Communications in Mathematics and Applications
Volume 1, Number 2 (2010), pp. 123–131

http://www.rgnpublications.com

Two Modes Bifurcation Solutions of Elastic Beams Equation with Nonlinear Approximation

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Abstract. In this paper we studied two modes bifurcation solutions of elastic beams equation by using Lyapunov-Schmidt method. The bifurcation equation corresponding to the elastic beams equation has been found. Also, we studied two modes of nonlinear approximation of bifurcation solutions of a specified equation and we found the Key function corresponding to the functional related to this equation.

1. Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

$$f(x,\lambda)=b, \quad x\in O\subset X,\ b\in Y,\ \lambda\in R^n, \tag{1.1}$$

where f is a smooth Fredholm map of index zero and X, Y are Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi,\lambda)=\beta, \quad \xi\in M,\ \beta\in N, \tag{1.2}$$

can be used, where M and N are smooth finite dimensional manifolds.

Passage from equation (1.1) into equation (1.2) (variant local scheme of Lyapunov-Schmidt) with the conditions, that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc.) dealing with [3], [8], [11], [12].

Definition 1.1. Suppose that E and F are Banach spaces and $A: E\to F$ be a linear continuous operator. The operator A is called Fredholm operator, if

(i) The kernel of A, $\mathrm{Ker}(A)$, is finite dimensional,
(ii) The range of A, $\mathrm{Im}(A)$, is closed in F,
(iii) The Cokernel of A, $\mathrm{Coker}(A)$, is finite dimensional.

Key words and phrases. Bifurcation solutions; Local scheme of Lyapunov-Schmidt; Singularities of smooth maps.

The number

$$\dim(\mathrm{Ker}A) - \dim(\mathrm{Coker}A)$$

is called Fredholm index of the operator A.

Suppose that $f : \Omega \to F$ is a nonlinear Fredholm map of index zero. A smooth map $f : \Omega \to F$ has variational property, if there exist a functional $V : \Omega \to R$ such that $f = \mathrm{grad}_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H, \quad \forall\ x \in \Omega,\ \ h \in E,$$

where ($\langle \cdot, \cdot \rangle_H$ is the scalar product in Hilbert space H). In this case the solutions of equation $f(x,\lambda) = 0$ are the critical points of functional $V(x,\lambda)$. Suppose that $f : E \to F$ is a smooth Fredholm map of index zero, E, F are Banach spaces and

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H, \quad h \in E,$$

where V is a smooth functional on E. Also we assume that $E \subset F \subset H$, H is a Hilbert space, then by using method of finite dimensional reduction (*Local scheme of Lyapunov-Schmidt*) the problem,

$$V(x,\lambda) \to \mathrm{extr}, \quad x \in E,\ \lambda \in R^n$$

can be reduced into equivalent problem,

$$W(\xi,\lambda) \to \mathrm{extr}, \quad \xi \in R^n.$$

The function $W(\xi,\lambda)$ is called Key function.

If $N = \mathrm{span}\{e_1, \ldots, e_n\}$ is a subspace of E, where $e_1, \ldots, e_n$ are orthonormal basis, then the Key function $W(\xi,\lambda)$ can be defined in the form,

$$W(\xi,\lambda) = \inf_{x:\langle x,e_i\rangle = \xi_i\ \forall\ i} V(x,\lambda), \quad \xi = (\xi_1, \ldots, \xi_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc.) [10]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of Key function. If f has variational property, then it is easy to check that,

$$\theta(\xi,\lambda) = \mathrm{grad}\, W(\xi,\lambda).$$

Equation $\theta(\xi,\lambda) = 0$ is called bifurcation equation.

Definition 1.2. The set of all λ for which the function $W(\xi,\lambda)$ has degenerate critical points, is called *Caustic*.

