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DEA with a Perfect Object: Analytical Solutions

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Abstract. For the main DEA models, adding a Perfect Object — that is, a virtual object that has the smallest inputs and greatest outputs — to a collection of actual objects permits obtaining solutions analytically. The paper derives formulas for the solutions and demonstrates that computations with them comprise simple operations with ratios of inputs and outputs while avoiding the use of *linear programming* (LP) algorithms. A numerical example illustrates the utility of the approach.

1. Introduction

Data Envelopment Analysis (DEA), developed in the late 1970s and the 1980s by Charnes *et al.* (1978) and Banker *et al.* (1984), has now become a well-established means of estimating the relative efficiency of a group of objects referred to as *Decision-Making Units* (DMUs). The objects use inputs $\boldsymbol{X} = (X_j, j = 1, \dots, r) > 0$ to produce outputs $\boldsymbol{Y} = (Y_i, i = 1, \dots, s) > 0$, and DEA combines all of the indicators of each object into a single efficiency score scaled to an interval $[0, 1]$. An object is considered *efficient* if it receives a score equal to 1, and *inefficient* if it receives a score of less than 1. The DEA efficiency measure is based on the efficiency ratio suggested by Farrell (1957):

$$E = \frac{\sum_{i=1}^{s} u_i Y_i}{\sum_{j=1}^{r} v_j X_j}\,, \tag{1}$$

where $\boldsymbol{u} = (u_1, \dots, u_s)$ and $\boldsymbol{v} = (v_1, \dots, v_r)$ are nonnegative weights assigned to outputs and inputs, respectively. To estimate the weights, DEA sets up a series of optimization problems similar to the following one:

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For each DMU_k, $k = 1, \ldots, N$, find nonnegative vectors $\boldsymbol{u}_k = (u_{k1}, \ldots, u_{ks})$ and $\boldsymbol{v}_k = (v_{k1}, \ldots, v_{kr})$ such that:

$$E_k = \frac{\sum_{i=1}^{s} u_{ki} Y_{ki}}{\sum_{j=1}^{r} v_{kj} X_{kj}} \to \max\,, \tag{2}$$

subject to $E_m \leq 1$ for all DMU_m, $m = 1, \ldots, N$, in the group with the same weight coefficients $\boldsymbol{u}_k = (u_{k1}, \ldots, u_{ks})$ and $\boldsymbol{v}_k = (v_{k1}, \ldots, v_{kr})$.

Specific DEA problems call for different modifications of the basic efficiency ratio (1) and different constraints to ensure the existence of the solution. Cooper *et al.* (2006), Thanassoulis (2001), and Zhu (2008) provide details and reviews of contemporary methodology and techniques related to DEA.

The main advantage of DEA is its capacity to assign values to weight coefficients $\boldsymbol{u}$ and $\boldsymbol{v}$ objectively. Conceptually, DEA allows each DMU to assign its own weight coefficients to each input and output favorably. However, the potential of a given DMU to achieve the maximal efficiency score is restricted by the requirement that with the weight coefficients assigned to itself, no other DMU in the group receives an efficiency score greater than 1. This means that a poorly performing DMU cannot achieve a high efficiency score for itself by a manipulation of the weight coefficients. In such a case, an object that performs extremely well would have received an efficiency score greater than 1. Charnes *et al.* (1978), as well as Banker *et al.* (1984) showed that maximization of the efficiency ratio (2) is equivalent to solving a series of *linear programming* (LP) problems, one for each DMU in a group.

Applying a statement of the DEA problem suggested by Vaninsky (2008), this paper adds a *Perfect Object* (PO) to the group of actual DMUs. The PO is given the smallest inputs and greatest outputs in the group. As a result, it serves as an objective benchmark for efficiency comparisons. The paper demonstrates that the presence of the PO permits obtaining solutions of the main DEA problems analytically, in explicit form. The formulas that are obtained have two important advantages: they require only a moderate number of simple operations with ratios of inputs and outputs, and they obviate the need for using LP algorithms.

The DEA models that this paper considers are the *input minimization* (IM) and *output maximization* (OM) models, each with *constant and variable returns to scale* (CRS and VRS, respectively). In addition, the paper considers a DEA model with *undesirable outputs* (UO) for a case in which some outputs are undesired though unavoidable — such as CO_2 emissions in energy generation. Färe *et al.* (1989) stated this problem, Seiford and Zhu (2002, 2005) suggested a DEA model for its solution, and Zhou *et al.* (2008) offered details and discussion of it.

This paper presents results that permit the solving of DEA problems through the use of simple spreadsheet formulas. For problems of small to moderate size, scientific calculators suffice. The remainder of the paper presents results obtained for the main DEA models and offers a numerical example that illustrates the general usefulness of the results in solving DEA problems. In addition to discussing the results for the models, section 2 gives a theorem substantiating the explicit form of solutions in the case of each model, and section 3 presents and explores an example.

