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Research Article

Oscillation Theorems for Certain Forced Nonlinear Discrete Fractional Order Equations

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Abstract. The main objective of this work is to obtain some new sufficient conditions that are essential for the oscillation of the solutions of forced nonlinear discrete fractional equations of the form

$$\Delta\left[\Delta^{\mu}(u(j))\right]+\eta(j)\Phi(u(j))=\psi(j), \quad j\in N_0$$

where $\Delta^{\mu-1}u(0)=u_0$; $\Delta u(j)=u(j+1)-u(j)$ and Δ^{μ} is defined as the difference operator of the Riemann-Liouville (R-L) derivative of order $\mu\in(0,1]$ and $N_0=\{0,1,2,\cdots\}$. Numerical examples are presented to show the validity of the theoretical results.

Keywords. Oscillation; Fractional order difference equations; Forcing term

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1. Introduction

Fractional calculus, which is an evolving field of applied mathematics, considers integrals and derivatives of an arbitrary order. Moreover, students from engineering, economics, applied mathematics and science encounter differential calculus operators $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, etc., but only few of them deliberate whether it is essential for the derivative to be only of an integer order. The fundamental concept of fractional calculus is widely believed to rise from a question raised by Marquis to Leibniz in the year 1695.

Fractional calculus has one of the most novel approaches to engineering, science and technology with a broad range of applications. In the past few decades, many researchers studied the stability of solutions, existence and uniqueness of solutions of fractional differential equations. Recently, some papers examine oscillatory theorems for forced nonlinear fractional order differential equations (for details, see [10], [16], [18], [23], [25]), and there are books that summarize and organize the theories and applications of fractional differential equations (for details, see [19], [22]).

Fractional differential equations are apt to model physical processes which vary with time and space and their non-local property enables to model systems with increased memory effectiveness. Fractional calculus endorses integration and differentiation to a fractional order whose values could be real and extended to imaginary. The discrete version of fractional differential equations is the fractional difference equations with fractional order sum and difference operators as basic notions. Ever since Kuttner (for details, see [20]) mentioned the fractional order differences for the first time in 1956, the theory of difference equations of fractional order has been evolving (for details, see [1], [3], [8], [15], [17]). In recent years, the investigation of qualitative properties of discrete fractional equations has risen to prominence, with the study of oscillation of solutions of fractional difference equations drawing the interest of many researchers (for details, see [6], [11], [12], [13], [14], [24] and the references therein). Recently paper [21] dealt with the oscillatory properties of certain nonlinear fractional nabla difference equations of the form

$$\nabla\left(\nabla_a^\alpha x(t)\right)+q(t)f(x(t))=g(t),\quad t\in N_a$$

where $N_a=\{a,a+1,a+2,\cdots\}$ and $\nabla_a^{-(1-\alpha)}x(t)|_{t=a}=c$ and ∇^α is the Liouvillie fractional nabla difference operator of order α of $x, a\ge 0$ is a real number. Only a few papers have been published on the oscillatory behavior of forced nonlinear fractional order difference equations. The present work is motivated by [21]. We examine the new oscillation criteria for forced fractional order discrete nonlinear equations of the form,

$$\Delta\left[\Delta^\mu(u(j))\right]+\eta(j)\Phi(u(j))=\psi(j),\qquad j\in N_0 \tag{1}$$

where $N_0=\{0,1,2,\cdots\}$ and $\Delta^{\mu-1}u(0)=x_0$; $\Delta u(j)=u(j+1)-u(j)$ and Δ^μ defined as the difference operator of the R-L derivative of order $\mu\in(0,1]$. The following assumptions hold throughout this paper:

(A) $\Phi:R\to R$ and $u\Phi(u)>0$ for $u\neq 0$;

(B) $\psi:N_0\to R$, and $\eta(j)\ge 0$, $j\in N_0$.

Some new sufficient conditions for oscillation of solutions of forced fractional order discrete nonlinear equations of (1) are established with the assistance of the properties of R-L sum and difference operators. Numerical examples are shown in order to prove the applicability of the given theoretical results. Hence, we hope this work will contribute to the progress of the study of oscillation theory for discrete fractional order forced nonlinear equations. Some preliminary definitions of fractional derivatives and basic lemmas have been given in Section 2. Further the oscillation theorems of the solutions of forced nonlinear discrete fractional order equations (1) are established in Section 3. Finally, numerical applications demonstrate the applicability of the given theoretical results.

