



# Degree-Magic Labelings on the Join and Composition of Complete Tripartite Graphs

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**Abstract.** A graph is called supermagic if there is a labeling of edges, where all edges are differently labeled with consecutive positive integers such that the sum of the labels of all edges, which are incident to each vertex of this graph, is a constant. A graph  $G$  is called degree-magic if all edges can be labeled by integers  $1, 2, \dots, |E(G)|$  so that the sum of the labels of the edges which are incident to any vertex  $v$  is equal to  $(1 + |E(G)|)\deg(v)/2$ . Degree-magic graphs extend supermagic regular graphs. In this paper, the necessary and sufficient conditions for the existence of degree-magic labelings of graphs obtained by taking the join and composition of complete tripartite graphs are found.

**Keywords.** Tripartite graph; Supermagic graph; Degree-magic graph; Balanced degree-magic graph

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## 1. Introduction

Consider simple graphs without isolated vertices. If  $G$  is a graph, then  $V(G)$  and  $E(G)$  stand for the vertex set and the edge set of  $G$ , respectively. Cardinalities of these sets are called the *order* and *size* of  $G$ .

Let a graph  $G$  and a mapping  $f$  from  $E(G)$  into positive integers be given. The *index mapping* of  $f$  is the mapping  $f^*$  from  $V(G)$  into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G), \quad (1.1)$$

where  $\eta(v, e)$  is equal to 1 when  $e$  is an edge incident with a vertex  $v$ , and 0 otherwise. An injective mapping  $f$  from  $E(G)$  into positive integers is called a *magic labeling* of  $G$  for an *index*  $\lambda$  if its index mapping  $f^*$  satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labeling  $f$  of a graph  $G$  is called a *supermagic labeling* if the set  $\{f(e) : e \in E(G)\}$  consists of consecutive positive integers. A graph  $G$  is *supermagic (magic)* whenever a supermagic (magic) labeling of  $G$  exists.

A bijective mapping  $f$  from  $E(G)$  into  $\{1, 2, \dots, |E(G)|\}$  is called a *degree-magic labeling* (or only *d-magic labeling*) of  $G$  if its index mapping  $f^*$  satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G).$$

A degree-magic labeling  $f$  of a graph  $G$  is called *balanced* if for all  $v \in V(G)$ , the following equation is satisfied

$$|\{e \in E(G) : \eta(v, e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor\}| = |\{e \in E(G) : \eta(v, e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}|.$$

A graph  $G$  is *degree-magic (balanced degree-magic)* or only *d-magic* when a *d-magic* (balanced *d-magic*) labeling of  $G$  exists.

A graph  $G$  is called *tripartite graph* if  $V(G)$  can be partitioned into three disjoint subsets  $V_1, V_2$  and  $V_3$ , called *partite sets*, such that  $uv$  is an edge of  $G$  if  $u$  and  $v$  belong to different partite sets. If every two vertices in different partite sets are joined by an edge, then  $G$  is a *complete tripartite graph*.

For any two vertex-disjoint graphs  $G$  and  $H$ , the *join* of graphs  $G$  and  $H$ , denoted by  $G + H$ , consists of  $G \cup H$  and all edges joining a vertex of  $G$  and a vertex of  $H$ . The *composition* of graphs  $G$  and  $H$ , denoted by  $G \cdot H$ , is a graph such that the vertex set of  $G \cdot H$  is the Cartesian product  $V(G) \times V(H)$  and any two vertices  $(u, v)$  and  $(x, y)$  are adjacent in  $G \cdot H$  if and only if either  $u$  is adjacent to  $x$  in  $G$  or  $u = x$  and  $v$  is adjacent to  $y$  in  $H$ .

The concept of magic graphs was put forward by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. Currently, numerous papers are published on magic and supermagic graphs (see [6, 7, 10] for more comprehensive references). The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in [1, 3, 5]. Hereinafter, the auxiliary results from these studies will be used.

**Theorem 1.1** ([2]). *Let  $G$  be a regular graph. Then  $G$  is supermagic if and only if it is d-magic.*

**Theorem 1.2** ([2]). *Let  $G$  be a d-magic graph of even size. Then every vertex of  $G$  has an even degree and every component of  $G$  has an even size.*

**Theorem 1.3** ([2]). *Let  $G$  be a balanced  $d$ -magic graph. Then  $G$  has an even number of edges and every vertex has an even degree.*

**Theorem 1.4** ([2]). *Let  $G$  be a  $d$ -magic graph having a half-factor. Then  $2G$  is a balanced  $d$ -magic graph.*

**Theorem 1.5** ([2]). *Let  $H_1$  and  $H_2$  be edge-disjoint subgraphs of a graph  $G$  which form its decomposition. If  $H_1$  is  $d$ -magic and  $H_2$  is balanced  $d$ -magic, then  $G$  is a  $d$ -magic graph. Moreover, if  $H_1$  and  $H_2$  are both balanced  $d$ -magic, then  $G$  is a balanced  $d$ -magic graph.*

