Communications in Mathematics and Applications
Vol. 10, No. 3, pp. 571–584, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i3.1154

Research Article

A Mathematical Analysis and Modelling of Hepatitis B Model with Non-Integer Time Fractional Derivative

Muhammad Farman[1,*], Aqeel Ahmad[1], M. Umer Saleem[2] and Amina Hafeez[1]

[1] *Department of Mathematics and Statistics, University of Lahore, Lahore, Pakistan*
[2] *Department of Mathematics, University of Education, Lahore, Pakistan*
***Corresponding author:** farmanlink@gmail.com

Abstract. In this paper, we develop time fractional Hepatitis B model and analyze qualitatively. The Caputo fractional derivative operator of order $\alpha \in (0,1]$ is employed to obtain the system of fractional differential equations. The stability and sensitivity analysis of fractional order model has been made and verify the non-negative unique solution. LADM was successfully used for solving different problems. Laplace transform method is a useful technique in different field of biological science, engineering and applied mathematics. The latest technique (LADM) is employed on the developed fractional order model for numerical solutions. Finally, numerical simulations are also established to investigate the influence of the system parameter on the spread of the disease and which show the effect of fractional parameter α on our obtained solutions.

Keywords. Epidemic model; Stability analysis; Fractional derivative; LADM

MSC. 37M05; 92B05

Received: December 14, 2018 **Accepted:** February 1, 2019

1. Introduction

Mathematical biology or biomathematics is a quickly developing and most energizing present day utilization of science. This is an interdisciplinary research zone with a scope of uses in biology, biotechnology and biomedical science. The recorded might be called as the mathematical biology or biomathematics to pressure the mathematical side or hypothetical biology to

pressure the natural side. Mathematical zones, for example, analytic, likelihood hypothesis, measurements, direct variable based math, diagram hypothesis, combinatory, logarithmic geometry, topology, dynamical frameworks, differential conditions and coding hypothesis are presently being connected in this field [7].

Viral hepatitis is an overall general medical issue influencing a huge number of individuals. Hepatitis is an irritation of the liver caused by one of the five hepatitis infections, alluded to as sorts A, B, C, D and E. While these infections cause liver malady, they shift regarding the study of disease transmission, characteristic history, anticipation, analysis and treatment [11].

Hepatitis B is a potentially life-threatening liver infection caused by the hepatitis B virus. It is a major global health problem. It can cause chronic liver disease and chronic infection and puts people at high risk of death from liver cancer. Infections of hepatitis B occur only if the virus is able to enter the blood stream and reach the liver. Once in the liver, the virus reproduces and releases large numbers of new viruses into the blood stream. This infection has two possible phases: (1) acute and (2) chronic. Acute hepatitis B infection lasts less than six months. If the disease is acute, your immune system is usually able to clear the virus from your body, and you should recover completely within a few months. About two-thirds of people with chronic HBV infection are chronic carriers. These people do not develop symptoms, even though they harbor the virus and can transmit it to other people. More than 240 million people have chronic liver infections [1, 3, 6].

Fractional Calculus is a branch of mathematical examination that reviews the a few unique conceivable outcomes of characterizing genuine number or complex number intensity of the separation administrator. In connected arithmetic and mathematical examination, fractional subsidiary is a subordinate of any subjective request, genuine or complex. The primary appearance of the idea of a fractional subordinate is found in a letter kept in touch with Guillaume de l'Hpital by Gotfried Wilhelm Leibniz in 1695 [8, 9].

Laplace change is utilized to limit the subsequent condition to a standard mathematical condition. Converse Laplace and opposite limited sine changes are used to get the coveted results. Laplace change technique is a valuable method in various branches of biological science, building and connected arithmetic. The go with of ADM and Laplace change prompts a great technique known as Laplace Adomain deterioration strategy. With the assistance of Lap-bind change, we can change over a differential condition to arithmetical conditions and the non-direct terms are deteriorated as far as Adomain polynomials. The given numerical strategy works effectively for an arrangement of settled and also stochastic differential conditions. More obvious, it can be utilized for traditional and also partial request arrangement of straight and nonlinear common. Assist it has no need of officially characterized advance size like RK_4. Additionally, this strategy does not rely on a parameter like required for *homotopy perturbation technique* (HPM) and *homotopy analysis method* (HAM). In spite of the fact that the arrangements got by means of this technique are the same as result got from ADM [2, 4, 10, 12].

Firstly, we proposed a fractional order SEICR epidemic model is in-corporate to investigate Hepatitis B model with non-integer time fractional derivative "A Mathematical analysis". S, E, I, C, R are known as variables and these variables represent the number of Susceptible, Exposed

population, Acute infected population, Chronic infected population and Recovered cells from viruses in the body. The Caputo fractional derivative operator of order $\alpha\epsilon(0,1]$ is employed to obtain the system of fractional differential equations. The basic reproductive number is derived for a general viral production rate which determines the local stability of the infection free equilibrium. The stability and sensitivity analysis of fractional order has been made and verify the non-negative unique solution. The solution of the time fractional model has been procured by employing *Laplace Adomian decomposition method* (LADM) and the accuracy of the scheme is presented by convergence analysis. Finally, numerical solutions are also established to investigate the in influence of system parameter on the spread of disease and which show the effect of fractional parameter on our obtained solution.

2. Mathematical Model

$$\frac{dS}{dt} = \nu - \nu p_1 C - \nu p_2 R - \rho(I+\theta C)S - \nu S - \mu_1 S + \lambda_4 R\,, \tag{1}$$

$$\frac{dE}{dt} = \rho(I+\theta C)S - (\nu+\lambda_1)E\,, \tag{2}$$

$$\frac{dI}{dt} = \lambda_1 E - (\nu+\lambda_2)I\,, \tag{3}$$

$$\frac{dC}{dt} = \nu p_1 C + P_3\lambda_2 I - (\nu+\lambda_3)C - \mu_2 C\,, \tag{4}$$

$$\frac{dR}{dt} = \nu p_2 R + (1-P_3)\lambda_2 I + \lambda_3 C - \nu R - \lambda_4 R + \mu_1 S + \mu_2 C \tag{5}$$