The linear Ritz approximation of the functional V is a function W given by the formula,

$$W(\xi,\lambda) = V\left(\sum_{i=1}^{n} \xi_i e_i\right), \quad \xi = (\xi_1, \ldots, \xi_n).$$

The oscillations and motion of waves of the elastic beams on elastic foundations can be described by means of the following PDE,

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \alpha \frac{\partial^2 y}{\partial x^2} + \beta y + y^3 = 0,$$

where y is the deflection of beam. It is known that, to study the oscillations of beams, stationary state $(u(x) = y(x,t))$ should be monitored which is describes by the equation,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0. \tag{1.3}$$

In this work equation (1.3) has been studied with the following boundary conditions,

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0 \tag{1.4}$$

Equation (1.3) has been studied by Thompson and Stewart [4] they showed numerically the existence of periodic solutions of equation (1.3) for some values of parameters. Bardin and Furta [1] used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (1.3), also they are introduced the solutions of equation (1.3) in the form of power series. Furta and Piccione [9] showed the existence of periodic travelling wave solutions of equation (1.3) describing oscillations of an infinite beam, which lies on a non-linearly elastic support with non-small amplitudes. Sapronov ([2], [10], [11], [12]) applied the local method of Lyapunov-Schmidt and found the bifurcation solutions of equation (1.3). Abdul Hussain ([5], [6]) studied equation (1.3) with small perturbation when the nonlinear part has quadratic term and Mohammed [7] studied equation (1.3) in the variational case when the nonlinear part has quadratic term.

The goal of this paper to study two modes bifurcation solutions of equation (1.3) with boundary conditions (1.4) by using the procedure of Bardin and Furta [1] and then we used the result of this procedure to study the two modes nonlinear bifurcation solutions of equation (1.3) by using the work of Sapronov.

2. **Nonlinear Approximation Solutions**

Suppose that $f : E \to F$ is a nonlinear Fredholm operator of index zero from Banach space E to Banach space F defined by,

$$f(u,\lambda) = \frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3, \tag{2.1}$$

where $E = C^4([0,\pi],R)$ is the space of all continuous functions which have derivative of order at most four, $F = C([0,\pi],R)$ is the space of all continuous functions where $u = u(x)$, $x \in [0,\pi]$, $\lambda = (\alpha,\beta)$. In this case the solutions of equation (1.3) is equivalent to the solutions of the operator equation,

$$f(u,\lambda) = 0. \tag{2.2}$$

We note that the operator f has variational property that is; there exist a functional V such that $f(u,\lambda) = \mathrm{grad}_H V(u,\lambda)$ or equivalently,

$$\frac{\partial V}{\partial u}(u,\lambda)h = \langle f(u,\lambda),h\rangle_H, \quad \forall\, u \in \Omega,\ h \in E,$$

where ($\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space H) and

$$V(u,\lambda)=\int_0^{\pi}\left(\frac{(u'')^2}{2}-\alpha\frac{(u')^2}{2}+\beta\frac{u^2}{2}+\frac{u^4}{4}\right)dx.$$

In this case the solutions of equation (2.2) are the critical points of the functional $V(u,\lambda)$, where the critical points of the functional $V(u,\lambda)$ are the solutions of Euler-Lagrange equation,

$$\frac{\partial V}{\partial u}(u,\lambda)h=\int_0^{\pi}(u^{iv}+\alpha u''+\beta u+u^3)h\,dx=0$$

and $\frac{\partial V}{\partial u}(u,\lambda)$ is the Frechet derivative of the functional $V(u,\lambda)$.