**Proposition 1.1** ([2]). *For  $p, q > 1$ , the complete bipartite graph  $K_{p,q}$  is  $d$ -magic if and only if  $p \equiv q \pmod{2}$  and  $(p, q) \neq (2, 2)$ .*

**Theorem 1.6** ([2]). *The complete bipartite graph  $K_{p,q}$  is balanced  $d$ -magic if and only if the following statements hold:*

- (i)  $p \equiv q \equiv 0 \pmod{2}$ ;
- (ii) if  $p \equiv q \equiv 2 \pmod{4}$ , then  $\min\{p, q\} \geq 6$ .

**Lemma 1.1** ([4]). *Let  $p, q$  and  $r$  be even positive integers. Then the complete tripartite graph  $K_{p,q,r}$  is balanced  $d$ -magic.*

**Lemma 1.2** ([4]). *Let  $q \geq r$  be odd positive integers and let  $p$  be even positive integer such that  $p \equiv 0 \pmod{4}$  whenever  $q = 1$ . Then the complete tripartite graph  $K_{p,q,r}$  is  $d$ -magic.*

## 2. Labelings in the Join of Complete Tripartite Graphs

For any positive integers  $o, p, q, r, s$  and  $t$ , consider the join  $K_{o,p,q} + K_{r,s,t}$  of complete tripartite graphs. Let  $K_{o,p,q} + K_{r,s,t}$  be a  $d$ -magic graph. Since  $\deg(v)$  is  $o + p + r + s + t$ ,  $o + q + r + s + t$ ,  $p + q + r + s + t$ ,  $o + p + q + r + s$ ,  $o + p + q + r + t$  or  $o + p + q + s + t$  and  $f^*(v) = (1 + op + oq + pq + rs + rt + st + (o + p + q)(r + s + t))\deg(v)/2$  for any vertex  $v \in V(K_{o,p,q} + K_{r,s,t})$ , resulting in

**Proposition 2.1.** *Let  $K_{o,p,q} + K_{r,s,t}$  be a  $d$ -magic graph. Then the following statements hold:*

- (i) only three of  $o, p, q, r, s$  and  $t$  are even or
- (ii) only four of  $o, p, q, r, s$  and  $t$  are even or
- (iii) all of  $o, p, q, r, s$  and  $t$  are either odd or even.

*Proof.* Suppose that  $f$  is a  $d$ -magic labeling of  $K_{o,p,q} + K_{r,s,t}$ . Assume to the contrary that only one of  $o, p, q, r, s$  and  $t$  is even, only two of  $o, p, q, r, s$  and  $t$  are even or only five of  $o, p, q, r, s$  and  $t$  are even. Thus,  $o + p + r + s + t$ ,  $o + q + r + s + t$ ,  $p + q + r + s + t$ ,  $o + p + q + r + s$ ,  $o + p + q + r + t$  or  $o + p + q + s + t$  is odd and  $1 + op + oq + pq + rs + rt + st + (o + p + q)(r + s + t)$  is odd. Since  $f$  satisfies  $f^*(v) = (1 + op + oq + pq + rs + rt + st + (o + p + q)(r + s + t))\deg(v)/2$  and it is not an integer for some vertex  $v \in V(K_{o,p,q} + K_{r,s,t})$ , by (1.1),  $f^*(v)$  is a sum of integers, a contradiction.  $\square$

**Proposition 2.2.** *Let  $K_{o,p,q} + K_{r,s,t}$  be a balanced  $d$ -magic graph. Then  $o, p, q, r, s$  and  $t$  are even.*

*Proof.* Assume to the contrary that there exists one of  $o, p, q, r, s$  and  $t$  is odd. Thus,  $o+p+r+s+t$ ,  $o+q+r+s+t$ ,  $p+q+r+s+t$ ,  $o+p+q+r+s$ ,  $o+p+q+r+t$  or  $o+p+q+s+t$  is odd. This means that some vertices of  $K_{o,p,q} + K_{r,s,t}$  have odd degrees. Since every vertex of balanced  $d$ -magic graph has an even degree, a contradiction.  $\square$

The next result shows the sufficient condition for the existence of  $d$ -magic labelings of the join of complete tripartite graphs.