with initial conditions $S(0)\geq 0$, $E(0)\geq 0$, $I(0)\geq 0$, $C(0)\geq 0$, $R(0)\geq 0$. In these equations, all the parameters are nonnegative with equal per capita birth and death rate ν, λ_1 is the rate of exposed individuals becoming infections, λ_2 is the rate at which individuals leave the acute infection class, λ_3 is the rate of carrier individuals who recover from the disease by natural way (spontaneous recovery) and move from carrier to recovered, λ_4 is represent the loss of recovery rate, θ is the infectiousness of carries relative to acute infection, ρ is represent the transmission rate, p_1 is represent the probability of infected newborns, p_2 is represent the probability of immune newborns, p_3 is represent the proportion of acute infection individual becoming carries, μ_1 represents the proportion of the susceptible that is vaccinated per unit, and μ_2 is the proportion of the chronic Hepatitis B virus. The new system of differential equation is represented by the fractional system of differential equations (FDEs) is given as follows:

$$D^{\alpha_1}S = \nu - \nu p_1 C - \nu p_2 R - \rho(I+\theta C)S - \nu S - \mu_1 S + \lambda_4 R\,, \tag{6}$$

$$D^{\alpha_2}E = \rho(I+\theta C)S - (\nu+\lambda_1)E\,, \tag{7}$$

$$D^{\alpha_3}I = \lambda_1 E - (\nu+\lambda_2)I\,, \tag{8}$$

$$D^{\alpha_4}C = \nu p_1 C + P_3\lambda_2 I - (\nu+\lambda_3)C - \mu_2 C\,, \tag{9}$$

$$D^{\alpha_5}R = \nu p_2 R + (1-P_3)\lambda_2 I + \lambda_3 C - \nu R - \lambda_4 R + \mu_1 S + \mu_2 C\,. \tag{10}$$

Since the initial conditions must satisfied. We will assume the total population is constant for size N. i.e. $S+E+I+C+R=N$ The system is qualitatively analyzed by two ways i.e. disease

Free Equilibrium and endemic Equilibrium or $N = n_1 + n_2 + n_3 + n_4 + n_5$.

$$S(0) = n_1 > 0,\ E(0) = n_2 > 0,\ I(0) = n_3 > 0,\ C(0) = n_4 > 0,\ R(0) = n_5 > 0. \tag{11}$$

3. Equilibria and Analysis

The system is qualitatively analyzed by two ways i.e. Disease Free Equilibrium and Endemic Equilibrium.

3.1 The Disease Free Point

Most of the epidemiological have a free equilibrium point at which population remains absent of disease. In this model, for equilibrium point put

$$\frac{dS^{\alpha_1}}{dt} = 0,\ \frac{dE_2^{\alpha}}{dt} = 0,\ \frac{dI_3^{\alpha}}{dt} = 0,\ \frac{dC_4^{\alpha}}{dt} = 0,\ \frac{dR_5^{\alpha}}{dt} = 0.$$

By substituting the values of parameters in given system of differential equations and the rate of change with respect to time is zero. We get the values of S,E,I,C,R so the disease free point $(S_0, E_0, I_0, C_0, R_0) = (1,0,0,0,0)$.

3.2 The Endemic Point

Endemic equilibrium are found in terms of one of the infected compartment, denoted by E^* i.e., and we get the endemic point.

So, the endemic point $(S,E,I,C,R) = \left(\frac{\nu + \lambda_4 - \nu p_2}{\nu + \lambda_4 + \mu_1 - \nu p_2}, 0, 0, 0, \frac{\mu}{\lambda_4 + \mu_1 + \nu - \nu p_2}\right)$.

3.3 Stability Analysis

As we know that the Jacobian Matrix is:

$$J(S,E,I,C,R) = \begin{bmatrix} \rho I - \rho\theta C - \nu - \mu_1 & 0 & -pS & \nu p_1 - \rho\theta S & -\nu p_2 + \lambda_4 \\ \rho I + \rho\theta C & -\nu - \lambda_1 & \rho S & \rho\theta S & 0 \\ 0 & \lambda_1 & -\nu - \lambda_2 & 0 & 0 \\ 0 & 0 & p_3\lambda_2 & \nu p_1 - \nu - \lambda_3 - \mu_2 & 0 \\ \mu_1 & 0 & (1-p_3)\lambda_2 & \lambda_3 + \mu_2 & \nu p_2 - \nu - \lambda_4 \end{bmatrix}.$$

Theorem 3.1. *The disease free equilibrium E_0 is locally asymptotically stable if $Re(\lambda) < 0$, otherwise unstable.*

Proof. E_0 of the given system is locally asymptotically stable if $Re(\lambda) < 0$ where can be evaluated from the relation $|(J_0 - \lambda I)| = 0$. Consider the Jacobian matrix and substituting the values of disease free point E_0, we get

$$J_0 = \begin{bmatrix} -\nu - \mu_1 & 0 & -p & \nu p_1 - \rho\theta & -\nu p_2 + \lambda_4 \\ 0 & -\nu - \lambda_1 & \rho & \rho\theta & 0 \\ 0 & \lambda_1 & -\nu - \lambda_2 & 0 & 0 \\ 0 & 0 & p_3\lambda_2 & \nu p_1 - \nu - \lambda_3 - \mu_2 & 0 \\ \mu_1 & 0 & (1-p_3)\lambda_2 & \lambda_3 + \mu_2 & \nu p_2 - \nu - \lambda_4 \end{bmatrix}$$

by using $|J_0 - \lambda I| = 0$, we get

$$\lambda = -0.13879 < 0,$$

$\lambda = -0.0121 < 0,$

$\lambda = -3.35332 < 0.$

So, our system is stable. □

3.4 Reproductive Number

The Jacobian Matrix at the equilibrium point that is J_0 is:

$$J_0 = \begin{bmatrix} -\nu-\mu_1 & 0 & -p & \nu p_1 - \rho\theta & -\nu p_2 + \lambda_4 \\ 0 & -\nu-\lambda_1 & \rho & \rho\theta & 0 \\ 0 & \lambda_1 & -\nu-\lambda_2 & 0 & 0 \\ 0 & 0 & p_3\lambda_2 & \nu p_1 - \nu - \lambda_3 - \mu_2 & 0 \\ \mu_1 & 0 & (1-p_3)\lambda_2 & \lambda_3+\mu_2 & \nu p_2 - \nu - \lambda_4 \end{bmatrix}.$$

