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Research Article

A New Combination of Lagrangean Relaxation, Dantzig-Wolfe Decomposition and Benders Decomposition Methods for Exact Solution of the Mixed Integer Programming Problems

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Abstract. The combinational technique of cross decomposition is a suitable one for exact solution of the mixed integer programming problems which uses simultaneously the advantages of Lagrangean relaxation, Dantzig-Wolfe decomposition and Benders decomposition methods for Minimization problem that each reinforces one another. The basic idea for this technique is the generation of suitable upper and lower bounds for the optimal value of the original problem at each iteration. In this paper, new cross decomposition algorithm, with the combination of Lagrangean relaxation method (the combination of three concepts of cutting-plane, sub-gradient and trust region), Dantzig-Wolfe decomposition and Benders decomposition methods are used in order to reinforce bounds and to speed up convergence. By increasing the problem scale and regarding the use of the Lagrangean relaxation method in this technique, the lower bound with more strength and efficacy, and by the aid of Dantzig-Wolfe decomposition method more suitable upper bound (if exists) and furthermore less number of iterations for achieving optimal solution is obtained. The convergence of this technique regarding the convergence of Benders decomposition method in finite iteration numbers is guaranteed.

Keywords. Cross decomposition; Benders decomposition; Lagrangean relaxation; Dantzig Wolfe decomposition; Cutting planes; Sub-gradient; Trust region; Column generation

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1. Introduction

The problems of mixed-integer programming are the most applied ones for optimization in the real world. Some of these large scale problems are special restriction matrix structures that we can use these structures for ineffective way solution. The decomposition methods are strong tools for solving them. A problem consists of independent or close to independent sub-problems a decomposition method can be benefited from. The idea of decomposition method is based on the set of variables is partitioned into two subsets of sub-problem and master problem. The solution of a sub-problem is often easier than the solution of a master problem. The decomposition method reiterates between master problem and sub-problem, and to solve the original problem for obtained optimal solution. Often these problems can be viewed as a combination of several sub-problems which have several common rows or columns. Therefore, restriction matrix structures of these problems consisting of several blocks are related. Among the important decomposition methods we can refer to the decomposition methods of Dantzig-Wolfe, Benders and Lagrange. The important problems that have been solved by this method include: facility location problems, cutting stock problem, airlines crew scheduling problem, vehicle routing problem etc. The cross decomposition method exploiting simultaneously both the primal and the dual structure of the problem, thus combining the advantages of Dantzig-Wolfe decomposition, Benders decomposition and Lagrangean relaxation. Finite convergence of the algorithm equipped with some simple convergence tests has been proved. Stronger convergence tests have been proposed, but not shown to yield finite convergence.

The cross decomposition algorithm, first proposed by Van Roy [24] along with the convergence testing for avoidance of solving master problems (the easier solving with the aid of sub-problems) that compared with the available methods produced a reduction in approximation of 20 percent in the time of problem solving. Holmberg [10] investigated all kinds of this method convergence testing and proposed the average value of the cross decomposition method for the use of the master problem via using iteration of the previous solutions. Regarding the advancement of MIP solvers, more expedient computers, and the parallel problem solution, Mitra [14] proposed the use of solving the master problem and the omission of convergence test. Ogbe [4] used Dantzig-Wolfe method instead of Lagrangean relaxation method along with Benders decomposition method for convergence expedition and upper bound better with feasible guarantee.

To the best of our knowledge, there is no work that this new cross decomposition method for mixed integer programming problems. We briefly explain each of the exact methods of Benders, Lagrange and Danzig-Wolfefor solving the mixed integer programming problems in Section 2, 3 and 4 then we submit the proposed algorithm for the new cross decomposition method in Section 5. Section6-9 addresses the implementation of facility location problem in terms of case studies and various exact solutions. Section 10 presents the conclusion of the whole review paper.