**Proposition 2.3.** *Let  $o, p$  and  $q$  be even positive integers, let  $s \geq t$  be odd positive integers and let  $r$  be even positive integer such that  $r \equiv 0 \pmod{4}$  whenever  $s = 1$ . Then  $K_{o,p,q} + K_{r,s,t}$  is a  $d$ -magic graph.*

*Proof.* Let  $o, p$  and  $q$  be even positive integers, let  $s \geq t$  be odd positive integers and let  $r$  be even positive integer such that  $r \equiv 0 \pmod{4}$  whenever  $s = 1$ . Then the graph  $K_{o,p,q}$  is balanced  $d$ -magic by Lemma 1.1 and the graph  $K_{r,s,t}$  is  $d$ -magic by Lemma 1.2. Since  $o+p+q \geq 6$  and  $r+s+t \geq 6$  are both even, the graph  $K_{o+p+q,r+s+t}$  is balanced  $d$ -magic by Theorem 1.6. Since  $K_{o,p,q} + K_{r,s,t}$  is the graph such that  $K_{o,p,q}$ ,  $K_{r,s,t}$  and  $K_{o+p+q,r+s+t}$  form its decomposition,  $K_{o,p,q} + K_{r,s,t}$  is a  $d$ -magic graph by Theorem 1.5.  $\square$

**Proposition 2.4.** *Let  $o, p, q, r, s$  and  $t$  be even positive integers. Then  $K_{o,p,q} + K_{r,s,t}$  is a balanced  $d$ -magic graph.*

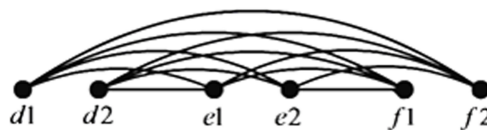
*Proof.* Let  $o, p, q, r, s$  and  $t$  be even positive integers. Then the graph  $K_{o,p,q}$  and  $K_{r,s,t}$  are balanced  $d$ -magic by Lemma 1.1. Since  $o+p+q \geq 6$  and  $r+s+t \geq 6$  are both even, the graph  $K_{o+p+q,r+s+t}$  is balanced  $d$ -magic by Theorem 1.6. Since  $K_{o,p,q} + K_{r,s,t}$  is the graph such that  $K_{o,p,q}$ ,  $K_{r,s,t}$  and  $K_{o+p+q,r+s+t}$  form its decomposition,  $K_{o,p,q} + K_{r,s,t}$  is a balanced  $d$ -magic graph by Theorem 1.5.  $\square$

**Corollary 2.1.** *Let  $o, p, q, r, s$  and  $t$  be even positive integers. If  $o = p = q = r = s = t$ , then  $K_{o,p,q} + K_{r,s,t}$  is a supermagic graph.*

*Proof.* Follows from Theorem 1.1 and Proposition 2.4.  $\square$

**Example 2.1.** Consider the join of complete tripartite graphs  $K_{2,2,2}$  and  $K_{4,1,1}$ . A  $d$ -magic graph  $K_{2,2,2} + K_{4,1,1}$  with the labels on edges is constructed as the following steps.

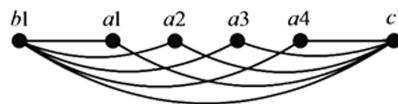
**Step I.** A balanced  $d$ -magic labeling  $f_1$  from  $E(K_{2,2,2})$  into  $\{1, 2, \dots, 12\}$  (see Figure 1 and Table 1) and a  $d$ -magic labeling  $f_2$  from  $E(K_{4,1,1})$  into  $\{1, 2, \dots, 9\}$  (see Figure 2 and Table 2) are defined as below.



**Figure 1.** A balanced  $d$ -magic graph  $K_{2,2,2}$

**Table 1.** The labels on edges of balanced  $d$ -magic graph  $K_{2,2,2}$

Vertex	$d1$	$d2$	$e1$	$e2$
$f1$	12	1	5	8
$f2$	9	4	7	6
$e1$	3	11	-	-
$e2$	2	10	-	-



**Figure 2.** A  $d$ -magic graph  $K_{4,1,1}$

**Step II.** Let  $A_1 = \{e \in E(K_{2,2,2}) : f_1(e) \leq 6\}$  and  $B_1 = \{e \in E(K_{2,2,2}) : f_1(e) > 6\}$  be given. Then a  $d$ -magic labeling  $g_1$  from  $E(K_{2,2,2} \cup K_{4,1,1})$  into  $\{1, 2, \dots, 21\}$  is defined by

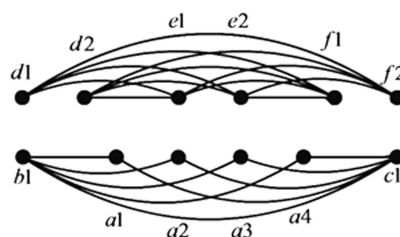
$$g_1(e) = \begin{cases} f_1(e) & : e \in A_1, \\ f_2(e) + 6 & : e \in E(K_{4,1,1}), \\ f_1(e) + 9 & : e \in B_1. \end{cases}$$

Thus, the  $d$ -magic graph  $K_{2,2,2} \cup K_{4,1,1}$  and its labels on edges are shown in Figure 3 and Table 3, respectively.