Since the Jacobian matrix is $J = F - V$ then the matrix F and V can be written as

$$F = \begin{bmatrix} 0 & 0 & 0 & \nu p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu p_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \nu+\mu_1 & 0 & p & \rho\theta & \nu p_2 - \lambda_4 \\ 0 & \nu+\lambda_1 & -\rho & -\rho\theta & 0 \\ 0 & -\lambda_1 & \nu+\lambda_2 & 0 & 0 \\ 0 & 0 & -p_3\lambda_2 & \nu+\lambda_3+\mu_2 & 0 \\ -\mu_1 & 0 & -(1-p_3)\lambda_2 & -\lambda_3-\mu_2 & -\nu p_2 + \nu + \lambda_4 \end{bmatrix}$$

by using the relation $|K - \lambda I| = 0$, where $K = F.V^{-1}$ got the eigen value

$$\lambda = \frac{p_1\nu(\nu^2 + \lambda_2(\lambda_1+\nu) + \lambda_1(-\rho+\nu))}{\lambda_2(p_3\theta\rho\lambda_1 + \lambda_1\nu + \nu^2 + \lambda_3(\lambda_1+\nu) + \mu_2(\lambda_1+\nu)) + (\lambda_3+\mu_2+\nu)(\nu^2+\lambda_1(-\rho+\nu))}$$

that is also called reproductive number. So, the reproductive number is:

$$R_0 = \frac{p_1\nu(\nu^2 + \lambda_2(\lambda_1+\nu) + \lambda_1(-\rho+\nu))}{\lambda_2(p_3\theta\rho\lambda_1 + \lambda_1\nu + \nu^2 + \lambda_3(\lambda_1+\nu) + \mu_2(\lambda_1+\nu)) + (\lambda_3+\mu_2+\nu)(\nu^2+\lambda_1(-\rho+\nu))}$$

which is less than one

$$R_0 = 0.00265856 < 1.$$

Theorem 3.2. *There is a unique solution for the initial value problem given in system* (6)-(10)*, and the solution remains in* $R^5, x \geq 0\}$.

Proof. The existence and uniqueness of the solution of system (6)-(10), in $(0, \alpha)$. We need to show that the domain R^5, $x \geq 0$ is positively invariant. Since

$$D^\alpha S(t)|_{S=0} = \nu - \nu p_1 C - \nu p_2 R + \lambda_4 R \geq 0, \tag{12}$$

$$D^\alpha E(t)|_{E=0} = \rho(I + \theta C)S \geq 0, \tag{13}$$

$$D^\alpha I(t)|_{I=0} = \lambda_1 E \geq 0, \tag{14}$$

$$D^\alpha C(t)|_{C=0} = P_3\lambda_2 I \geq 0, \tag{15}$$

$$D^\alpha R(t)|_{R=0} = (1-P_3)\lambda_2 I + \lambda_3 C + \mu_1 S + \mu_2 C \geq 0. \tag{16}$$

The vector field points R_5^+ shows the nonnegative or that in each hyper-plane. □

4. Preliminaries

In this section, we give some fundamental results and definitions from fractional calculus. For detailed over view of the topic readers are referred to [8,9].

Definition 4.1. The Riemann-liouville fractional integration of order $\alpha \in (0,1)$ of the function $f \in L^1([0,T],R)$ is defined as as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo's definition owing to its convenience for initial conditions of the fractional differential equations.

Definition 4.2. The definitions of Laplace transform of Caputo's derivative is written as

$$L\{{}^{c}D^{\alpha} y(t)\} = s^{\alpha} h(s) - \sum_{k=0}^{n-1} s^{\alpha-i-1} y^{(k)}(0), \quad n-1 < \alpha \le n;\ n \in N$$

for arbitrary $c_i \in R$, $i = 0,1,2,\ldots,n-1$, where $n = [\alpha]+1$ and $[\alpha]$ shows the integer part of α.

5. The Laplace-Adomian Decomposition Method

Consider the fractional-order epidemic model (6)-(10) subject to the initial condition (11). The nonlinear term in this model is SI and CS. For $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$ the fractional order model converts to the classical epidemic model. by using Laplace transform on the system given in equations (6)-(10), we get

$$\begin{aligned}
\mathscr{L}\{D_t^{\alpha_1} S\} &= \nu \mathscr{L}\{1\} - \nu p_1 \mathscr{L}\{C\} - \nu p_2 \mathscr{L}\{R\} - \rho \mathscr{L}\{IS\} - \rho\theta \mathscr{L}\{CS\} \\
&\quad - \nu \mathscr{L}\{S\} - \mu_1 \mathscr{L}\{S\} + \lambda_4 \mathscr{L}\{R\}, \qquad (17)\\
\mathscr{L}\{D_t^{\alpha_2} E\} &= \rho \mathscr{L}\{SI\} + \rho\theta \mathscr{L}\{CS\} - (\nu + \lambda_1)\mathscr{L}\{E\}, \qquad (18)\\
\mathscr{L}\{D_t^{\alpha_3} I\} &= \lambda_1 \mathscr{L}\{E\} - (\nu + \lambda_2)\mathscr{L}\{I\}, \qquad (19)\\
\mathscr{L}\{D_t^{\alpha_4} C\} &= \nu p_1 \mathscr{L}\{C\} + p_3 \lambda_2 \mathscr{L}\{I\} - (\nu + \lambda_3)\mathscr{L}\{C\} - \mu_2 \mathscr{L}\{C\}, \qquad (20)\\
\mathscr{L}\{D_t^{\alpha_5} R\} &= \nu p_2 \mathscr{L}\{R\} + \lambda_2 \mathscr{L}\{I\} - p_3 \lambda_2 \mathscr{L}\{I\} + \lambda_3 \mathscr{L}\{C\} \\
&\quad - \nu \mathscr{L}\{R\} - \lambda_4 \mathscr{L}\{R\} + \mu_1 \mathscr{L}\{S\} + \mu_2 \mathscr{L}\{C\}. \qquad (21)
\end{aligned}$$