2. Preliminaries

In this section, we give some preliminary definitions of fractional derivatives and their basic results, in order to be used in the main results of this work.

Definition 2.1 ([9]). If a solution $u(j)$ of (1) has arbitrarily large zeros, then it is called *oscillatory solution*, otherwise the solution $u(j)$ of (1) is called *nonoscillatory*. Equation (1) is called oscillatory if all its solutions are oscillatory.

Definition 2.2 ([5]). The RL κ^{th}-order fractional sum $\Delta^{-\kappa}$ is defined by

$$\Delta^{-\kappa}u(j) = \frac{1}{\Gamma(\kappa)}\sum_{\ell=a}^{j-\kappa}(j-\ell-1)^{(\kappa-1)}u(\ell), \quad j \in N_0.$$

For any $\kappa \geq 0$, the falling factorial is defined by

$$j^{(\kappa)} = \frac{\Gamma(j+1)}{\Gamma(j+1-\kappa)}.$$

Definition 2.3 ([5]). The RL κ^{th}-order fractional difference Δ^{κ} is defined by

$$\Delta^{\kappa}u(j) = \Delta^{\beta}\Delta^{-(\beta-\kappa)}u(j), \quad j \in N_0$$

and so

$$\Delta^{\kappa}u(j) = \frac{\Delta^{\beta}}{\Gamma(\beta-\kappa)}\sum_{\ell=a}^{j-\beta+\kappa}(j-\ell-1)^{(\beta-\kappa-1)}u(\ell), \quad j \in N_0.$$

Hence, the law of exponent for fractional sum is

$$\Delta^{-\kappa}[\Delta^{-\mu}u(j)] = \Delta^{-(\kappa+\mu)}u(j) = \Delta^{-\mu}[\Delta^{-\kappa}u(j)].$$

Lemma 2.1 ([5]). *Commutative property of the fractional sum and difference operators. For any $\kappa > 0$, the below equality holds*

$$\Delta^{-\kappa}\Delta u(j) = \Delta\Delta^{-\kappa}u(j) - \frac{(j-a)^{(\kappa-1)}}{\Gamma(\kappa)}\Phi(a).$$

Lemma 2.2 (For details, see [7], [13], [24]). *Let solution of* (1) *be $u(j)$ and*

$$G(j) = \sum_{i=j_0}^{j-1+\mu}(j-1-i)^{(-\mu)}u(i),$$

then $\Delta G(j) = \Gamma(1-\mu)\Delta^{\mu}u(j)$.

Proof.

$$\begin{aligned} G(j) &= \sum_{i=j_0}^{j-1+\mu}(j-1-i)^{(-\mu)}u(i) \\ &= \sum_{i=j_0}^{j-(1-\mu)}(j-i-1)^{(1-\mu)-1}u(i) \\ &= \Gamma(1-\mu)\Delta^{-(1-\mu)}u(j). \end{aligned}$$

Thus

$$\Delta G(j) = \Gamma(1-\mu)\Delta^{\mu}u(j). \qquad \square$$

Lemma 2.3 ([4]). *The power rule for discrete fractional operator is given by*

$$\Delta^{-\kappa} j^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\kappa+1)} j^{(\mu+\kappa)}.$$

Lemma 2.4 ([2]). *For $\lambda > 0$, we have*

$$\lim_{j\to\infty} \frac{\Gamma(j) j^{\lambda}}{\Gamma(j+\lambda)} = 1.$$

3. Main Results

In this section, oscillation theorems of all solutions of forced nonlinear discrete fractional order equations (1) are established based on the properties of R-L sum and difference operators.

Theorem 3.1. *Suppose that* (A) *and* (B) *hold;*

1. *If $u(j) > 0$ is a solution of the equation* (1)*, then $u(j)$ satisfies the difference inequality*
$$\Delta\left[\Delta^{\mu} u(j)\right] \le \psi(j), \quad j \in N_0. \tag{2}$$
2. *If $u(j) < 0$ is a solution of the equation* (1)*, then $u(j)$ satisfies the difference inequality*
$$\Delta\left[\Delta^{\mu} u(j)\right] \ge \psi(j), \quad j \in N_0. \tag{3}$$

Proof. Assume that $u(j) > 0$, and (A) and (B) hold. Then, by (1) we have

$$\Delta\left(\Delta^{\mu} u(j)\right) = -\eta(j)\Phi(u(j)) + \psi(j) \le \psi(j)$$

which shows that $u(j) > 0$ is a solution of the inequality (2).