**Step III.** A balanced  $d$ -magic labeling  $f_3$  from  $E(K_{6,6})$  into  $\{1, 2, \dots, 36\}$  (see Figure 4 and Table 4) is defined as follows.

**Table 2.** The labels on edges of  $d$ -magic graph  $K_{4,1,1}$

Vertex	$a1$	$a2$	$a3$	$a4$	$c1$
$b1$	9	2	6	3	5
$c1$	1	8	4	7	-



**Figure 3.** A  $d$ -magic graph  $K_{2,2,2} \cup K_{4,1,1}$

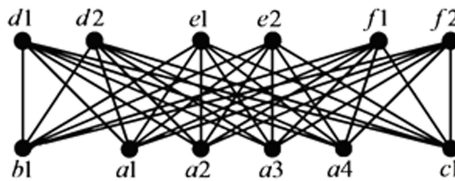
**Table 3.** The labels on edges of  $d$ -magic graph  $K_{2,2,2} \cup K_{4,1,1}$

Vertex	$a1$	$a2$	$a3$	$a4$	$c1$	Vertex	$d1$	$d2$	$e1$	$e2$
$b1$	15	8	12	9	11	$f1$	21	1	5	17
$c1$	7	14	10	13	-	$f2$	18	4	16	6
-	-	-	-	-	-	$e1$	3	20	-	-
-	-	-	-	-	-	$e2$	2	19	-	-

**Step IV.** The graph  $K_{2,2,2} + K_{4,1,1}$  is a graph such that  $K_{2,2,2} \cup K_{4,1,1}$  and  $K_{6,6}$  form its decomposition. Let  $A_2 = \{e \in E(K_{6,6}) : f_3(e) \leq 18\}$  and  $B_2 = \{e \in E(K_{6,6}) : f_3(e) > 18\}$  be given. Then a  $d$ -magic labeling  $g_2$  from  $E(K_{2,2,2} + K_{4,1,1})$  into  $\{1, 2, \dots, 57\}$  is defined by

$$g_2(e) = \begin{cases} f_3(e) & : e \in A_2, \\ g_1(e) + 18 & : e \in E(K_{2,2,2} \cup K_{4,1,1}), \\ f_3(e) + 21 & : e \in B_2. \end{cases}$$

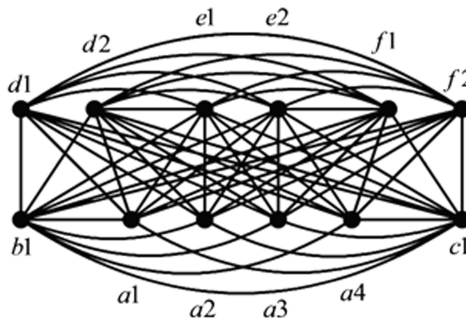
Therefore, the  $d$ -magic graph  $K_{2,2,2} + K_{4,1,1}$  and its labels on edges are shown in Figure 5 and Table 5, respectively.



**Figure 4.** A balanced  $d$ -magic graph  $K_{6,6}$

**Table 4.** The labels on edges of balanced  $d$ -magic graph  $K_{6,6}$

Vertex	b1	a1	a2	a3	a4	c1
d1	36	25	13	24	7	6
d2	5	29	20	14	11	32
e1	3	10	22	16	27	33
e2	34	9	21	15	28	4
f1	2	26	17	23	8	35
f2	31	12	18	19	30	1



**Figure 5.** A  $d$ -magic graph  $K_{2,2,2} + K_{4,1,1}$

**Table 5.** The labels on edges of  $d$ -magic graph  $K_{2,2,2} + K_{4,1,1}$

Vertex	b1	a1	a2	a3	a4	c1	e1	e2	f1	f2
d1	57	46	13	45	7	6	21	20	39	36
d2	5	50	41	14	11	53	38	37	19	22
e1	3	10	43	16	48	54	-	-	23	34
e2	55	9	42	15	49	4	-	-	35	24
f1	2	47	17	44	8	56	-	-	-	-
f2	52	12	18	40	51	1	-	-	-	-
b1	-	33	26	30	27	29	-	-	-	-
c1	-	25	32	28	31	-	-	-	-	-



### 3. Labelings in the Composition of Complete Tripartite Graphs

For any positive integers  $o, p, q, r, s$  and  $t$ , consider the composition  $K_{o,p,q} \cdot K_{r,s,t}$  of complete tripartite graphs. Let  $K_{o,p,q} \cdot K_{r,s,t}$  be a  $d$ -magic graph. Since  $\deg(v)$  is  $(o+p)(r+s+t)+r+s$ ,  $(o+p)(r+s+t)+r+t$ ,  $(o+p)(r+s+t)+s+t$ ,  $(o+q)(r+s+t)+r+s$ ,  $(o+q)(r+s+t)+r+t$ ,  $(o+q)(r+s+t)+s+t$ ,  $(p+q)(r+s+t)+r+s$ ,  $(p+q)(r+s+t)+r+t$  or  $(p+q)(r+s+t)+s+t$  and  $f^*(v) = (1 + (r+s+t)^2(op + oq + pq) + (rs + rt + st)(o + p + q))\deg(v)/2$  for any vertex  $v \in V(K_{o,p,q} \cdot K_{r,s,t})$ , giving the following results