By applying Laplace transform results,we get

$$\begin{aligned}
S^{\alpha_1}\mathscr{L}\{S\} - S^{\alpha_1-1}S(0) &= \nu \mathscr{L}\{1\} - \nu p_1 \mathscr{L}\{C\} - \nu p_2 \mathscr{L}\{R\} - \rho \mathscr{L}\{IS\} \\
&\quad - \rho\theta \mathscr{L}\{CS\} - \nu \mathscr{L}\{S\} - \mu_1 \mathscr{L}\{S\} + \lambda_4 \mathscr{L}\{R\}, \qquad (22)\\
S^{\alpha_2}\mathscr{L}\{E\} - S^{\alpha_2-1}E(0) &= \rho \mathscr{L}\{SI\} + \rho\theta \mathscr{L}\{CS\} - (\nu + \lambda_1)\mathscr{L}\{E\}, \qquad (23)\\
S^{\alpha_3}\mathscr{L}\{I\} - S^{\alpha_3-1}I(0) &= \lambda_1 \mathscr{L}\{E\} - (\nu + \lambda_2)\mathscr{L}\{I\}, \qquad (24)\\
S^{\alpha_4}\mathscr{L}\{C\} - S^{\alpha_4-1}C(0) &= \nu p_1 \mathscr{L}\{C\} + p_3 \lambda_2 \mathscr{L}\{I\} - (\nu + \lambda_3)\mathscr{L}\{C\} - \mu_2 \mathscr{L}\{C\}, \qquad (25)\\
S^{\alpha_5}\mathscr{L}\{R\} - S^{\alpha_5-1}R(0) &= \nu p_2 \mathscr{L}\{R\} + \lambda_2 \mathscr{L}\{I\} - p_3 \lambda_2 \mathscr{L}\{I\} + \lambda_3 \mathscr{L}\{C\} \\
&\quad - \nu \mathscr{L}\{R\} - \lambda_4 \mathscr{L}\{R\} + \mu_1 \mathscr{L}\{S\} + \mu_2 \mathscr{L}\{C\} \qquad (26)
\end{aligned}$$

by using the initial conditions;

$$\mathscr{L}\{S\} = \frac{n_1}{S} + \frac{\nu}{S^{\alpha_1+1}} - \frac{\nu p_1}{S^{\alpha_1}}\mathscr{L}\{C\} - \frac{\nu p_2}{S^{\alpha_1}}\mathscr{L}\{R\} - \frac{\rho}{S^{\alpha_1}}\mathscr{L}\{IS\} - \frac{\rho\theta}{S^{\alpha_1}}\mathscr{L}\{CS\} - \frac{\nu}{S^{\alpha_1}}\mathscr{L}\{S\} - \frac{\mu_1}{S^{\alpha_1}}\mathscr{L}\{S\} + \frac{\lambda_4}{S^{\alpha_1}}\mathscr{L}\{R\}, \tag{27}$$

$$\mathscr{L}\{E\} = \frac{n_2}{S} + \frac{\rho}{S^{\alpha_2}}\mathscr{L}\{SI\} + \frac{\rho\theta}{S^{\alpha_2}}\mathscr{L}\{CS\} - \frac{\nu+\lambda_1}{S^{\alpha_2}}\mathscr{L}\{E\}, \tag{28}$$

$$\mathscr{L}\{I\} = \frac{n_3}{S} + \frac{\lambda_1}{S^{\alpha_3}}\mathscr{L}\{E\} - \frac{\nu+\lambda_2}{S^{\alpha_3}}\mathscr{L}\{I\}, \tag{29}$$

$$\mathscr{L}\{C\} = \frac{n_4}{S} + \frac{\nu p_1}{S^{\alpha_4}}\mathscr{L}\{C\} + \frac{p_3\lambda_2}{S^{\alpha_4}}\mathscr{L}\{I\} - \frac{\nu+\lambda_3}{S^{\alpha_4}}\mathscr{L}\{C\} - \frac{\mu_2}{S^{\alpha_4}}\mathscr{L}\{C\}, \tag{30}$$

$$\mathscr{L}\{R\} = \frac{n_5}{S} + \frac{\nu p_2}{S^{\alpha_5}}\mathscr{L}\{R\} + \frac{\lambda_2}{S^{\alpha_5}}\mathscr{L}\{I\} - \frac{p_3\lambda_2}{S^{\alpha_5}}\mathscr{L}\{I\} + \frac{\lambda_3}{S^{\alpha_5}}\mathscr{L}\{C\} - \frac{\nu}{S^{\alpha_5}}\mathscr{L}\{R\} - \frac{\lambda_4}{S^{\alpha_5}}\mathscr{L}\{R\} + \frac{\mu_1}{S^{\alpha_5}}\mathscr{L}\{S\} + \frac{\mu_2}{S^{\alpha_5}}\mathscr{L}\{C\}. \tag{31}$$

It should be assumed that method gives the solution as an infinite series

$$S = \sum_{k=1}^{\infty} S_k,\ E = \sum_{k=1}^{\infty} (E)_k,\ I = \sum_{k=1}^{\infty} (I)_k,\ C = \sum_{k=1}^{\infty} C_k,\ R = \sum_{k=1}^{\infty} R_k\,.$$

The nonlinearity SI and CS can be written as

$$SI = \sum_{k=1}^{\infty} A_k,\ CS = \sum_{k=1}^{\infty} B_k\,,$$

where A_k and B_k is called the Adomian polynomials given as

$$A_k = \frac{1}{k!}\frac{d^k}{\lambda^k}\left[\sum_{j=0}^{k}\lambda^j S_j \sum_{j=0}^{k}\lambda^j I_j\right]\Bigg|_{\lambda=0},\quad B_k = \frac{1}{k!}\frac{d^k}{\lambda^k}\left[\sum_{j=0}^{k}\lambda^j S_j \sum_{j=0}^{k}\lambda^j C_j\right]\Bigg|_{\lambda=0},$$

$$\mathscr{L}\{S_0\} = \frac{n_1}{S} + \frac{\nu}{S^{\alpha_1+1}},\ \mathscr{L}\{E_0\} = \frac{n_2}{S},\ \mathscr{L}\{(I_0\} = \frac{n_3}{S},\ \mathscr{L}\{C_0\} = \frac{n_4}{S},\ \mathscr{L}\{R_0\} = \frac{n_5}{S}\,.$$