Thus, the study of equation (1.3) with the conditions (1.4) is equivalent to the study extremely problem,

$$V(u,\lambda)\to \text{extr},\quad u\in E.$$

Analysis of bifurcation can be finding by using method of Lyapunov-Schmidt to reduce into finite dimensional space. By localized parameters,

$$\alpha=\alpha_1+\delta_1,\ \ \beta=\beta_1+\delta_2,\qquad \delta_1,\delta_2 \text{ are small parameters.}$$

The reduction lead to the function in two variables,

$$W(\xi,\delta)=\inf_{\langle u,e_i\rangle=\xi_i,\ i=1,2}V(u,\delta),\quad \xi=(\xi_1,\xi_2),\ \delta=(\delta_1,\delta_2).$$

It is well known that in the reduction of Lyapunov-Schmidt the function $W(\xi,\delta)$ is smooth. This function has all the topological and analytical properties of functional V [10]. In particular, for small δ there is one-to-one corresponding between the critical points of functional V and smooth function W, preserving the type of critical points (multiplicity, index Morse, etc.) [10]. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (2.2) has the form:

$$h''''+\alpha h''+\beta h=0,\quad h\in E,$$
$$h(0)=h(\pi)=h''(0)=h''(\pi)=0.$$

This equation give in the $\alpha\beta$-plane characteristic lines. The point of characteristic lines are the points of (α,β) in which equation (2.2) has non-zero solutions. The point of intersection of characteristic lines in the $\alpha\beta$-plane is a bifurcation point [10]. The result of this intersection lead to bifurcation along the modes $e_1=c_1\sin(x)$, $e_2=c_2\sin(2x)$. For the equation (2.2) the point $(\alpha,\beta)=(5,4)$ is a bifurcation point [10]. Localized parameters,

$$\widetilde{\alpha}=5+\delta_1,\quad \widetilde{\beta}=4+\delta_2.$$

Lead to the bifurcation along the modes e_1, e_2, where $\|e_1\|=\|e_2\|=1$ and $c_1=c_2=\sqrt{2}$. Let $N=\mathrm{Ker}(A)=\mathrm{span}\{e_1,e_2\}$, where $A=f_u(0,\lambda)=\frac{d^4}{dx^4}+\alpha\frac{d^2}{dx^2}+\beta$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus \widehat{E}, \quad \widehat{E} = N^{\perp} \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space F decomposed in direct sum of two subspaces, N and orthogonal complement to N,

$$F = N \oplus \widehat{F}, \quad \widehat{F} = N^{\perp} \cap F = \{v \in F : v \perp N\}.$$

There exists projections $p : E \to N$ and $I - p : E \to \widehat{E}$ such that $pu = w$ and $(I - p)u = v$, (I is the identity operator). Hence every vector $u \in E$ can be written in the form,

$$u = w + v, \quad w = \sum_{i=1}^{2} \xi_i e_i \in N, \quad N \perp v \in \widehat{E}, \quad \xi_i = \langle u, e_i \rangle.$$

Similarly, there exists projections $Q : F \to N$ and $I - Q : F \to \widehat{F}$ such that

$$f(u,\lambda) = Qf(u,\lambda) + (I - Q)f(u,\lambda). \tag{2.3}$$

Accordingly, equation (2.2) can be written in the form,

$$Qf(w + v, \lambda) = 0,$$
$$(I - Q)f(w + v, \lambda) = 0.$$

By the implicit function theorem, there exist a smooth map $\Phi : N \to \widehat{E}$, such that

$$W(\xi,\delta) = V(\Phi(\xi,\lambda),\delta), \quad \delta = (\delta_1,\delta_2)$$

and then the linear Ritz approximation of the functional V is a function W given by,

$$W(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2, \delta) = \xi_1^4 + 4\xi_1^2\xi_2^2 + \xi_2^4 + \frac{q_1}{2}\xi_1^2 + \frac{q_2}{2}\xi_2^2 .$$

The nonlinear Ritz approximation of the functional V is a function W given by,

$$W(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2 + \Phi(\xi_1 e_1 + \xi_2 e_2, \delta), \delta), \quad v(x,\xi,\lambda) = \Phi(w,\delta).$$