**Proposition 3.1.** *Let  $K_{o,p,q} \cdot K_{r,s,t}$  be a  $d$ -magic graph. Then the following statements hold:*

- (i) *only one of  $o, p, q, r, s$  and  $t$  is even or*
- (ii) *only three of  $o, p, q, r, s$  and  $t$  are even or*
- (iii) *all of  $o, p, q, r, s$  and  $t$  are either odd or even or*
- (iv) *only one of  $o, p$  and  $q$  is even and all of  $r, s$  and  $t$  are even or*
- (v) *only two of  $o, p$  and  $q$  are even and all of  $r, s$  and  $t$  are even or*
- (vi) *all of  $o, p$  and  $q$  are odd and only two of  $r, s$  and  $t$  are even or*
- (vii) *only two of  $o, p$  and  $q$  are even and all of  $r, s$  and  $t$  are odd.*

*Proof.* Suppose that  $f$  is a  $d$ -magic labeling of  $K_{o,p,q} \cdot K_{r,s,t}$ . Assume to the contrary that only one of  $o, p$  and  $q$  is even and only one of  $r, s$  and  $t$  is even, only two of  $o, p$  and  $q$  are even and only two of  $r, s$  and  $t$  are even, all of  $o, p$  and  $q$  are even and only one of  $r, s$  and  $t$  is even or all of  $o, p$  and  $q$  are even and only two of  $r, s$  and  $t$  are even. Thus,  $(o+p)(r+s+t)+r+s$ ,  $(o+p)(r+s+t)+r+t$ ,  $(o+p)(r+s+t)+s+t$ ,  $(o+q)(r+s+t)+r+s$ ,  $(o+q)(r+s+t)+r+t$ ,  $(o+q)(r+s+t)+s+t$ ,  $(p+q)(r+s+t)+r+s$ ,  $(p+q)(r+s+t)+r+t$  or  $(p+q)(r+s+t)+s+t$  is odd and  $1 + (r+s+t)^2(op + oq + pq) + (rs + rt + st)(o + p + q)$  is odd. Since  $f$  satisfies  $f^*(v) = (1 + (r+s+t)^2(op + oq + pq) + (rs + rt + st)(o + p + q))\deg(v)/2$  and it is not an integer for some vertex  $v \in V(K_{o,p,q} \cdot K_{r,s,t})$ , by (1.1),  $f^*(v)$  is a sum of integers, a contradiction.  $\square$

**Proposition 3.2.** *Let  $K_{o,p,q} \cdot K_{r,s,t}$  be a balanced  $d$ -magic graph. Then the following statements hold:*

- (i) *all of  $r, s$  and  $t$  are even or*
- (ii) *all of  $o, p, q, r, s$  and  $t$  are odd or*
- (iii) *all of  $o, p$  and  $q$  are even and all of  $r, s$  and  $t$  are odd.*

*Proof.* Assume to the contrary that only one of  $o, p, q, r, s$  and  $t$  is even, only two of  $o, p, q, r, s$  and  $t$  are even, only three of  $o, p, q, r, s$  and  $t$  are even but all of  $o, p$  and  $q$  are neither odd nor even, only four of  $o, p, q, r, s$  and  $t$  are even but all of  $r, s$  and  $t$  are not even or only five of  $o, p, q, r, s$  and  $t$  are even but all of  $r, s$  and  $t$  are not even. Thus,  $(o+p)(r+s+t)+r+s$ ,  $(o+p)(r+s+t)+r+t$ ,  $(o+p)(r+s+t)+s+t$ ,  $(o+q)(r+s+t)+r+s$ ,  $(o+q)(r+s+t)+r+t$ ,  $(o+q)(r+s+t)+s+t$ ,  $(p+q)(r+s+t)+r+s$ ,  $(p+q)(r+s+t)+r+t$  or  $(p+q)(r+s+t)+s+t$  is odd. This means that some vertices of  $K_{o,p,q} \cdot K_{r,s,t}$  have odd degrees. Since every vertex of balanced  $d$ -magic graph has an even degree, a contradiction.  $\square$

In the next result, the sufficient condition for the existence of  $d$ -magic labelings of the composition of complete tripartite graphs is found.