Similarly, we have

$$\mathscr{L}\{S_1\} = -\frac{\nu p_1}{S^{\alpha_1}}\mathscr{L}\{C_0\} - \frac{\nu p_2}{S^{\alpha_1}}\mathscr{L}\{R_0\} - \frac{\rho}{S^{\alpha_1}}\mathscr{L}\{A_0\} - \frac{\rho\theta}{S^{\alpha_1}}\mathscr{L}\{B_0\} - \frac{\nu}{S^{\alpha_1}}\mathscr{L}\{S_0\} - \frac{\mu_1}{S^{\alpha_1}}\mathscr{L}\{S_0\} + \frac{\lambda_4}{S^{\alpha_1}}\mathscr{L}\{R_0\} \tag{32}$$

$$\vdots$$

$$\mathscr{L}\{S_k+1\} = -\frac{\nu p_1}{S^{\alpha_1}}\mathscr{L}\{C_{k-1}\} - \frac{\nu p_2}{S^{\alpha_1}}\mathscr{L}\{R_{k-1}\} - \frac{\rho}{S^{\alpha_1}}\mathscr{L}\{A_{k-1}\} - \frac{\rho\theta}{S^{\alpha_1}}\mathscr{L}\{B_{k-1}\} - \frac{\nu}{S^{\alpha_1}}\mathscr{L}\{S_{k-1}\} - \frac{\mu_1}{S^{\alpha_1}}\mathscr{L}\{S_{k-1}\} + \frac{\lambda_4}{S^{\alpha_1}}\mathscr{L}\{R_{k-1}\} \tag{33}$$

Now equations for Infected individuals (exposed):

$$\mathscr{L}\{E_1\} = \frac{\rho}{S^{\alpha_2}}\mathscr{L}\{A_0\} + \frac{\rho\theta}{S^{\alpha_2}}\mathscr{L}\{B_0\} - \frac{\nu+\lambda_1}{S^{\alpha_2}}\mathscr{L}\{E_0\} \tag{34}$$

$$\vdots$$

$$\mathscr{L}\{E_k+1\} = \frac{\rho}{S^{\alpha_2}}\mathscr{L}\{A_{k-1}\} + \frac{\rho\theta}{S^{\alpha_2}}\mathscr{L}\{B_{k-1}\} - \frac{\nu+\lambda_1}{S^{\alpha_2}}\mathscr{L}\{E_{k-1}\} \tag{35}$$

Now, equations for acute infection individuals:

$$\mathscr{L}\{I_1\} = \frac{\lambda_1}{S^{\alpha_3}}\mathscr{L}\{E_0\} - \frac{(\nu+\lambda_2)}{S^{\alpha_3}}\mathscr{L}\{I_0\} \tag{36}$$

$$\vdots$$

$$\mathscr{L}\{I_k+1\} = \frac{\lambda_1}{S^{\alpha_3}}\mathscr{L}\{E_{k-1}\} - \frac{(\nu+\lambda_2)}{S^{\alpha_3}}\mathscr{L}\{I_{k-1}\} \tag{37}$$

Now, equations for chronic hepatitis B virus carries;

$$\mathscr{L}\{C_1\} = \frac{\nu p_1}{S^{\alpha_4}}\mathscr{L}\{C_0\} + \frac{p_3\lambda_2}{S^{\alpha_4}}\mathscr{L}\{I_0\} - \frac{\nu+\lambda_3}{S^{\alpha_4}}\mathscr{L}\{C_0\} - \frac{\mu_2}{S^{\alpha_4}}\mathscr{L}\{C_0\} \tag{38}$$

$$\vdots$$

$$\mathscr{L}\{C_k+1\} = \frac{\nu p_1}{S^{\alpha_4}}\mathscr{L}\{C_{k-1}\} + \frac{p_3\lambda_2}{S^{\alpha_4}}\mathscr{L}\{I_{k-1}\} - \frac{\nu+\lambda_3}{S^{\alpha_4}}\mathscr{L}\{C_{k-1}\} - \frac{\mu_2}{S^{\alpha_4}}\mathscr{L}\{C_{k-1}\} \tag{39}$$

$$\begin{aligned}\mathscr{L}\{R_1\} = {} & \frac{\nu p_2}{S^{\alpha_5}}\mathscr{L}\{R_0\} + \frac{\lambda_2}{S^{\alpha_5}}\mathscr{L}\{I_0\} - \frac{p_3\lambda_2}{S^{\alpha_5}}\mathscr{L}\{I_0\} + \frac{\lambda_3}{S^{\alpha_5}}\mathscr{L}\{C_0\} - \frac{\nu}{S^{\alpha_5}}\mathscr{L}\{R_0\} \\ & - \frac{\lambda_4}{S^{\alpha_5}}\mathscr{L}\{R_0\} + \frac{\mu_1}{S^{\alpha_5}}\mathscr{L}\{S_0\} + \frac{\mu_2}{S^{\alpha_5}}\mathscr{L}\{C_0\}\end{aligned} \tag{40}$$

$$\vdots$$

$$\begin{aligned}\mathscr{L}\{R_k+1\} = {} & \frac{\nu p_2}{S^{\alpha_5}}\mathscr{L}\{R_{k-1}\} + \frac{(1-p_3)\lambda_2}{S^{\alpha_5}}\mathscr{L}\{I_{k-1}\} + \frac{\lambda_3}{S^{\alpha_5}}\mathscr{L}\{C_{k-1}\} \\ & - \frac{\nu}{S^{\alpha_5}}\mathscr{L}\{R_{k-1}\} - \frac{\lambda_4}{S^{\alpha_5}}\mathscr{L}\{R_{k-1}\} + \frac{\mu_1}{S^{\alpha_5}}\mathscr{L}\{S_{k-1}\} + \frac{\mu_2}{S^{\alpha_5}}\mathscr{L}\{C_{k-1}\}\end{aligned} \tag{41}$$