Assume that $u(j) < 0$, and (A) and (B) hold. Then, by (1) we have

$$\Delta\left(\Delta^{\mu} u(j)\right) = -\eta(j)\Phi(u(j)) + \psi(j) \ge \psi(j)$$

which shows that $u(j) < 0$ is a solution of the inequality (3). The proof of the theorem is complete. From the above theorem we can conclude the following statement as a proposition.

Proposition 3.1. *If the inequality* (2) *has no eventually positive solution and the inequality* (3) *has no eventually negative solution, then every solution $u(j)$ of* (1) *is oscillatory.*

Theorem 3.2. *Suppose that $u(j)$ is a solution of equation* (1) *and there exists $j_0 \in N_a$ such that $\Delta^{\mu} u(j_0) = c$ exists. If*

$$\liminf_{j\to\infty} \left\{ \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi) \right] \right\} = -\infty, \tag{4}$$

$$\limsup_{j\to\infty} \left\{ \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi) \right] \right\} = +\infty, \tag{5}$$

then $u(j)$ is oscillatory.

Proof. Assume, for the sake of contradiction, that $u(j)$ is a nonoscillatory solution of (1). Then $u(j)$ is eventually positive or eventually negative solution of (1). If $u(j) \ge 0$ for $j \ge j_0$ by Theorem 3.1, we obtain

$$\Delta\left(\Delta^{\mu} u(j)\right) \le \psi(j), \quad j \in N_0.$$

Now summing up from j_0 to $j-1$, we have

$$\sum_{i=j_0}^{j-1} \Delta\left[\Delta^{\mu} u(j)\right] \le \sum_{i=j_0}^{j-1} \psi(i)$$

or

$$\sum_{i=j_0}^{j-1} \left[\Delta^{\mu} u(j+1) - \Delta^{\mu} u(j)\right] \le \sum_{i=j_0}^{j-1} \psi(i)$$

or

$$\Delta^{\mu} u(j) - \Delta^{\mu} u(j_0) \le \sum_{i=j_0}^{j-1} \psi(i)$$

i.e.,

$$\Delta^{\mu} u(j) \le c + \sum_{i=j_0}^{j-1} \psi(i), \tag{6}$$

where $c = \Delta^{\mu} u(j_0)$ is a constant. Now by multiplying $\Delta^{-\mu}$ to the above inequality, we have

$$\Delta^{-\mu}\Delta^{\mu} u(j) \le \Delta^{-\mu}\left[c + \sum_{i=j_0}^{j-1} \psi(i)\right]. \tag{7}$$

Applying Lemma 2.1. to the above inequality, we get

$$\begin{aligned}\Delta^{-\mu}\Delta^{\mu} u(j) &= \Delta^{-\mu}\Delta\Delta^{-(1-\mu)} u(j)\\ &= \Delta\Delta^{-\mu}\Delta^{-(1-\mu)} u(j) - \frac{j^{(\mu-1)}}{\Gamma(\mu)}\Delta^{\mu-1} u(j_0)\\ &= \Delta\Delta^{-\mu}\Delta^{-(1-\mu)} u(j) - \frac{j_0}{\Gamma(\mu)} j^{(\mu-1)}\end{aligned}$$

or

$$\Delta^{-\mu}\Delta^{\mu} u(j) = u(j) - \frac{u_0}{\Gamma(\mu)} j^{(\mu-1)}. \tag{8}$$

On the other hand, we have

$$\Delta^{-\mu}\left[c + \sum_{i=j_0}^{j-1} \psi(i)\right] = \frac{1}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]. \tag{9}$$

Using (8) and (9) in (7), we obtain

$$u(j) - \frac{u_0}{\Gamma(\mu)} j^{(\mu-1)} \le \frac{1}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \phi(\xi)\right]$$

or

$$\Gamma(\mu) j^{1-\mu} u(j) \le u_0 j^{1-\mu} j^{(\mu-1)} + j^{1-\mu} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]$$

or

$$j^{1-\mu} u(j) \le \frac{u_0}{\Gamma(\mu)} j^{1-\mu} j^{(\mu-1)} + \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]. \tag{10}$$