To determine the nonlinear Ritz approximation of the functional V we must find the functions $v(x,\xi,\lambda) = O(\xi^3)$, $\mu(\xi) = O(\xi^2)$ and $\widetilde{\mu}(\xi) = O(\xi^2)$ in the form of power series in term of ξ, where $q_1 = \widetilde{q}_1 + \mu(\xi_1,\xi_2)$, $q_2 = \widetilde{q}_2 + \widetilde{\mu}(\xi_1,\xi_2)$ and $\xi = (\xi_1,\xi_2)$. Because the symmetry of the problem, the quadratic form in the function is equal to zero, so the functions $v(x,\xi,\lambda)$, $\mu(\xi)$ and $\widetilde{\mu}(\xi)$ can be written in the following form,

$$v(x,\xi,\lambda) = v_0(x,\lambda)\xi_1^3 + v_1(x,\lambda)\xi_1^2\xi_2 + v_2(x,\lambda)\xi_1\xi_2^2 + v_3(x,\lambda)\xi_2^3 + \ldots,$$
$$\mu(\xi) = \mu_0\xi_1^2 + \mu_1\xi_1\xi_2 + \mu_2\xi_2^2, \tag{2.4}$$
$$\widetilde{\mu}(\xi) = \widetilde{\mu}_0\xi_1^2 + \widetilde{\mu}_1\xi_1\xi_2 + \widetilde{\mu}_2\xi_2^2.$$

Equation (2.2) can be written in the form,

$$f(u,\lambda) = Au + Tu = 0, \quad Tu = u^3.$$

Since,

$$Qf(u,\lambda) = \sum_{i=1}^{2} \langle f(u,\lambda), e_i \rangle e_i = 0.$$

Then we have

$$\sum_{i=1}^{2}\langle Au+Tu,e_i\rangle e_i=0$$

and hence

$$q_1\xi_1e_1+q_2\xi_2e_2+\left(\int_0^\pi(\xi_1e_1+\xi_2e_2+v)^3e_1\,dx\right)e_1 + \left(\int_0^\pi(\xi_1e_1+\xi_2e_2+v)^3e_2\,dx\right)e_2=0\,. \tag{2.5}$$

From (2.3) and (2.5) we have

$$v^{iv}+\alpha v''+\beta v+(\xi_1e_1+\xi_2e_2+v)^3+q_1\xi_1e_1+q_2\xi_2e_2=0 \tag{2.6}$$

It follows that,

$$\begin{aligned}&\left[(\widetilde{q}_1+\mu(\xi_1,\xi_2))\xi_1+\xi_1^3\int_0^\pi e_1^4dx+3\xi_1^2\xi_2\int_0^\pi e_1^3e_2dx+3\xi_1\xi_2^2\int_0^\pi e_1^2e_2^2dx\right.\\ &\left.+\xi_2^3\int_0^\pi e_1e_2^3dx\right]e_1+\Big[(\widetilde{q}_2+\widetilde{\mu}(\xi_1,\xi_2))\xi_2\\ &\left.+\xi_1^3\int_0^\pi e_1^3e_2dx+3\xi_1^2\xi_2\int_0^\pi e_1^2e_2^2dx+3\xi_1\xi_2^2\int_0^\pi e_1e_2^3dx+\xi_2^3\int_0^\pi e_2^4\right]e_2=0,\\ &v^{iv}+\alpha v''+\beta v+\xi_1^3e_1^3+3\xi_1^2\xi_2e_1^2e_2+3\xi_1\xi_2^2e_1e_2^2+\xi_2^3e_2^3\\ &+v^3+3v^2\xi_1e_1+3v^2\xi_2e_2+3v\xi_1^2e_1^2+6v\xi_1\xi_2e_1e_2\\ &+3v\xi_2^2e_2^2+(\widetilde{q}_1+\mu(\xi_1,\xi_2))\xi_1e_1+(\widetilde{q}_2+\widetilde{\mu}(\xi_1,\xi_2))\xi_2e_2=0.\end{aligned} \tag{2.7}$$