The purpose of the work is to analysis the mathematical behavior of the solution $S(t)$, $E(t)$, $I(t)$, $C(t)$, $R(t)$ for the different values of α. By applying the inverse laplace transform to both sides of the equation (42), we get the values of S_0, E_0, E_0, I_0, R_0 and used for further process. Putting the values of S_0, E_0, I_0, C_0, R_0 and A_0, B_0 into the equations (43)-(47) and get the values of S_1, E_1, I_1, C_1, R_1. Similarly, we find the remaining term $S_2, S_3, S_4, \ldots$, $E_2, E_3, E_4, \ldots$, $I_2, I_3, I_4, \ldots$, $C_2, C_3, C_4, \ldots$ and $R_2, R_3, R_4, \ldots$ in the same manners. We have the series solution in the form

$$S(t) = S_0 + S_1 + S_2 + S_3 + S_4 + \ldots \tag{42}$$

$$E(t) = E_0 + E_1 + E_2 + E_3 + E_4 + \ldots \tag{43}$$

$$I(t) = I_0 + I_1 + I_2 + I_3 + I_4 + \ldots \tag{44}$$

$$C(t) = C_0 + C_1 + C_2 + C_3 + C_4 + \ldots \tag{45}$$

$$R(t) = R_0 + R_1 + R_2 + R_3 + R_4 + \ldots . \tag{46}$$

The Laplace Adomian decomposition method an analytical approximate solution in term of infinite power series. For numerical results, we used Table 1 values of parameters.

$$\begin{aligned}S(t) = 0.1 & + \frac{0.0070672t^{\alpha}}{\Gamma(\alpha_1+1)} - \frac{0.005060242t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} - \frac{0.000401241t^{3\alpha_1}}{\Gamma(3\alpha_1+1)} - \frac{0.000060253t^{\alpha_1+\alpha_4}}{\Gamma(\alpha_1+\alpha_4+1)} \\ & - \frac{0.000530977t^{\alpha_1+\alpha_5}}{\Gamma(\alpha_1+\alpha_5+1)} - \frac{0.0099758t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} - \frac{0.0012070718(\alpha_1+\alpha_3)!t^{2\alpha_1+\alpha_3}}{\Gamma(\alpha_1+1)\Gamma(\alpha_3+1)\Gamma(2\alpha_1+\alpha_3+1)}\end{aligned}$$

$$- \frac{0.000273880(\alpha_1+\alpha_4)!t^{2\alpha_1+\alpha_4}}{\Gamma(\alpha_1+1)\Gamma(\alpha_4+1)\Gamma(2\alpha_1+\alpha_4+1)} + \ldots \tag{47}$$

$$E(t) = 0.03 - \frac{0.163363t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{0.004074729t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{0.0099758t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + \frac{0.001207018(\alpha_1+\alpha_3)!t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+1)\Gamma(\alpha_3+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - \frac{0.000374580t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)} + \frac{0.000273880(\alpha_1+\alpha_4)!t^{\alpha_1+\alpha_2+\alpha_4}}{\Gamma(\alpha_1+1)\Gamma(\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_4+1)} + \frac{0.982154692t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \frac{0.012366890t^{2\alpha_2+\alpha_1}}{\Gamma(2\alpha_2+\alpha_1+1)} + \ldots$$

$$I(t) = 0.02 + \frac{0.099758t^{\alpha_3}}{\Gamma(\alpha_3+1)} - \frac{0.980178t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + \frac{0.012342^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - \frac{0.400239072t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} + \ldots \tag{48}$$

$$C(t) = 0.3 + \frac{0.0452693t^{\alpha_4}}{\Gamma(\alpha_4+1)} + \frac{0.2793224t^{\alpha_4+\alpha_3}}{\Gamma(\alpha_4+\alpha_3+1)} - \frac{0.001619238t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} + \ldots \tag{49}$$

$$R(t) = 0.35 + \frac{0.0136885t^{\alpha_5}}{\Gamma(\alpha_5+1)} - \frac{0.000696607t^{2\alpha_5}}{\Gamma(2\alpha_5+1)} + \frac{0.1197096^{\alpha_5+\alpha_3}}{\Gamma(\alpha_5+\alpha_3+1)} + \frac{0.001131733t^{\alpha_5+\alpha_4}}{\Gamma(\alpha_5+\alpha_4+1)} + \ldots . \tag{50}$$

Table 1. Values of physical parameters used in model when $R_0 < 1$

Parameter	Value	Parameter	Value
ν	0.0121	θ	0.5
ρ	1	λ_1	6
λ_2	4	λ_3	0.025
λ_4	0.04	p_1	0.11
p_2	0.1	p_3	0.7
μ_1	0	μ_2	0
n_1	0.1	n_2	0.03
n_3	0.02	n_4	0.3
n_5	0.35		

6. Numerical Results and Discussion

The mathematical analysis of SEICR epidemic model with nonlinear system of differential equation has been presented. Firstly, we investigate the diseases free equilibrium point of fractional order model. The numerical results of susceptible, Exposed, Acute infected, Chronic infected and recovered population for α_i, where $i = 1,2,3,4,5$ are established in Tables 2-6 by using LADM. For the reliable investigation, evaluation is made for different values of α. From Figures 1-5, we observe that fractional order SEICR epidemic model has more degree of freedom as compared to ordinary derivatives. By taking non-integer values of fractional parameter,

remarkable responses of the compartments of the proposed model are obtained. Solution gives better converges to steady state for Susceptible and Exposed by increasing the fractional values of α, while other compartments converges fastly by decreasing the fractional value of α. The Laplace Adomian decomposition method is an analytical approximate solution in term of infinite power series.

Table 2. Susceptible population $S(t)$ at different values of α_i where $i = 1,2,3,4,5$

t	$\alpha_i = 1$	$\alpha_i = 0.95$	$\alpha_i = 0.9$	$\alpha_i = 0.85$
0	0.1	0.1	0.1	0.1
0.2	0.101118	0.101182	0.101237	0.101278
0.4	0.101628	0.10157	0.101486	0.101362
0.6	0.101497	0.101254	0.100978	0.100656
0.8	0.100706	0.100243	0.0997668	0.099253
1	0.099224	0.0985335	0.097874	0.097202

Table 3. Exposed population $E(t)$ at different values of α_i where $i = 1,2,3,4,5$

t	$\alpha_i = 1$	$\alpha_i = 0.95$	$\alpha_i = 0.9$	$\alpha_i = 0.85$
0	0.03	0.03	0.03	0.03
0.2	0.0172565	0.0192523	0.0229032	0.0278307
0.4	0.0443189	0.0540838	0.0697246	0.0865029
0.6	0.111109	0.128443	0.15955	0.188482
0.8	0.217549	0.239356	0.28833	0.327824
1	0.36356	0.384623	0.453557	0.500997