Taking limit as $j \to \infty$ in (10) and using Lemma 2.4, we have

$$\liminf_{j\to\infty}\left[j^{1-\mu} u(j)\right] \le \liminf_{j\to\infty}\left[\frac{u_0}{\Gamma(\mu)} j^{1-\mu} j^{(\mu-1)} + \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]\right]$$

$$\lim_{j\to\infty} j^{\mu-1} j^{(\mu-1)} = \lim_{j\to\infty} \frac{j^{1-\mu}\Gamma(j+1)}{\Gamma(j+2-\mu)} = \lim_{j\to\infty} \frac{\Gamma(j) j^{1-\mu}}{\left(1+\frac{1-\mu}{j}\right)\Gamma(j+1-\mu)}$$

$$= \lim_{j\to\infty} \frac{\Gamma(j) j^{1-\mu}}{\Gamma(j+1-\mu)} = 1$$

$$\liminf_{j\to\infty} [j^{1-\mu} u(j)] < -\infty$$

which contradicts $u(j) > 0$.

If $u(j) < 0$ for $j \ge j_0$ by Theorem 3.1, we get

$$\Delta\left(\Delta^{\mu} u(j)\right) \ge \psi(j), \quad j \in N_0.$$

Summing up from j_0 to $j-1$, we have

$$\Delta^{\mu} u(j) \ge c + \sum_{i=j_0}^{j-1} \psi(i), \tag{11}$$

where $c = \Delta^{\mu} u(j_0)$ is a constant. Now by multiplying $\Delta^{-\mu}$ to the above inequality, we have

$$\Delta^{-\mu}\Delta^{\mu} u(j) \ge \Delta^{-\mu}\left[c + \sum_{i=j_0}^{j-1} \psi(i)\right]. \tag{12}$$

Using (8) and (9) in (12), we have

$$u(j) - \frac{u_0}{\Gamma(\mu)} j^{(\mu-1)} \ge \frac{1}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right],$$

$$\Gamma(\mu) j^{1-\mu} u(j) \ge u_0 j^{1-\mu} j^{(\mu-1)} + j^{1-\mu} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right],$$

$$j^{1-\mu} u(j) \ge \frac{u_0}{\Gamma(\mu)} j^{1-\mu} j^{(\mu-1)} + \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]. \tag{13}$$

Taking limit as $j \to \infty$ in (13) and using Lemma 2.4, we have

$$\limsup_{j\to\infty} [j^{1-\mu} u(j)] \ge \limsup_{j\to\infty} \left[\frac{u_0}{\Gamma(\mu)} j^{1-\mu} j^{(\mu-1)} + \frac{j^{1-\mu}}{\Gamma(\mu)} \sum_{i=0}^{j-\mu} (j-i-1)^{(\mu-1)} \left[c + \sum_{\xi=j_0}^{i-1} \psi(\xi)\right]\right]$$

i.e.,

$$\limsup_{j\to\infty} [j^{1-\mu} u(j)] > +\infty$$

which contradicts $u(j) < 0$. The proof of the theorem is complete. □

Theorem 3.3. *Assume that $u(j)$ is a solution of equation* (1) *and there exists $j_0 \in N_0$ such that $\Delta^{\mu} u(j_0) = c$ exists. If*

$$\liminf_{j\to\infty} \sum_{i=0}^{j-1} \left[\left(1 - \frac{i+1}{j}\right)\psi(i)\right] = -\infty, \tag{14}$$

$$\limsup_{j\to\infty} \sum_{i=0}^{j-1} \left[\left(1 - \frac{i+1}{j}\right)\psi(i)\right] = +\infty, \tag{15}$$

then $u(j)$ is oscillatory.