To determine the functions $v(x,\xi,\lambda)$, $\mu(\xi)$ and $\widetilde{\mu}(\xi)$ we first substitute (2.4) in (2.7) and then we find the coefficients μ_0, μ_1, μ_2, $\widetilde{\mu}_0$, $\widetilde{\mu}_1$, $\widetilde{\mu}_2$, v_0, v_1, v_2 and v_3 by equating the terms of ξ_1 and ξ_2 as follows:

Equating the coefficients of ξ_1^3 we have the following tow equations,

$$\begin{aligned}&\left[\mu_0+\int_0^\pi e_1^4dx\right]e_1+\left[\int_0^\pi e_1^3e_2dx\right]e_2=0,\\ &v_0^{iv}+\alpha v_0''+\beta v_0+e_1^3+\mu_0e_1=0.\end{aligned} \tag{2.8}$$

From the first equation of (2.8) we have

$$\mu_0=-\frac{3}{2\pi}\,.$$

Substitute the value of μ_0 in the second equation of (2.8) we have the following linear ODE,

$$v_0^{iv}+\alpha v_0''+\beta v_0+e_1^3-\frac{3}{2\pi}e_1=0.$$

And then we have

$$v_0^{iv}+\alpha v_0''+\beta v_0-\frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\sin(3x)=0. \tag{2.9}$$

Solve equation (2.9) we have

$$v_0(x,\lambda) = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(81-9\alpha+\beta)}\ \sin(3x).$$

Similarly, equating the coefficients of $\xi_1^2\xi_2$ we have

$$\left[\mu_1 + 3\int_0^\pi e_1^3 e_2 dx\right] e_1 + \left[\widetilde{\mu}_0 + 3\int_0^\pi e_1^2 e_2^2 dx\right] e_2 = 0,$$

$$v_1^{iv} + \alpha v_1'' + \beta v_1 + 3e_1^2 e_2 + \mu_1 e_1 + \widetilde{\mu}_0 e_2 = 0. \qquad (2.10)$$

From the first equation of (2.10) we have $\mu_1 = 0$ and $\widetilde{\mu}_0 = -\frac{3}{\pi}$. Substitute these values in the second equation of (2.10) we have

$$v_1^{iv} + \alpha v_1'' + \beta v_1 + 3e_1^2 e_2 - \frac{3}{\pi}e_2 = 0. \qquad (2.11)$$

Solve equation (2.11) we have

$$v_1(x,\lambda) = \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(256-16\alpha+\beta)}\ \sin(4x).$$

Equating the coefficients of $\xi_1\xi_2^2$ we have

$$\left[\mu_2 + 3\int_0^\pi e_1^2 e_2^2 dx\right] e_1 + \left[\widetilde{\mu}_1 + 3\int_0^\pi e_1 e_2^3 dx\right] e_2 = 0,$$

$$v_2^{iv} + \alpha v_2'' + \beta v_2 + 3e_1 e_2^2 + \mu_2 e_1 + \widetilde{\mu}_1 e_2 = 0. \qquad (2.12)$$

From the first equation of (2.12) we have $\widetilde{\mu}_1 = 0$ and $\mu_2 = -\frac{3}{\pi}$. Substitute these values in the second equation of (2.12) we have

$$v_2^{iv} + \alpha v_2'' + \beta v_2 + \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}(\sin(3x) - \sin(5x)) = 0. \qquad (2.13)$$

Solve equation (2.13) we have

$$v_2(x,\lambda) = -\frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(81-9\alpha+\beta)}\ \sin(3x) + \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(625-25\alpha+\beta)}\ \sin(5x).$$