Table 4. Acute infected population $I(t)$ at different values of α_i where $i = 1,2,3,4,5$

t	$\alpha_i = 1$	$\alpha_i = 0.95$	$\alpha_i = 0.9$	$\alpha_i = 0.85$
0	0.02	0.02	0.02	0.02
0.03	0.0225584	0.0228434	0.023051	0.0231031
0.06	0.0235835	0.0233707	0.027777	0.0216111
0.09	0.0230758	0.0219162	0.020014	0.0170808
0.12	0.0210356	0.0185934	0.015027	0.00997227
0.15	0.017463	0.0134715	0.00797374	0.000548022

Table 5. Chronic population $C(t)$ at different values of α_i where $i = 1,2,3,4,5$

t	$\alpha_i = 1$	$\alpha_i = 0.95$	$\alpha_i = 0.9$	$\alpha_i = 0.85$
0	0.3	0.3	0.3	0.3
0.2	0.314608	0.317154	0.320199	0.323843
0.4	0.340324	0.345994	0.352468	0.359836
0.6	0.377148	0.386013	0.395768	0.406451
0.8	0.42508	0.436829	0.449356	0.462627
1	0.484121	0.498169	0.512714	0.527653

Table 6. Recovered population $R(t)$ at different values of α_i where $i = 1,2,3,4,5$

t	$\alpha_i = 1$	$\alpha_i = 0.95$	$\alpha_i = 0.9$	$\alpha_i = 0.85$
0	0.35	0.35	0.35	0.35
0.2	0.355141	0.356117	0.357299	0.358728
0.4	0.365087	0.367379	0.370012	0.373025
0.6	0.379839	0.383508	0.387561	0.392015
0.8	0.399397	0.404329	0.409602	0.4152
1	0.423761	0.429717	0.435897	0.442255

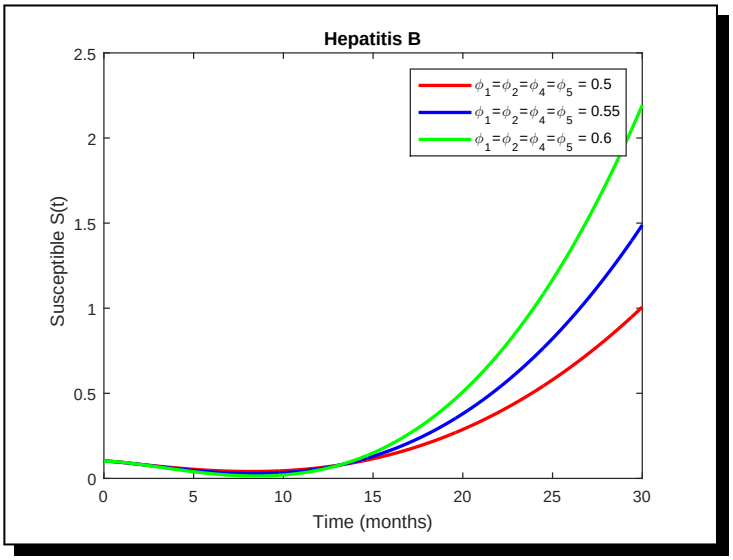

Figure 1. Numerical solutions for Susceptible $S(t)$ population in a time t (months) at different values of α

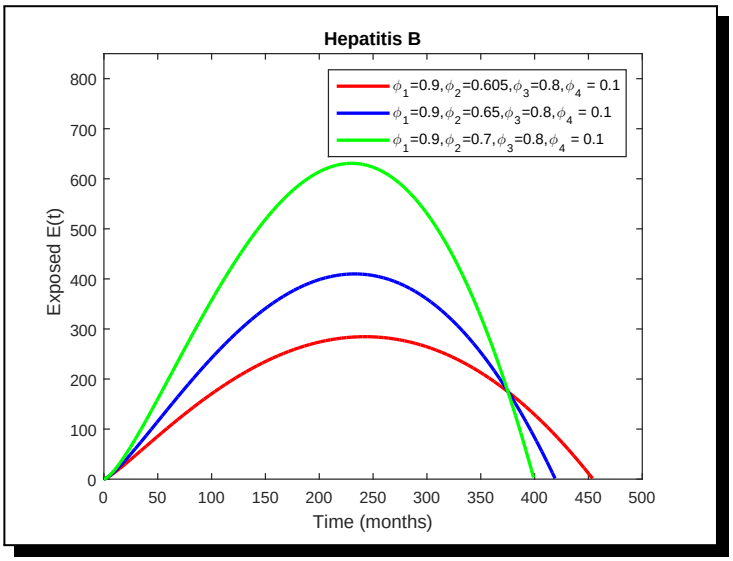

Figure 2. Numerical solutions for Exposed $S(t)$ population in a time t (months) at different values of α

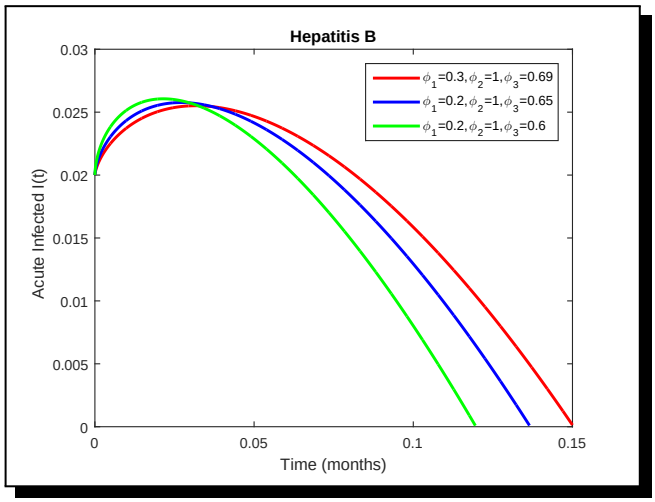

Figure 3. Numerical solutions for Acute infection $I(t)$ population in a time t (months) at different values of α