Assume, for the sake of contradiction, that $u(j)$ is a nonoscillatory solution of (1). Then $u(j)$ is eventually positive or eventually negative. If $u(j) > 0$, $j \geq j_0$ by Theorem 3.1 and proceeding as Theorem 3.2, we obtain

$$\Delta^{\mu} u(j) \leq c + \sum_{i=j_0}^{j-1} \psi(i).$$

Using Lemma 2.2, we get

$$\Delta G(j) \leq \Gamma(1-\mu)\left[c + \sum_{i=j_0}^{j-1} \psi(i)\right].$$

Summing up the last inequality from j_0 to $j-1$, we get

$$\sum_{i=j_0}^{j-1} \Delta G(j) < \Gamma(1-\mu) \sum_{i=j_0}^{j-1} \left[c + \sum_{\xi=j_0}^{j-1} \psi(\xi)\right]$$

or

$$G(j) < G(j_0) + \Gamma(1-\mu) \sum_{i=j_0}^{j-1} \left[c + \sum_{\xi=j_0}^{j-1} \psi(\xi)\right]$$

or

$$G(j) < G(j_0) + c\Gamma(1-\mu)(j-1-j_0) + \Gamma(1-\mu) \sum_{i=j_0}^{j-1} (j-i-1)\psi(i)$$

i.e.,

$$\frac{G(j)}{j} < \frac{G(j_0)}{j} + c\Gamma(1-\mu)\left(1 - \frac{j_0+1}{j}\right) + \Gamma(1-\mu) \sum_{i=j_0}^{j-1} \left(1 - \frac{i+1}{j}\right)\psi(i).$$

Consequently,

$$\lim_{j\to\infty} \frac{G(j)}{j} < -\infty$$

which contradict $G(j) > 0$. Similarly, we can prove that $u(j) < 0$, $j \geq j_0$. The proof of the theorem is complete. □

4. Examples

In this section, we present some examples to show the validity of the theoretical results.

Example 4.1. Consider the following forced nonlinear discrete fractional equation

$$\begin{cases} \Delta\left(\Delta^{\frac{2}{3}} u(j)\right) + \frac{2\Gamma(j+\frac{1}{3})}{3\Gamma(j+1)} u(j) = \frac{2}{3}, \quad j \in N_0 \\ \Delta^{-\frac{1}{3}} u(0) = \frac{2}{3}\Gamma\left(\frac{2}{3}\right). \end{cases} \tag{16}$$

Here $\mu = \frac{2}{3}$, $\eta(j) = \frac{2\Gamma(j+\frac{1}{3})}{3\Gamma(j+1)}$, $\Phi(u) = u$ and $\psi(j) = \frac{2}{3}$. We show that $u(j) = j^{(\frac{2}{3})}$ is a nonoscillatory solution of (16) by simple and careful calculations. By using Lemma 2.3, we have

$$\begin{aligned} \Delta^{\frac{2}{3}} u(j) &= \Delta^{\frac{2}{3}} j^{(\frac{2}{3})} = \Delta^{1-\frac{1}{3}} j^{(\frac{2}{3})} = \Delta\Delta^{-\frac{1}{3}} j^{(\frac{2}{3})} \\ &= \Delta\left[\frac{\Gamma\left(\frac{2}{3}+1\right)}{\Gamma\left(\frac{2}{3}+1+\frac{1}{3}\right)} j^{(\frac{2}{3}+\frac{1}{3})}\right] = \Delta\left[\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)} j^{(1)}\right] = \frac{2}{3}\Gamma\left(\frac{2}{3}\right)\Delta[j] \end{aligned}$$

or

$$\Delta^{\frac{2}{3}}u(j) = \frac{2}{3}\Gamma\left(\frac{2}{3}\right). \tag{17}$$

Therefore,

$$\Delta\left[\Delta^{\frac{2}{3}}u(j)\right] = \Delta\left[\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\right] = 0. \tag{18}$$

By using the result of Definition 2.2, we obtain

$$u(j) = j^{\left(\frac{2}{3}\right)} = \frac{j\Gamma(j)}{\Gamma\left(j+1-\frac{2}{3}\right)} = \frac{j\Gamma(j)}{\Gamma\left(j+\frac{1}{3}\right)}. \tag{19}$$

Combining (17)-(19), we conclude that $u(j) = j^{\left(\frac{2}{3}\right)}$ is a solution of eq. (16), as shown in Figure 1.