2. **Analytical Solutions for the Main DEA Models**

This section considers a group of N objects referred to as DMU_k, $k = 1, \ldots, N$, using inputs $\boldsymbol{X}_k = (X_{kj}, j = 1, \ldots, r) > 0$ to produce outputs $\boldsymbol{Y}_k = (Y_{ki}, i = 1, \ldots, s) > 0$. We append to the group a Perfect Object, referred to as DMU_0, such that

$$0 < X_{0j} \leq \min_j (X_{kj}, j = 1, \ldots, r), \quad Y_{0i} \geq \max_i (Y_{ki}, i = 1, \ldots, s) > 0. \tag{3}$$

Figure 1, borrowed from Vaninsky (2008), shows a geometric interpretation of the impact of the PO on CRS and VRS efficiency frontiers. In Figure 1, a Perfect Object is located at point F so that $X_{0j} = \min X_{kj}$, $j = 1, \ldots, r$; $Y_{0i} = \max Y_{ki}$, $i = 1, \ldots, s$; $k = 1, \ldots, N$.

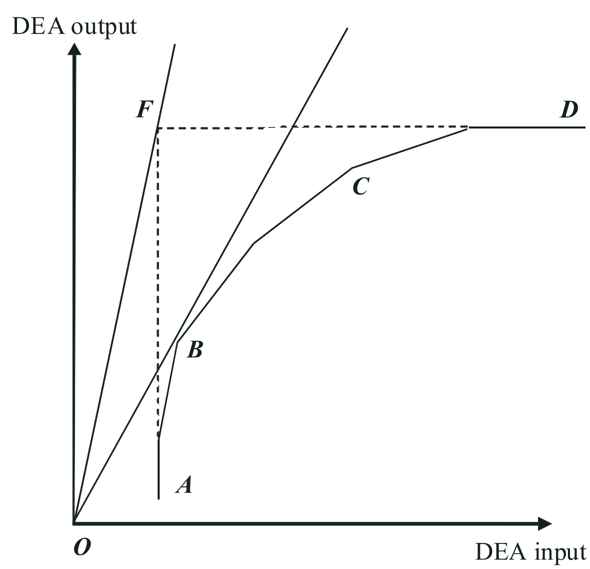

Figure 1. DEA frontiers, Vaninsky (2008).
A Perfect Object (DMU_0) is located at point F: $X_{01} = \min X_{k1}$, $Y_{01} = \max Y_{k1}$, $k = 1, \ldots, N$.
Frontiers: OB — constant returns to scale;
$ABCD$ — variable returns to scale;
OF — constant returns to scale with the Perfect Object;
AFD — variable returns to scale with the Perfect Object.

The following DEA models are considered in this section:

1. Input minimization model, constant returns to scale (IM CRS).
For each DMU_k, $k = 1,\ldots,N$, find a nonnegative vector $\boldsymbol{\lambda}_k = (\lambda_{k0}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kN}) = 0$ and scalar θ_k such that

$$\theta_k \to \min\,,$$

subject to

$$\begin{aligned} &\sum_{m=0}^{N} \lambda_{km} X_{mj} \le \theta_k X_{kj}, \quad j = 1,\ldots,r; \\ &\sum_{m=1}^{N} \lambda_{km} Y_{mi} \ge Y_{ki}, \quad i = 1,\ldots,s; \\ &\lambda_{km} \ge 0, \quad m = 1,\ldots,N\,. \end{aligned} \tag{4}$$

2. Input minimization model, variable returns to scale (IM VRS).
The IM VRS model resembles the IM CRS model with an additional constraint imposed on the vector $\boldsymbol{\lambda}_k$:

$$\sum_{m=0}^{m=N} \lambda_{km} = 1, \quad k = 1,\ldots,N. \tag{5}$$

3. Output maximization model, constant returns to scale (OM CRS).
For each DMU_k, $k = 1,\ldots,N$, find a nonnegative vector $\boldsymbol{\lambda}_k = (\lambda_{k0}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kN}) = 0$ and scalar ω_k such that

$$\omega_k \to \max\,,$$

subject to

$$\begin{aligned} &\sum_{m=0}^{N} \lambda_{km} X_{mj} \le X_{kj}, \qquad j = 1,\ldots,r; \\ &\sum_{m=1}^{N} \lambda_{km} Y_{mi} \ge \omega_k Y_{ki}, \quad i = 1,\ldots,s; \\ &\lambda_{km} \ge 0, \qquad\qquad\quad m = 1,\ldots,N; \end{aligned} \tag{6}$$

4. Output maximization model, variable returns to scale (OM VRS).
The OM VRS model resembles the OM CRS model with an additional constraint (5) imposed on vector the $\boldsymbol{\lambda}_k$.