Equating the coefficients of ξ_2^3 we have the following two equations,

$$\left[\widetilde{\mu}_2 + 3\int_0^\pi e_2^4 dx\right] e_2 + \left[\int_0^\pi e_1 e_2^3 dx\right] e_1 = 0,$$

$$v_3^{iv} + \alpha v_3'' + \beta v_3 + e_2^3 + \widetilde{\mu}_2 e_2 = 0. \qquad (2.14)$$

From the first equation of (2.14) we have $\widetilde{\mu}_2 = -\frac{3}{2\pi}$ Substitute the value of $\widetilde{\mu}_2$ in the second equation of (2.14) we have the following linear ODE,

$$v_3^{iv} + \alpha v_3'' + \beta v_3 + e_2^3 - \frac{3}{2\pi}e_2 = 0.$$

And then we have

$$v_3^{iv} + \alpha v_3'' + \beta v_3 - \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\ \sin(6x) = 0\ . \qquad (2.15)$$

Solve equation (2.15) we have

$$v_3(x,\lambda) = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{(1296-36\alpha+\beta)}\ \sin(6x).$$

Now substitute the values of μ_0, μ_1, μ_2, $\widetilde{\mu}_0$, $\widetilde{\mu}_1$, $\widetilde{\mu}_2$, v_0, v_1, v_2 and v_3 in (2.4) we have the nonlinear approximation solutions of equation (2.2) in the form,

$$\begin{aligned}
u(x,\xi) &= \xi_1\sin(x) + \xi_2\sin(2x) \\
&\quad + \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{\xi_1^3}{(81-9\alpha+\beta)}\ \sin(3x) \\
&\quad + \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\frac{\xi_1^2\xi_2}{(256-16\alpha+\beta)}\ \sin(4x) \\
&\quad + \frac{3}{2\pi}\sqrt{\frac{2}{\pi}}\ \xi_1\xi_2^2\left[\frac{\sin(5x)}{(625-25\alpha+\beta)} - \frac{\sin(3x)}{(81-9\alpha+\beta)}\right] \\
&\quad + \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{\xi_2^3}{(1296-36\alpha+\beta)}\ \sin(6x) + O(\xi^5), \\
q_1 &= \widetilde{q}_1 - \frac{3}{2\pi}\xi_1^2 - \frac{3}{\pi}\xi_2^2 + O(\xi^3), \\
q_2 &= \widetilde{q}_2 - \frac{3}{\pi}\xi_1^2 - \frac{3}{2\pi}\xi_2^2 + O(\xi^3), \\
\xi &= (\xi_1,\xi_2).
\end{aligned} \tag{2.16}$$

By using (2.16) we have stated the following theorem,

Theorem 2.1. *The Key function of the functional V has the following form,*

$$\begin{aligned}
\widetilde{W}(\xi,\delta) \;=\; & \xi_1^{12} + \xi_2^{12} + \lambda_1\xi_1^2\xi_2^{10} + \lambda_2\xi_1^4\xi_2^8 + \lambda_3\xi_1^6\xi_2^6 + \lambda_4\xi_1^8\xi_2^4 \\
& + \lambda_5\xi_1^{10}\xi_2^2 + \lambda_6\xi_1^2\xi_2^8 + \lambda_7\xi_1^8\xi_2^2 + \lambda_8\xi_1^6\xi_2^4 + \lambda_9\xi_1^4\xi_2^6 \\
& + \lambda_{10}\xi_1^8 + \lambda_{11}\xi_2^8 + \lambda_{12}\xi_1^6\xi_2^2 + \lambda_{13}\xi_1^2\xi_2^6 + \lambda_{14}\xi_1^4\xi_2^4 \\
& + \lambda_{15}\xi_1^6 + \lambda_{16}\xi_2^6 + \lambda_{17}\xi_1^4\xi_2^2 + \lambda_{18}\xi_1^2\xi_2^4 + \lambda_{19}\xi_1^4 \\
& + \lambda_{20}\xi_2^4 + \lambda_{21}\xi_1^2\xi_2^2 + \lambda_{22}\xi_1^2 + \lambda_{23}\xi_2^2 \\
& + o(|\xi|^{12}) + O(|\xi|^{12})O(|\delta|),
\end{aligned} \tag{2.17}$$

$\lambda_i = \lambda_i(\alpha,\beta)$, $i = 1,2,\ldots,23$.