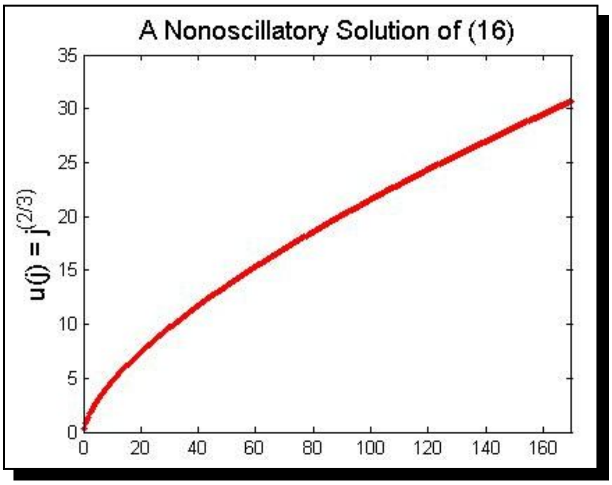

Figure 1. Nonoscillatory behavior of (16)

For the solution $u(j) = j^{\left(\frac{2}{3}\right)}$ of eq. (16), it is easy to see that there exists $j_0 \in N_0$ such that $\Delta^{-\frac{1}{3}}u(0) = c = \frac{2}{3}\Gamma\left(\frac{2}{3}\right)$ and

$$\frac{j^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\sum_{i=0}^{j-\frac{2}{3}}(j-i-1)^{\left(-\frac{1}{3}\right)}\left[c+\sum_{\epsilon=j_0}^{i-1}\psi(\epsilon)\right] = \frac{j^{\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\sum_{i=0}^{j-\frac{2}{3}}(j-i-1)^{\left(-\frac{1}{3}\right)}\left[\frac{2}{3}\Gamma\left(\frac{2}{3}\right)+\sum_{\epsilon=j_0}^{i-1}\frac{2}{3}\right] > 0,\ j \in N_0, j \geq 1$$

which proves that the condition (4) of Theorem 3.2 does not hold.

Example 4.2. Consider the following forced nonlinear discrete fractional equation

$$\begin{cases} \Delta\left(\Delta^{\frac{1}{4}}u(j)\right) + \frac{\Gamma\left(j+\frac{3}{4}\right)}{4\Gamma(j+1)}u(j) = \frac{1}{4}, \quad j \in N_0 \\ \Delta^{-\frac{3}{4}}u(0) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right). \end{cases} \tag{20}$$

Here $\mu = \frac{1}{4}$, $\eta(j) = \frac{\Gamma\left(j+\frac{3}{4}\right)}{4\Gamma(j+1)}$, $\Phi(u) = u$ and $\psi(j) = \frac{1}{4}$. Obviously, there exists $j_0 \in N_0$, such that

$$\sum_{i=0}^{j-1}\left(1-\frac{i+1}{j}\right)\psi(i) = \frac{1}{4}\sum_{i=0}^{j-1}\left(1-\frac{i+1}{j}\right) > 0.$$

Thus the condition (14) of Theorem 3.3 does not hold. In fact, by using a similar method as in Example 4.1, we can verify that $u(j) = j^{\left(\frac{1}{4}\right)} > 0$ is a nonoscillatory solution of (20), as shown in Figure 2.

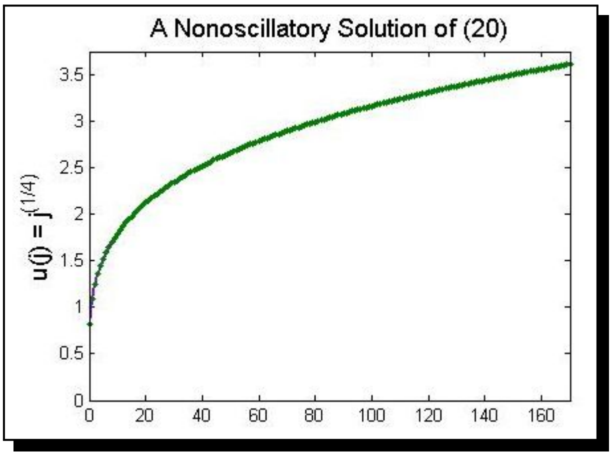

Figure 2. Nonoscillatory behavior of (20)

5. Conclusion

This paper is assigned to establishing some new sufficient conditions that are essential for the oscillation of the solutions of forced nonlinear discrete fractional equations. In particular, we proved two oscillation theorems for the proposed equations, in order to used the properties of R-L sum and difference operators, comparison results and some mathematical inequalities. Also we presented two numerical examples with figures that demonstrate the applicability of the given theoretical results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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