5. Output maximization model with undesirable outputs, variable returns to scale (OM UO VRS).
For each DMU_k, $k = 1,\ldots,N$, find a nonnegative vector $\boldsymbol{\lambda}_k = (\lambda_{k0}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kN}) = 0$ and scalar ω_k such that

$$\omega_k \to \max,$$

subject to

$$\begin{aligned}
&\sum_{m=0}^{N} \lambda_{km} X_{mj} \le X_{kj}, \qquad j = 1,\ldots,r;\\
&\sum_{m=1}^{N} \lambda_{km} Y_{mi} \ge \omega_k Y_{ki}, \quad i = 1,\ldots,p;\\
&\sum_{m=1}^{N} \lambda_{km} \overline{Y}_{mi} \ge \omega_k Y_{ki}, \quad i = p+1,\ldots,s;\\
&\overline{Y}_{mi} = W_i - Y_{mi} > 0, \qquad i = p+1,\ldots,s;\\
&\sum_{m=0}^{N} \lambda_{km} = 1;\\
&\lambda_{km} \ge 0, \qquad m = 1,\ldots,N;\\
&W_i > \max_{0\le m\le N} Y_{mi}, \qquad i = p+1,\ldots,s;
\end{aligned} \tag{7}$$

In this model, the outputs Y_{mi}, $i = p+1,\ldots,s$; $m = 0,\ldots,N$ — that is, those from the $(p+1)$-th through the s-th — are undesirable. The model actually tends to decrease these outputs by increasing corresponding translated outputs $\overline{Y}_{mi}$, their complements to the values of W_i, $i = p+1,\ldots,s$. The values chosen for W_i, are large enough to make all of the translated outputs positive. As Seiford and Zhu (2002) showed, positiveness of the translated outputs is guaranteed for the VRS model only. It is also true that the extent to which the choice of specific values of W_i affects the efficiency scores has yet to be completely investigated.

The discussion below in this section presents considerations regarding DEA with a Perfect Object. In the DEA models given by formulas (4)-(7), the efficiency scores are defined as follows:

$$E_{\text{IM}} = \theta, \quad E_{\text{OM}} = \frac{1}{\omega}\,, \tag{8}$$

for all models, respectively. As is known,

$$E_{\text{IM CRS}} = E_{\text{OM CRS}}; \quad E_{\text{CRS}} \le E_{\text{VRS}}\,. \tag{9}$$

The inequality in formulas (9) holds for both IM and OM models. Cooper *et al.* (2006) discuss in detail the equivalence of the efficiency scores given by formulas (8) to those given by formulas (1) and (2). As in Banker *et al.* (1984), $E_{\text{IM CRS}}$ or $E_{\text{OM CRS}}$ are referred to as *total efficiency*, corresponding values of $E_{\text{IM VRS}}$ or $E_{\text{OM VRS}}$ are called *technical efficiency*, and their ratios, $\frac{E_{\text{IM VRS}}}{E_{\text{IM CRS}}}$ or $\frac{E_{\text{OM VRS}}}{E_{\text{OM CRS}}}$ are known as *scale efficiency*, respectively.

The DEA models given by formulas (4)-(8) have the following interpretations. For a given DMU$_k$, an LP algorithm tends to construct a linear combination of all DMUs in a group, including the DMU$_k$ itself, which provides either the smallest possible inputs combined with outputs that are at least the same (IM models), or the greatest possible outputs combined with inputs that are at most the same

(OM models). Because the DMU_k itself is included, the problems have solutions $\theta_k = 1$ or $\omega_k = 1$, correspondingly. However, better solutions can be found for some DMUs with $\theta_k < 1$ or $\omega_k > 1$, respectively. DMUs with $\min\theta_k = 1$ or $\max\omega_k = 1$ are referred to as efficient; otherwise, they are considered inefficient. The performance of the inefficient DMUs can be improved by acquiring technological, managerial, or other features of other DMUs in a group. It is known that a linear combination providing $\min\theta_k$ or $\max\omega_k$ may be selected so that only efficient DMUs are included. In other words, in a vector $\boldsymbol{\lambda}_k = (\lambda_{k0}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kN}) \geq 0$ providing an optimal solution for a given DMU_k, only λ_{km} corresponding to efficient DMU_m are non-zero. Such DMUs are referred to as peer DMUs for a given DMU_k.