**Proposition 3.3.** *Let  $o, p$  and  $q$  be positive integers and let  $r, s$  and  $t$  be even positive integers. Then  $K_{o,p,q} \cdot K_{r,s,t}$  is a balanced  $d$ -magic graph.*

*Proof.* Let  $o, p$  and  $q$  be positive integers and let  $r, s$  and  $t$  be even positive integers. Then the graph  $K_{r,s,t}$  is balanced  $d$ -magic by Lemma 1.1. Since  $r + s + t \geq 6$  are even, the graph  $K_{r+s+t,r+s+t}$  is balanced  $d$ -magic by Theorem 1.6. The graph  $K_{o,p,q} \cdot K_{r,s,t}$  is decomposed into  $op + oq + pq$  balanced  $d$ -magic subgraphs isomorphic to  $K_{r+s+t,r+s+t}$  and  $o + p + q$  balanced  $d$ -magic subgraphs isomorphic to  $K_{r,s,t}$ . According Theorem 1.5,  $K_{o,p,q} \cdot K_{r,s,t}$  is a balanced  $d$ -magic graph.  $\square$

**Corollary 3.1.** *Let  $o, p$  and  $q$  be positive integers and let  $r, s$  and  $t$  be even positive integers. If  $o = p = q$  and  $r = s = t$ , then  $K_{o,p,q} \cdot K_{r,s,t}$  is a supermagic graph.*

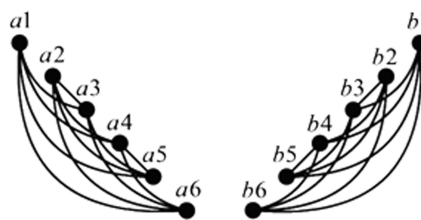
*Proof.* Follows from Theorem 1.1 and Proposition 3.3.  $\square$

**Example 3.1.** Consider the composition of complete tripartite graphs  $K_{1,1,1}$  and  $K_{2,2,2}$ . A balanced  $d$ -magic graph  $K_{1,1,1} \cdot K_{2,2,2}$  with the labels on edges is constructed as the following steps.

**Step I.** A balanced  $d$ -magic graph  $K_{2,2,2}$  is defined in Figure 1 and Table 1. Let  $A_3 = \{e \in E(K_{2,2,2}) : f_1(e) \leq 6\}$  and  $B_3 = \{e \in E(K_{2,2,2}) : f_1(e) > 6\}$  be given. For  $i \in \{1, 2\}$ , let  $K_{2,2,2}^i$  be a copy of  $K_{2,2,2}$  and let  $e_i$  be its edge corresponding to  $e \in E(K_{2,2,2})$ . Suppose that  $2K_{2,2,2} = K_{2,2,2}^1 \cup K_{2,2,2}^2$ . Then a balanced  $d$ -magic labeling  $g_3$  from  $E(2K_{2,2,2})$  into  $\{1, 2, \dots, 24\}$  is defined by

$$g_3(e_i) = \begin{cases} f_1(e) : i = 1 \text{ and } e \in A_3, \\ f_1(e) + 12 : i = 1 \text{ and } e \in B_3, \\ f_1(e) + 12 : i = 2 \text{ and } e \in A_3, \\ f_1(e) : i = 2 \text{ and } e \in B_3. \end{cases}$$

Thus, the balanced  $d$ -magic graph  $2K_{2,2,2}$  and its labels on edges are shown in Figure 6 and Table 6, respectively.



**Figure 6.** A balanced  $d$ -magic graph  $2K_{2,2,2}$

**Table 6.** The labels on edges of balanced  $d$ -magic graph  $2K_{2,2,2}$

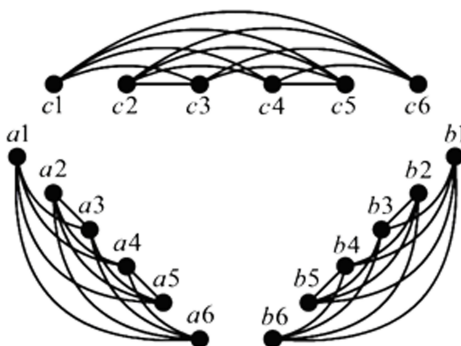
Vertex	a1	a2	a3	a4	Vertex	b1	b2	b3	b4
a5	24	1	5	20	b5	12	13	17	8
a6	21	4	19	6	b6	9	16	7	18
a3	3	23	-	-	b3	15	11	-	-
a4	2	22	-	-	b4	14	10	-	-



**Step II.** The graph  $3K_{2,2,2}$  is a graph such that  $2K_{2,2,2}$  and  $K_{2,2,2}$  form its decomposition. Let  $A_4 = \{e \in E(2K_{2,2,2}) : g_3(e) \leq 12\}$  and  $B_4 = \{e \in E(2K_{2,2,2}) : g_3(e) > 12\}$  be given. Then a balanced  $d$ -magic labeling  $g_4$  from  $E(3K_{2,2,2})$  into  $\{1, 2, \dots, 36\}$  is defined by

$$g_4(e) = \begin{cases} g_3(e) & : e \in A_4, \\ f_1(e) + 12 & : e \in E(K_{2,2,2}), \\ g_3(e) + 12 & : e \in B_4. \end{cases}$$

Thus, the balanced  $d$ -magic graph  $3K_{2,2,2}$  and its labels on edges are shown in Figure 7 and Table 7, respectively.