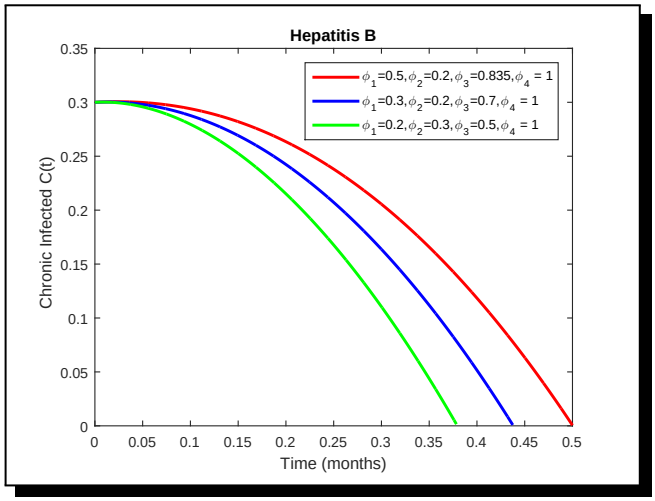

Figure 4. Numerical solutions for Chronic infection $C(t)$ population in a time t (months) at different values of α

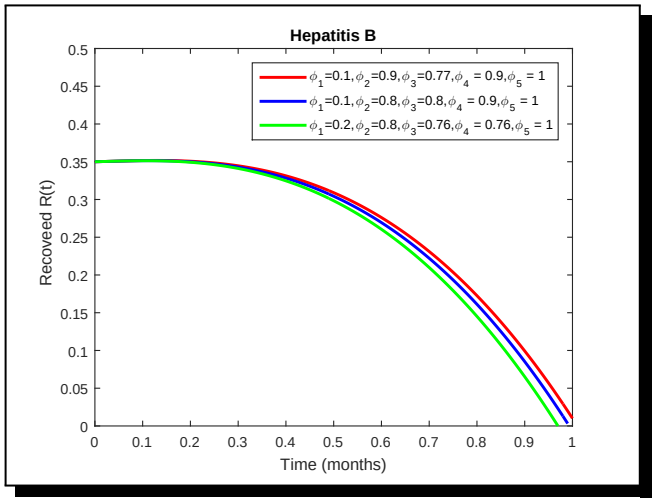

Figure 5. Numerical solutions for Recovered $R(t)$ population in a time t (months) at different values of α

7. Conclusion

We developed a scheme for analytical solution of epidemic fractional SEICR model by using Laplace Adomian decomposition method. The well known epidemic model namely *Susceptible, Exposed Population, Infected population, Chronic population and Recovered* (SEICR) is considered with and without demographic effects. The model represents population dynamic during the disease as a set of non-linear coupled ordinary differential equation. It is observed that the infection rate and reproductive number play a key role for an epidemic to occur and the epidemic can be controlled by vaccination. It is also observed that to eliminate the disease, It is not necessary to vaccinate whole of the population. The effect of fractional parameter on our obtained solutions is presented through Tables and Graphs. It is worthy to observe that fractional derivatives show significant changes and memory effect as compared to ordinary derivatives.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Ahmad, M. Farman, F. Yasin and M. O. Ahmad, Dynamical transmission and effect of smoking in society, *Int. J. Adv. Appl. Sci.* **5**(2) (2018), 71 – 75, DOI: 10.21833/ijaas.2018.02.012.

[2] A. Ahmad, M. Farman, F. Yasin and M. O. Ahmad, Dynamical transmission and effect of smoking in society, *Int. J. Adv. Appl. Sci.* **5**(2) (2018), 71 – 75, DOI: 10.21833/ijaas.2018.02.012.

[3] A. Ahmad, M. Farman, M. O Ahmad, N. Raza and Abdullah, Dynamical behavior of SIR epidemic model with non-integer time fractional derivatives: A mathematical analysis, *Int. J. Adv. Appl. Sci.* **5**(1) (2018), 123 – 129, DOI: 10.21833/ijaas.2018.01.016.

[4] F. Ashraf and M.O. Ahmad, Nonstandard finite difference scheme for control of measles epidemiology, *Int. J. Adv. Appl. Sci.* **6**(3) (2019), 79 – 85, DOI: 10.21833/ijaas.2019.03.012.

[5] F. Ashraf, A. Ahmad, M. U. Saleem, M. Farman and M. O. Ahmad, Dynamical behavior of HIV immunology model with non-integer time fractional derivatives, *Int. J. Adv. Appl. Sci.* **5**(3) (2018), 39 – 45, DOI: 10.21833/ijaas.2018.03.006.

[6] B. Buonomo and D. Lacitignola, On the dynamics of an SEIR epidemic model with a convex incidence rate, *Ricerche Mat.* **57** (2008), 261 – 281, DOI: 10.1007/s11587-008-0039-4.

[7] P. Elango, The role of mathematics in biology, *5th International Symposium 2015 – IntSym 2015*, Department of Mathematical Sciences, Faculty of Applied Sciences, South Eastern University of Sri Lanka (2015), `http://ir.lib.seu.ac.lk/handle/123456789/1299`.

[8] F. Haq, K. Shah, G. Rahman and M. Shahzad, Numerical solution of fractional order smoking model via Laplace Adomian decomposition method, *Alex. Eng. J.* 1110-0168.(2017), DOI: 10.1016/j.aej.2017.02.015.

[9] H. Jafari, C. M. Khalique and M. Nazari, Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion wave equations, *Applied Mathematics Letters* **24**(11) (2011), 1799 – 1805, DOI: 10.1016/j.aml.2011.04.037.

[10] A. V. Kamyad, R. Akbari, A. A. Heydari and A. Heydari, Mathematical modeling of transmission dynamics and optimal control of vaccination and treatment for hepatitis B virus, *Computational and Mathematical Methods in Medicine* **2014** (2014), Article ID 475451, 15 pages, DOI: 10.1155/2014/475451.

[11] World Health Organization (WHO), *Prevention and Control of Viral Hepatitis Infection, Framework for Global Action* (2012), `http://www.who.int/iris/handle/10665/130012`.

[12] T. Zhu, B. T. Korber, A. J. Nahmias, E. Hooper, P. M. Sharp and D. D. Ho, An African HIV-1 sequence and implications for the origin of the epidemic, *Nature* **391** (1998), 594 – 597, DOI: 10.1038/35400.