In the presence of a PO, the situation can be refined. This paper shows below that an optimal solution always exists with the PO as the sole peer DMU. In other words, an optimal solution exists with $\lambda_{k0} > 0$, and all the remaining $\lambda_{km} = 0$, $m = 1, \ldots, N$. If the PO is chosen so that $X_{oj} < \min X_{kj}$, $j = 1, \ldots, s$; $Y_{oi} > \max Y_{ki}$, $i = 1, \ldots, r$; $k = 1, \ldots, N$; then only the PO is efficient. In this situation, the conclusion follows from the general theory of DEA. But the case in which $X_{oj} = \min X_{kj}$, $j = 1, \ldots, s$; $Y_{oi} = \max Y_{ki}$, $i = 1, \ldots, r$; $k = 1, \ldots, N$ — actually the most important case for applications — requires special consideration.

Lemma. *Given a Perfect Object — that is, a* DMU_0 *satisfying conditions* (3) *— an optimal solution exists with only* $\lambda_{k0} > 0$*, and all the remaining* $\boldsymbol{\lambda}_{km} = 0$*,* $m = 1, \ldots, N$*.*

Proof. To prove the lemma, we will consider each DEA model separately.

1. IM CRS model.

Let $\boldsymbol{\lambda}_k^* = (\lambda_{km}^*, m = 1, \ldots, N)$ and θ_k^* be an optimal solution for a given DMU_k. Then, in accordance with formula (4), we have

$$\begin{aligned} &\sum_{m=0}^{N} \lambda_{km}^* X_{mj} \leq \theta_k^* X_{kj}, \quad && j = 1, \ldots, r; \\ &\sum_{m=1}^{N} \lambda_{km}^* Y_{mi} \geq Y_{ki}, && i = 1, \ldots, s; \\ &\lambda_{km}^* \geq 0, && m = 1, \ldots, N. \end{aligned} \tag{10}$$

By exchanging each X_{mj} and Y_{mi} in formula (10) for corresponding values of X_{0j} and Y_{0i} and taking formula (3) into consideration, we obtain

$$\begin{aligned} &\left(\sum_{m=0}^{N} \lambda_{km}^*\right) X_{0j} = \sum_{m=0}^{N} \lambda_{km}^* X_{0j} \leq \sum_{m=0}^{N} \lambda_{km}^* X_{mj} \leq \theta_k^* X_{kj}, \quad && j = 1, \ldots, r; \\ &\left(\sum_{m=1}^{N} \lambda_{km}^*\right) Y_{0i} = \sum_{m=1}^{N} \lambda_{km}^* Y_{0i} \geq \sum_{m=1}^{N} \lambda_{km}^* Y_{mi} \geq Y_{ki}, && i = 1, \ldots, s. \end{aligned} \tag{11}$$

Denoting $\left(\sum\limits_{m=0}^{N} \lambda_{km}^{*}\right)_{k0}^{**}$ as λ_{k0}^{**}, we obtain

$$\begin{aligned} &\lambda_{k0}^{**} X_{0j} \le \theta_k^* X_{kj}, \quad j = 1, \dots, r, \\ &\lambda_{k0}^{**} Y_{0i} \ge Y_{ki}, \qquad i = 1, \dots, s, \end{aligned} \tag{12}$$

as desired.

2. IM VRS model.

The proof resembles that for the IM CRS model with

$$\lambda_{k0}^{**} = \left(\sum_{m=0}^{N} \lambda_{km}^{*}\right) = 1, \tag{13}$$

as follows from formula (5). In this case, we obtain:

$$\begin{aligned} &X_{0j} \le \theta_k^* X_{kj}, \quad j = 1, \dots, r, \\ &Y_{0i} \ge Y_{ki}, \qquad i = 1, \dots, s, \end{aligned} \tag{14}$$

with the PO being the only peer DMU.

3. OM CRS model.

In a similar manner as for the IM CRS model, let $\boldsymbol{\lambda}_k^* = (\lambda_{km}^*, m = 1, \dots, N)$ and ω_k^* be an optimal solution for a given DMU_k. Then, in accordance with formula (6), we have

$$\begin{aligned} &\left(\sum_{m=0}^{N} \lambda_{km}^{*}\right) X_{0j} = \sum_{m=0}^{N} \lambda_{km}^{*} X_{0j} \le \sum_{m=0}^{N} \lambda_{km}^{*} X_{mj} \le X_{kj}, \quad j = 1, \dots, r; \\ &\left(\sum_{m=1}^{N} \lambda_{km}^{*}\right) Y_{0i} = \sum_{m=1}^{N} \lambda_{km}^{*} Y_{0i} \ge \sum_{m=1}^{N} \lambda_{km}^{*} Y_{mi} \ge \omega_k^* Y_{ki}, \quad i = 1, \dots, s. \end{aligned} \tag{15}$$

As previously,

$$\begin{aligned} &\lambda_{k0}^{**} X_{0j} \le X_{kj}, \qquad j = 1, \dots, r, \\ &\lambda_{k0}^{**} Y_{0i} \ge \omega_k^* Y_{ki}, \quad i = 1, \dots, s, \end{aligned} \tag{16}$$

where λ_{k0}^{**} is given by formula (13). As above, the PO is the only peer object.