**Figure 7.** A balanced  $d$ -magic graph  $3K_{2,2,2}$

**Table 7.** The labels on edges of balanced  $d$ -magic graph  $3K_{2,2,2}$

Vertex	a1	a2	a3	a4	Vertex	b1	b2	b3	b4	Vertex	c1	c2	c3	c4
a5	36	1	5	32	b5	12	25	29	8	c5	24	13	17	20
a6	33	4	31	6	b6	9	28	7	30	c6	21	16	19	18
a3	3	35	-	-	b3	27	11	-	-	c3	15	23	-	-
a4	2	34	-	-	b4	26	10	-	-	c4	14	22	-	-

**Step III.** A balanced  $d$ -magic graph  $K_{6,6}$  is defined in Figure 4 and Table 4. Let  $K_{6,6}^1 = K_{6,6}^2 = K_{6,6}$  be given. The graph  $K_{6,12}$  is a graph such that  $K_{6,6}^1$  and  $K_{6,6}^2$  form its decomposition. Let  $A_5 = \{e \in E(K_{6,6}^1) : f_3(e) \leq 18\}$  and  $B_5 = \{e \in E(K_{6,6}^1) : f_3(e) > 18\}$  be given. Then a balanced  $d$ -magic labeling  $g_5$  from  $E(K_{6,12})$  into  $\{1, 2, \dots, 72\}$  is defined by

$$g_5(e) = \begin{cases} f_3(e) & : e \in A_5, \\ f_3(e) + 18 & : e \in E(K_{6,6}^2), \\ f_3(e) + 36 & : e \in B_5. \end{cases}$$

Thus, the balanced  $d$ -magic graph  $K_{6,12}$  and its labels on edges are shown in Figure 8 and Table 8, respectively.

**Step IV.** The graph  $K_{6,6,6}$  is a graph such that  $K_{6,6}$  and  $K_{6,12}$  form its decomposition. Let  $A_6 = \{e \in E(K_{6,12}) : g_5(e) \leq 36\}$  and  $B_6 = \{e \in E(K_{6,12}) : g_5(e) > 36\}$  be given. Then a balanced  $d$ -magic labeling  $g_6$  from  $E(K_{6,6,6})$  into  $\{1, 2, \dots, 108\}$  is defined by

$$g_6(e) = \begin{cases} g_5(e) & : e \in A_6, \\ f_3(e) + 36 & : e \in E(K_{6,6}), \\ g_5(e) + 36 & : e \in B_6. \end{cases}$$

Thus, the balanced  $d$ -magic graph  $K_{6,6,6}$  and its labels on edges are shown in Figure 9 and Table 9, respectively.

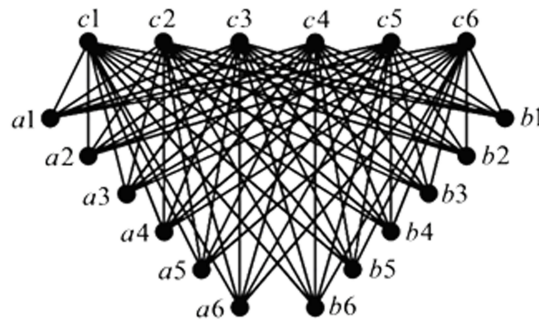


Figure 8. A balanced  $d$ -magic graph  $K_{6,12}$

Table 8. The labels on edges of balanced  $d$ -magic graph  $K_{6,12}$

Vertex	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$
$c_1$	72	61	13	60	7	6	54	43	31	42	25	24
$c_2$	5	65	56	14	11	68	23	47	38	32	29	50
$c_3$	3	10	58	16	63	69	21	28	40	34	45	51
$c_4$	70	9	57	15	64	4	52	27	39	33	46	22
$c_5$	2	62	17	59	8	71	20	44	35	41	26	53
$c_6$	67	12	18	55	66	1	49	30	36	37	48	19

**Step V.** The graph  $K_{1,1,1} \cdot K_{2,2,2}$  is a graph such that  $3K_{2,2,2}$  and  $K_{6,6,6}$  form its decomposition. Let  $A_7 = \{e \in E(K_{6,6,6}) : g_6(e) \leq 54\}$  and  $B_7 = \{e \in E(K_{6,6,6}) : g_6(e) > 54\}$  be given. Then a balanced  $d$ -magic labeling  $g_7$  from  $E(K_{1,1,1} \cdot K_{2,2,2})$  into  $\{1, 2, \dots, 144\}$  is defined by

$$g_7(e) = \begin{cases} cg_6(e) & : e \in A_7, \\ g_4(e) + 54 & : e \in E(3K_{2,2,2}), \\ g_6(e) + 36 & : e \in B_7. \end{cases}$$

Therefore, the balanced  $d$ -magic graph  $K_{1,1,1} \cdot K_{2,2,2}$  and its labels on edges are shown in Figure 10, Table 10 and Table 11, respectively.