The prove of Theorem 2.1 is directly from the formula,

$$\widetilde{W}(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2, \Phi(\xi_1 e_1 + \xi_2 e_2, \delta), \delta), \quad v(x,\xi,\lambda) = \Phi(w,\delta).$$

Function (2.17) has all the topological and analytical properties of functional V. Also, the function is symmetric in the variables ξ_1 and ξ_2 ($\widetilde{W}(\xi_1,\xi_2) = \widetilde{W}(-\xi_1,-\xi_2)$) it have 121 critical points. So it is not easy to determine the Caustic of function (2.17) and study the bifurcation solutions of this function. The point $u(x) = \xi_1 e_1 + \xi_2 e_2 + v(x,\xi,\lambda)$ is a critical point of the functional $V(u,\lambda)$ iff the point ξ is a critical point of the function $\widetilde{W}(\xi,\delta)$ [10]. This mean that the existence of the solutions of equation (2.2) depend on the existence of the critical points of the functional $V(u,\lambda)$ and then on the existence of the critical points of the function $\widetilde{W}(\xi,\delta)$. From this notation, we can find a nonlinear approximation of the solutions of equation (2.2) corresponding to each critical point of the function $\widetilde{W}(\xi,\delta)$. Caustic of the function $\widetilde{W}(\xi,\delta)$ and the distribution of the critical points in the plane of parameters (Bifurcation diagram) depending on the corner singularities of smooth maps will be discuses in other paper.

References

[1] B.C. Bardin and S.D. Furta, Periodic travelling waves of an infinite beam on a nonlinear elastic support, *Institute B f r Mechanik, Universit t Stuttgart, Institutsbericht IB-36. Januar 2001*.

[2] B.M. Darinskii, C.L. Tcarev and Yu.I. Sapronov, Bifurcation of extremely of Fredholm functionals, *Voronezh*, 2004.

[3] B.V. Loginov, Theory of Branching nonlinear equations in theconditions of invariance group, *Tashkent*: Fan, 1985.

[4] J.M.T. Thompson and H.B. Stewart, *Nonlinear Dynamics and Chaos*, J. Wiley and Sons, Singapore — Chichester, 1986.

[5] M.A. Abdul Hussain, *Corner singularities of smooth functions in the analysis of bifurcations balance of the elastic beams and periodic waves*, Ph.D. thesis, Voronezh, Russia. 2005.

[6] M.A. Abdul Hussain, Bifurcation solutions of elastic beams equation with small perturbation, *Int. Journal of Mathematical Analysis* **3** (18) (2009), 879–888.

[7] M.J. Mohammed, *Bifurcation solutions of nonlinear wave equation*, M.Sc. thesis, Basrah University, Iraq, 2007.

[8] M.M. Vainberg and V.A. Trenogin, Theory of Branching solutions of nonlinear equations, *M.-Science*, 1969.

[9] P. Piccione and S.D. Furta, Global existence of periodic traveling waves of an infinite non-linearly supported beam, *Regular and Chaotic Dynamics* **7** (1) (2002).

[10] V.R. Zachepa and Yu.I. Sapronov, Local analysis of Fredholm equation, *Voronezh*, 2002.

[11] Yu.I. Sapronov, Regular perturbation of Fredholm maps and theorem about odd field, *Works Dept. of Math.*, Voronezh Univ., **10** (1973), 82–88.

[12] Yu. I. Sapronov, Finite dimensional reduction in the smooth extremely problems, *Uspehi Math. Science* **51** (1) (1996), 101–132.

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Received October 2, 2009
Accepted September 30, 2010