4. OM VRS model.

The following proof resembles that for the case of the OM CRS model with $\lambda_{k0}^{**} = 1$, as given by formula (5). In a similar manner as for the OM CRS model, we obtain

$$\begin{aligned} &X_{0j} \le X_{kj}, \qquad j = 1, \dots, r, \\ &Y_{0i} \ge \omega_k^* Y_{ki}, \quad i = 1, \dots, s, \end{aligned} \tag{17}$$

as required.

5. The model with undesirable outputs, OM UO VRS.

A proof in terms of translated outputs follows directly from the case of the OM VRS model. The only issue is to show that the optimal values obtained for the undesirable outputs, Y^*_{ki}, $i = p+1,\dots,s$, are positive. From formula (17) we have

$$\overline{Y}_{0i} = W_i - Y_{0i} \geq \omega^*_k \overline{Y}_{ki} = W_i - Y^*_{ki}, \quad i = p+1,\dots,s, \tag{18}$$

where the last equality is the definition of Y^*_{ki}. From formula (18) we obtain

$$Y^*_{ki} \geq Y_{0i} > 0, \quad i = p+1,\dots,s, \tag{19}$$

as required.

Therefore, the lemma is proved. □

Now we can prove a theorem that provides analytical solutions for the DEA models considered above. We assume that for a given DMU$_k$, $k = 1,\dots,N$, $\boldsymbol{\lambda}^*_k = (\lambda^*_{km}, m = 1,\dots,N) = (\lambda^*_{k0}, 0,\dots,0)$ is a vector providing optimal solutions $\theta^*_k = \min \theta_k$ or $\omega^*_k = \max \omega_k$ for θ_k or ω_k, respectively. For VRS problems, as mentioned above, $\lambda^*_{k0} = 1$.

Theorem. *Efficiency scores for the DEA models under consideration are as follows:*

(a) *Input minimization model, constant returns to scale* (IM CRS)*:*

$$E_{k\ IM\ CRS} = \theta^*_k = \max_{0\leq j\leq r} \frac{X_{0j}}{X_{kj}} \times \max_{0\leq i\leq s} \frac{Y_{ki}}{Y_{oi}}\ . \tag{20}$$

(b) *Input minimization model, variable returns to scale* (IM VRS)*:*

$$E_{k\ IM\ VRS} = \theta^*_k = \max_{0\leq j\leq r} \frac{X_{0j}}{X_{kj}}\ . \tag{21}$$

(c) *Output maximization model, constant returns to scale* (OM CRS)*:*

$$E_{k\ OM\ CRS} = \frac{1}{\omega^*_k} = \max_{0\leq j\leq r} \frac{X_{0j}}{X_{kj}} \times \max_{0\leq i\leq s} \frac{Y_{ki}}{Y_{0i}}, \tag{22}$$

which is the same as E_{kIMCRS}.

(d) *Output maximization model, variable returns to scale* (OM VRS)*:*

$$E_{k\ IM\ VRS} = \frac{1}{\omega^*_k} = \max_{0\leq i\leq s} \frac{Y_{ki}}{Y_{0i}} \tag{23}$$

(e) *Output maximization model with undesirable outputs, variable returns to scale* (OM UO VRS)*:*

$$\begin{aligned} E_{k\ OM\ VRS\ UO} = \frac{1}{\omega^*_k} &= \max\left(\max_{0\leq i\leq p} \frac{Y_{ki}}{Y_{0i}}, \max_{p+1\leq i\leq s} \frac{\overline{Y}_{ki}}{\overline{Y}_{0i}} \right) \\ &= \max\left(\max_{0\leq i\leq p} \frac{Y_{ki}}{Y_{0i}}, \max_{p+1\leq i\leq s} \frac{W_i - Y_{ki}}{W_i - Y_{0i}} \right). \end{aligned} \tag{24}$$

Proof. (a) IM CRS model.

