta_1, \delta_2].$$

Then we obtain that for all $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that

$$F(x,z) \leq \epsilon |z|^p + c_\epsilon |z|^{p^*}$$

for all $z \in R$ and a.e. $x \in \Omega$.

We deduce from (2.12), condition (1.2) and the embedding $E \hookrightarrow L^{p^*}(\Omega)$ that

$$J(u) = \frac{1}{p} \int_\Omega (h(x)|\nabla u|^p + q(x)|u|^p) dx - \int_\Omega F(x,u) dx$$

$$\geq \frac{1}{p}\|u\|_H^p - \epsilon \int_\Omega |u|^p dx - c_\epsilon \int_\Omega |u|^{p^*} dx$$
$$\geq \frac{1}{p}\|u\|_H^p - \frac{\epsilon}{q_0} \int_\Omega q(x)|u|^p dx - c_\epsilon \int_\Omega |u|^{p^*} dx$$
$$\geq \left(\frac{1}{p} - \frac{\epsilon}{q_0}\right)\|u\|_H^p - \bar{c}_\epsilon \|u\|_H^{p^*}$$

where $\bar{c}_\epsilon$ is a positive constant.

Thus, for all $\epsilon > 0$ there exists a constant $\bar{c}_\epsilon > 0$ such that

$$J(u) \geq \left(\frac{1}{p} - \frac{\epsilon}{q_0} - \bar{c}_\epsilon \|u\|_H^{p^*-p}\right)\|u\|_H^p \quad \forall\, u \in H.$$

Therefore, letting $\epsilon \in \left(0, \frac{q_0}{2}\right)$ and for $r > 0$ small enough such that

$$\left(\frac{1}{p} - \frac{\epsilon}{q_0} - \bar{c}_\epsilon r^{p^*-p}\right) > 0.$$

We obtain that for all $u \in H$ with $\|u\|_H = r$.

$$J(u) \geq \left(\frac{1}{p} - \frac{\epsilon}{q_0} - \bar{c}_\epsilon r^{p^*-p}\right) r^p = \alpha > 0.$$

(ii) By condition (F_3) we have

$$F(x,z) > \lambda |z|^\mu \quad \text{for all } |z| \geq \eta \text{ and a.e. } x \in \Omega.$$

where λ and η are two positive constants.

Now let $\varphi_0(x) \in C_0^\infty(\Omega)$ be such that

$$\text{meas}\{x \in \Omega : |\varphi_0(x)| \geq \eta\} > 0.$$

Then for $t > 0$ we have

$$J(t\varphi_0) = \frac{t^p}{p} \int_\Omega (h(x)|\nabla \varphi_0|^p + q(x)|\varphi_0|^p) dx - \int_\Omega F(x, t\varphi_0) dx$$
$$= \frac{t^p}{p}\|\varphi_0\|_H^p - \int_{x\in\Omega:|\varphi_0(x)|\geq\eta} F(x,t\varphi_0)dx - \int_{x\in\Omega:|\varphi_0(x)|\leq\eta} F(x,t\varphi_0)dx$$
$$\leq \frac{t^p}{p}\|\varphi_0\|_H^p - t^\mu \lambda \int_{x\in\Omega:|\varphi_0(x)|\geq\eta} |\varphi_0|^\mu dx. \tag{2.12}$$

Since $\mu > p$ the right-hand side of (2.13) converges to $-\infty$ when $t \to +\infty$. Then there exists $t_0 > 0$ such that $\|t_0\varphi_0\|_H > r$ and $J(t_0\varphi_0) < 0$. Set $u_0 = t_0\varphi_0$ we have $J(u_0) < 0$ and $\|u_0\|_H > r$. The proof of Proposition 2.6 is complete. □

Proposition 2.7.

(i) $J(0) = 0$

(ii) *The acceptable set*

$$G = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = u_0\} \text{ is not empty.}$$

Proof. (i) It follows from the definition of functional J that $J(0) = 0$.
(ii) Let $\gamma(t) = tu_0$ where u_0 is given in Proposition 2.6. It is clear that$\gamma(t) \in C([0,1], H)$ and $\gamma(0) = 0, \gamma(1) = u_0$. Hence $\gamma \in G$ and G is not empty. □

***Proof of Theorem* 2.1.** By Propositions 2.3-2.7, all assumptions of the variation of the mountain pass theorem introduced in [3] are satisfied. Therefore, there exists $\hat{u} \in H$ such that

$$0 < \alpha \leq J(\hat{u}) = \inf\{\max J(\gamma([0,1])) : \gamma \in G\}$$

and

$$\langle J'(\hat{u}), v\rangle = 0 \quad \text{for all } v \in H,$$

i.e. $\hat{u}$ is a weak solution of problem (1.1). The solution $\hat{u}$ is not trivial since $J(\hat{u}) > 0 = J(0)$.Theorem 2.1 is completely proved. □

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References

[1] A. Abakhti-Mchachti and J. Fleckinger-Pellé, Existence of positive solutions for non-cooperative semilinear elliptic systems defined on an unbounded domain, *Pitman Research Notes in Maths.* **266** (1992), 92–106.
[2] D.G. Costa, On a class of elliptic systems in R^n, *Electron. J. Differential Equations* **7** (1994), 1–14.
[3] D.M. Duc, Nonlinear singular elliptic equations, in *Mathematical Analysis and Applications*, *J. London. Math. Soc.* **40** (2) (1989), 420–440.
[4] H.Q. Toan and N.T. Chung, Existence of weak solutions for a class of nonuniformly nonlinear elliptic equations in unbounded domains, *Nonlinear Anal.* **70** (2009), 3987–3996.
[5] M. Mihilescu, Existence and multiplicity of weak solutions for a class of degenerate nonlinear elliptic equations, *Boud. Value Probl. Art.ID*, **41295** (2006), 1–17.
[6] N.T. Vu and D.M. Duc, Nonuniformly elliptic equations of p-Laplacian type, *Nonlinear Anal.* **61** (2005), 1483–1495.

G.A. Afrouzi, *University of Mazandaran, Babolsar, Iran*.
E-mail: afrouzi@umz.ac.ir

Z. Naghizadeh, *University of Mazandaran, Babolsar, Iran*.
E-mail: z.naghizadeh@umz.ac.ir

F. Mirzadeh, *Islamic Azad University, Ghaemshahr Branch, P.O.Box 163, Ghaemshahr, Iran*.

M. Amirian, *Islamic Azad University, Ghaemshahr Branch, P.O.Box 163, Ghaemshahr, Iran*.

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