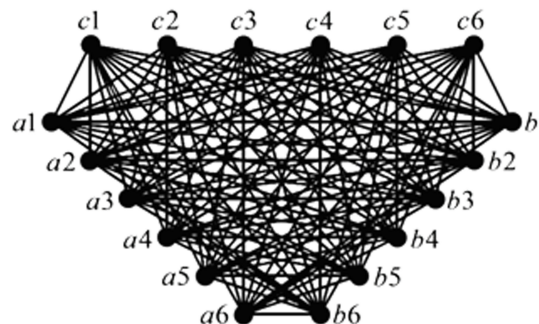


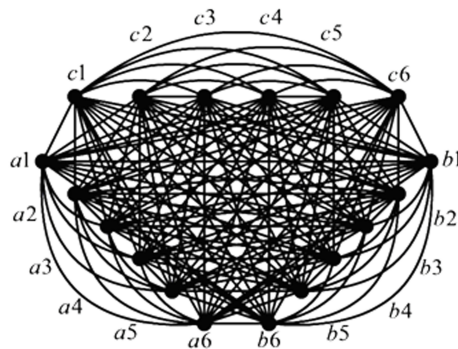
Figure 9. A balanced  $d$ -magic graph  $K_{6,6,6}$

**Table 9.** The labels on edges of balanced  $d$ -magic graph  $K_{6,6,6}$

Vertex	a1	a2	a3	a4	a5	a6						
b1	72	61	49	60	43	42						
b2	41	65	56	50	47	68						
b3	39	46	58	52	63	69						
b4	70	45	57	51	64	40						
b5	38	62	53	59	44	71						
b6	67	48	54	55	66	37						

Vertex	a1	a2	a3	a4	a5	a6	b1	b2	b3	b4	b5	b6
c1	108	97	13	96	7	6	90	79	31	78	25	24
c2	5	101	92	14	11	104	23	83	74	32	29	86
c3	3	10	94	16	99	105	21	28	76	34	81	87
c4	106	9	93	15	100	4	88	27	75	33	82	22
c5	2	98	17	95	8	107	20	80	35	77	26	89
c6	103	12	18	91	102	1	85	30	36	73	84	19



**Figure 10.** A balanced  $d$ -magic graph  $K_{1,1,1} \cdot K_{2,2,2}$

**Table 10.** The labels on edges of balanced  $d$ -magic graph  $K_{1,1,1} \cdot K_{2,2,2}$

Vertex	a1	a2	a3	a4	a5	a6						
b1	108	97	49	96	43	42						
b2	41	101	92	50	47	104						
b3	39	46	94	52	99	105						
b4	106	45	93	51	100	40						
b5	38	98	53	95	44	107						
b6	103	48	54	91	102	37						

Vertex	a1	a2	a3	a4	a5	a6	b1	b2	b3	b4	b5	b6
c1	144	133	13	132	7	6	126	115	31	114	25	24
c2	5	137	128	14	11	140	23	119	110	32	29	122
c3	3	10	130	16	135	141	21	28	112	34	117	123
c4	142	9	129	15	136	4	124	27	111	33	118	22
c5	2	134	17	131	8	143	20	116	35	113	26	125
c6	139	12	18	127	138	1	121	30	36	109	120	19

**Table 11.** The labels on edges of balanced  $d$ -magic graph  $K_{1,1,1} \cdot K_{2,2,2}$

Vertex	a1	a2	a3	a4	Vertex	b1	b2	b3	b4	Vertex	c1	c2	c3	c4
a5	90	55	59	86	b5	66	79	83	62	c5	78	67	71	74
a6	87	58	85	60	b6	63	82	61	84	c6	75	70	73	72
a3	57	89	-	-	b3	81	65	-	-	c3	69	77	-	-
a4	56	88	-	-	b4	80	64	-	-	c4	68	76	-	-

## 4. Conclusion

In this paper, the necessary and sufficient conditions for the existence of degree-magic labelings on tripartite graphs under some binary operations have been examined and proven. However, the labeling of discrete structures is an extensive field of study, so a further open area of research would be to investigate and derive similar results for different families in the context of varying graph-labeling problems.

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## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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