Given

$$\begin{aligned} &\lambda^*_{k0} X_{0j} \leq \theta^*_k X_{kj}, \quad j = 1,\dots,r, \\ &\lambda^*_{k0} Y_{0i} \geq Y_{ki}, \qquad i = 1,\dots,s, \end{aligned} \tag{25}$$

we get

$$\theta_k^* \geq \lambda_{k0}^* \frac{X_{0j}}{X_{kj}}, \quad j = 1, \ldots, r,$$
$$\lambda_{k0}^* \geq \frac{Y_{ki}}{Y_{0i}}, \qquad i = 1, \ldots, s. \tag{26}$$

Since θ_k^* is the minimum of θ_k, minimal possible values should be assigned to both of them:

$$\lambda_{k0}^* = \max_{1 \leq i \leq s} \frac{Y_{ki}}{Y_{0i}} \, ,$$
$$\theta_k^* = \lambda_{k0}^* \times \max_{1 \leq j \leq r} \frac{X_{0j}}{X_{kj}} = \max_{1 \leq i \leq s} \frac{Y_{ki}}{Y_{0i}} \times \max_{1 \leq j \leq r} \frac{X_{0j}}{X_{kj}} \, . \tag{27}$$

The last equality in formulas (27) proves the case, since $E_{k \text{ IM CRS}} = \theta_k^*$.

(b) IM VRS model.

The proof is similar to that for the case of the IM CRS model with $\lambda_{k0}^* = 1$. We obtain

$$E_{k \text{ IM VRS}} = \theta_k^* = \max_{1 \leq j \leq r} \frac{X_{0j}}{X_{kj}} \, . \tag{28}$$

(c) OM CRS model.

Given

$$\lambda_{k0}^* X_{0j} \leq X_{kj}, \qquad j = 1, \ldots, r,$$
$$\lambda_{k0}^* Y_{0i} \geq \omega_k^* Y_{ki}, \quad i = 1, \ldots, s, \tag{29}$$

we obtain

$$\lambda_{k0}^* \leq \frac{X_{kj}}{X_{0j}}, \qquad j = 1, \ldots, r,$$
$$\omega_k^* \leq \lambda_{k0}^* \frac{Y_{0i}}{Y_{ki}}, \quad i = 1, \ldots, s. \tag{30}$$

Since ω_k^* is the maximum of ω_k, we should take the maximal values for both of them:

$$\lambda_{k0}^* = \min_{1 \leq j \leq r} \frac{X_{kj}}{X_{0j}},$$
$$\omega_k^* = \lambda_{k0}^* \min_{1 \leq i \leq s} \frac{Y_{0i}}{Y_{ki}} = \min_{1 \leq j \leq r} \frac{X_{kj}}{X_{0j}} \times \min_{1 \leq i \leq s} \frac{Y_{0i}}{Y_{ki}} \, , \tag{31}$$

so

$$E_{k \text{ OM CRS}} = \frac{1}{\omega_k^*} = \frac{1}{\min\limits_{1 \leq j \leq r} \frac{X_{kj}}{X_{0j}} \times \min\limits_{1 \leq i \leq s} \frac{Y_{0i}}{Y_{ki}}} = \max_{1 \leq j \leq r} \frac{X_{0j}}{X_{kj}} \times \max_{1 \leq i \leq s} \frac{Y_{ki}}{Y_{0i}} \, . \tag{32}$$

It follows from the last formula that $E_{k \text{ OM CRS}} = E_{k \text{ IM CRS}}$, in accordance with the general theory of DEA.

(d) OM VRS model.

The proof is similar to that in the case of the OM CRS model with $\lambda^*_{k0} = 1$. We obtain

$$\omega^*_k = \max_{1 \le i \le s} \frac{Y_{0i}}{Y_{ki}} \,,$$
$$E_{k\text{ OM VRS}} = \frac{1}{\omega^*_k} = \max_{1 \le i \le s} \frac{Y_{ki}}{Y_{0i}} \,, \tag{33}$$

as required.

(e) The model with undesirable outputs, OM UO VRS.

Adjusting for undesirable outputs, we can treat this model simply as OM VRS. So, as previously, we have,

$$E_{k\text{ OM VRS UO}} = \frac{1}{\omega^*_k} = \max\left(\max_{0 \le i \le p} \frac{Y_{ki}}{Y_{0i}}, \max_{p+1 \le i \le s} \frac{\overline{Y}_{ki}}{\overline{Y}_{0i}} \right), \tag{34}$$

with undesirable outputs separated for convenience. We can rewrite this formula in terms of the original outputs and parameters W_i, $i = p+1, \ldots, s$, as follows:

$$E_{k\text{ OM VRS UO}} = \frac{1}{\omega^*_k} = \max\left(\max_{0 \le i \le p} \frac{Y_{ki}}{Y_{0i}}, \max_{p+1 \le i \le s} \frac{W_i - Y_{ki}}{W_i - Y_{0i}} \right). \tag{35}$$

The last formula proves the case.

Therefore, the theorem is proved. □

Corollary. *Scale efficiencies for the IM and OM DEA models are as follows:*

$$E^{Scale}_{k\ IM} = \frac{E_{k\ IM\ CRS}}{E_{k\ IM\ VRS}} = \max_{0 \le i \le s} \frac{Y_{ki}}{Y_{oi}},$$
$$E^{Scale}_{k\ OM} = \frac{E_{k\ OM\ CRS}}{E_{k\ OM\ VRS}} = \max_{0 \le j \le r} \frac{X_{0j}}{X_{kj}} \,. \tag{36}$$

A proof follows directly from taking the ratios of the CRS and VRS efficiencies given by formulas (20) and (21) for the IM model and formulas (22) and (23) for the OM model. Therefore, the corollary is proved.

Given formula (24), we can address the problem of the impact of W_i on efficiency scores in cases where a PO is included. For each i, the component related to parameters W_i, $i = p+1, \ldots, s$ is

$$E_k(W_i) = \frac{W_i - Y_{ki}}{W_i - Y_{0i}} \,. \tag{37}$$

Consider the derivative of $E_k\ (W_i)$ with respect to W_i:

$$\frac{dE_k}{dW_i} = \frac{Y_{ki} - Y_{0i}}{(W_i - Y_{0i})^2} \ge 0. \tag{38}$$

The derivative is nonnegative because $\overline{Y}_{0i} \ge \overline{Y}_{ki}$, and thus, $Y_{ki} = Y_{0i}$. If $Y_{ki} > Y_{0i}$, then the derivative is strictly positive, so in this case the term related to undesirable outputs increases with the increase in W_i. With large values of W_i it becomes dominant in formula (24). This observation may serve as a guideline for the choice of W_i: undesirable outputs that are preferable to other undesirable outputs should be given larger values of this parameter.

3. **Example**

An analysis of the environmental efficiency of the electric power industry of the United States can provide an example of the application of the approach that this paper suggests. Working with information available on the Web site of the Energy Information Administration (EIA; `http://www.eia.doe.gov`), we can calculate efficiency scores by using Electrical Energy Loss (input X_1), Electrical Energy GDP Intensity (input X_2), Fuel Utilization (output Y_1), and CO_2 Emission Rate (output Y_2, undesirable). Electrical Energy Loss is measured as the share of generated energy that is lost in transition and distribution processes. Electrical Energy GDP Intensity is measured as the amount of energy (kWh) that is expended in the production of \$1 GDP. Fuel Utilization is the ratio of the amount of electrical energy generated to the amount of thermal energy required for its generation. CO_2 Emission Rate is the amount of CO_2 emitted per unit of generated electrical power (g/1 kWh).

Table 1 presents calculations performed with formulas (20)-(24). The last row is assigned to the Perfect Object, referred to as the PO or DMU_0, and contains minimal inputs or maximal outputs, respectively. Column 1 lists the DMUs — in this case, the years from 1990 through 2006. Columns 2-5 contain DEA inputs and outputs calculated on the basis of information from the Web site.

Columns 6 and 7 show the ratios of inputs $\mathrm{PO}/\mathrm{DMU}_k$ for X_1 and X_2, respectively, and column 8 displays the maximum of each pair of values. Likewise, columns 9 and 10 show the ratios of outputs $\mathrm{DMU}_k/\mathrm{PO}$ for Y_1 and Y_2, respectively, and column 11 displays the maximum of each pair of values. Column 12 displays the product of the values in columns 8 and 11.

As follows from the theorem — specifically, formulas (20) and (22) — the entries in column 12 are CRS efficiencies: $E_{k\ \mathrm{IM\ CRS}} = E_{k\ \mathrm{OM\ CRS}}$. The entries in columns 8 and 11 are VRS efficiencies $E_{k\ \mathrm{IM\ VRS}}$ and E_{kOMVRS}, as given by formulas (21) and (23) respectively. These columns can be also treated as scale efficiencies: $E^{\mathrm{Scale}}_{k\ \mathrm{OM}}$ and $E^{\mathrm{Scale}}_{k\ \mathrm{IM}}$, respectively; see formulas (36).

Up to this point, the objective was to demonstrate technique of calculations, and output Y_2, CO_2 Emissions Rate, was considered as a regular one. To make the example practical, we continued our computations further with this output as an undesirable (UO). Column 5 shows the output Y_2, and column 13 shows $\overline{Y}_2$, the translated undesirable output (TUO). Since the values of Y_2 are less than 1, we used $W_2 = 1$, so $\mathrm{TUO} = \overline{Y}_2 = W_2 - Y_2 = 1 - Y_2$.

Column 14 represents ratios of the translated undesirable output $\mathrm{TUO}_k/\mathrm{TUO}_0$, and column 15 displays the maximum of each pair of values shown in columns

Table 1. Example of calculations

DMU_k	X_1	X_2	Y_1	Y_2	$\frac{X_{01}}{X_{k1}}$	$\frac{X_{02}}{X_{k2}}$	$\max(\frac{X_0}{X_k})^*$	$\frac{Y_{k1}}{Y_{01}}$	$\frac{Y_{k2}}{Y_{02}}$	$\max(\frac{Y_k}{Y_0})^\sharp$	$\max X \times \max Y$
DEA role	Inp-1	Inp-2	Out-1	Out-2 (UO)			E_{IMVRS}			$E_{\text{OM VRS}}$	$E_{\text{IM}} = E_{\text{OM}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Y1990	0.0669	0.3989	0.4773	0.6357	0.8064	0.8460	0.8460	0.9921	0.9899	0.9921	0.8393
Y1991	0.0675	0.4065	0.4811	0.6272	0.7996	0.8302	0.8302	1.0000	0.9767	1.0000	0.8302
Y1992	0.0688	0.3949	0.4758	0.6322	0.7843	0.8545	0.8545	0.9890	0.9844	0.9890	0.8451
Y1993	0.0702	0.3984	0.4745	0.6359	0.7689	0.8471	0.8471	0.9863	0.9902	0.9902	0.8388
Y1994	0.0651	0.3932	0.4702	0.6355	0.8284	0.8582	0.8582	0.9774	0.9896	0.9896	0.8493
Y1995	0.0682	0.3939	0.4779	0.6213	0.7908	0.8566	0.8566	0.9935	0.9674	0.9935	0.8510
Y1996	0.0670	0.3907	0.4782	0.6275	0.8056	0.8638	0.8638	0.9941	0.9771	0.9941	0.8587
Y1997	0.0643	0.3794	0.4700	0.6393	0.8395	0.8895	0.8895	0.9770	0.9956	0.9956	0.8855
Y1998	0.0611	0.3778	0.4669	0.6420	0.8834	0.8933	0.8933	0.9706	0.9996	0.9996	0.8930
Y1999	0.0650	0.3679	0.4717	0.6330	0.8301	0.9173	0.9173	0.9806	0.9856	0.9856	0.9041
Y2000	0.0640	0.3659	0.4638	0.6422	0.8422	0.9222	0.9222	0.9642	1.0000	1.0000	0.9222
Y2001	0.0539	0.3596	0.4597	0.6395	1.0000	0.9383	1.0000	0.9556	0.9959	0.9959	0.9959
Y2002	0.0642	0.3614	0.4648	0.6207	0.8400	0.9337	0.9337	0.9661	0.9666	0.9666	0.9025
Y2003	0.0586	0.3555	0.4656	0.6221	0.9204	0.9492	0.9492	0.9679	0.9687	0.9687	0.9195
Y2004	0.0670	0.3481	0.4631	0.6188	0.8054	0.9695	0.9695	0.9626	0.9635	0.9635	0.9341
Y2005	0.0652	0.3468	0.4617	0.6198	0.8271	0.9731	0.9731	0.9598	0.9651	0.9651	0.9392
Y2006	0.0619	0.3375	0.4640	0.6069	0.8713	1.0000	1.0000	0.9645	0.9450	0.9645	0.9645
PO (DMU_0)	0.0539	0.3375	0.4811	0.6422							

DMU_k	$TUO^\S$	TUO_k/TUO_0	$\max YU^\ddagger$
DEA role	Translated UO		$E_{\text{OM UO VRS}}$
(1)	(13)	(14)	(15)
Y1990	0.3643	0.9268	0.9921
Y1991	0.3728	0.9483	1.0000
Y1992	0.3678	0.9357	0.9890
Y1993	0.3641	0.9262	0.9863
Y1994	0.3645	0.9272	0.9774
Y1995	0.3787	0.9634	0.9935
Y1996	0.3725	0.9476	0.9941
Y1997	0.3607	0.9175	0.9770
Y1998	0.3580	0.9108	0.9706
Y1999	0.3670	0.9337	0.9806
Y2000	0.3578	0.9102	0.9642
Y2001	0.3605	0.9170	0.9556
Y2002	0.3793	0.9649	0.9661
Y2003	0.3779	0.9614	0.9679
Y2004	0.3812	0.9698	0.9698
Y2005	0.3802	0.9672	0.9672
Y2006	0.3931	1.0000	1.0000
PO (DMU_0)	0.3931		

Notes: * Equal to E_{OM}^{Scale}; $^\sharp$ Equal to $E_{\text{IM}}^{\text{Scale}}$; § $TUO = W_2 - Y_2 = 1 - Y_2$;
‡ $\max YU = \max(Y_{k1}/Y_{01}, TUO_k/TUO_0)$, columns 9 and 14, respectively.

9 and 14. As follows from formula (24), the entries in column 15 are efficiency scores corresponding to the DEA model with undesirable outputs, $E_{k\text{ OM UO VRS}}$.

The results obtained by applying the suggested approach of adding a Perfect Object were checked with a DEA program using an LP algorithm.

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