2. Benders Decomposition Method

Benders decomposition method [1] is one of the most important and effective methods for the solving of algorithm mixed integer programming problems and has been applied in various fields such as network, transportation, supply chain, scheduling etc. In this method by fixing the complicated variables, the primary problem partitioned into a sub-problem and a master problem which in the minimization problem,it creates an upper bound by sub-problem and a lower bound by master problem for obtaining optimal solution. In Benders iteration, dual sub-problem objective function, by means of the master problem optimal solution from the previous iteration is updated and in any iterations the feasibility and optimality cuts are added to the master problem. These cuts from extreme points (extreme rays) give the space of dual sub-problem solution. The main idea of Benders decomposition method is on the basis that prior to use all the cuts, the conditions of optimality and feasibility and convergence to be established.

This method is confronted with a computational bottleneck. In any Benders algorithm iteration, the master problem must be once solved, but this problem is a mixed integer programming whose solving in any iteration causes rising the required time for the solving problem. Moreover, the effective method for the sensitivity analysis model, after adding new cuts to the master problem is not in access. Regarding this bottleneck, the most investigations in the field of the improvement of this method have been focused on reducing this bottleneck and on accelerating algorithms. In fact the studies carried out in this respect have been divided into two main parts:submission of a more suitable and effective algorithm for solving the master problem and sub-problem, submission of suitable strategies for reducing the required numbers of iterations in Benders decomposition algorithm

2.1 Classic Benders decomposition method

Supposing that we have the following mixed integer linear programming problem:

$$\text{IP}: \quad \text{Min}\, c^T x + f^T y \tag{1}$$

$$\text{s.t. } Ax + By = b \tag{2}$$

$$Dy = d \tag{3}$$

$$x \geq 0 \tag{4}$$

$$y \geq 0, \ \text{ integer} \tag{5}$$

where $c \in R^{m_1}$, $f \in R^{m_2}$, $b \in R^{n_1}$ and $d \in R^{n_2}$ and also A, B and D are matrices $n_1 \times m_1$, $n_1 \times m_2$ and $n_2 \times m_2$, respectively. The variables of $x \in R^{m_1}$ are continuous and the integer and complicating variables of $y \in R^{m_2}$, are the objective function of minimizing the total costs. If y is fixed to a feasible integer configuration $(\overline{y})$, we can rewrite this model as the following:

$$\text{PSP}: \quad \underset{\overline{y} \in Y}{\text{Min}} \{ f^T \overline{y} + \underset{x \geq 0}{\text{Min}} \{ c^T x : Ax = b - B\overline{y} \} \} \tag{6}$$

where therein is:

$$Y = \{ y \mid Dy = d, y \geq 0, \text{integer} \}$$

Let $\hat{x}^k$ be the optimal solution of PSP^k. The inner minimization is a continuous linear problem that can be dualized by dual variables u as the following:

$$\text{DSP}: \underset{u \in R^{n_1}}{\text{Max}} \{ u^T (b - B\overline{y}) : u^T A \leq c \}. \tag{7}$$

Let u^k be the optimal solution of DSP^k. In the Benders decomposition algorithm, we consider the relationship (7) as Benders sub-problem. With regard to the duality theorem we can write the relation (6) as the following:

$$\underset{\overline{y} \in Y}{\text{Min}} \{ f^T \overline{y} + \underset{u \in R^{n_1}}{\text{Max}} \{ u^T (b - B\overline{y}) : u^T A \leq c \} \}. \tag{8}$$

The feasible space of the inner maximizing problem depends on selecting $\overline{y}$ which if it is non empty for any selection of $\overline{y}$, the problem of inner maximizing is feasible or unbounded. Therefore by adding optimality cuts (11) and feasibility cuts (12), the master problem of Benders decomposition is as the following ($u_i^T s$ is the vector that corresponds to the extreme points I of the dual of PSP and v^{T_j}s is the vector that corresponds to the extreme rays J of the dual of PSP):

$$\text{BMP}: \quad \text{Min}\, f^T y + z \tag{9}$$

$$\text{s.t.}\ \ Dy = d \tag{10}$$

$$u_i^T(b - By) \le z, \qquad \forall\ \ i \in I \tag{11}$$

$$v_j^T(b - By) \le 0, \quad \forall\ \ j \in J \tag{12}$$

$$y \ge 0, \quad \text{integer} \tag{13}$$

Let y^k be the optimal solution of BMP^k. In Benders decomposition method we consider the above model as the Benders master problem. Benders decomposition algorithm due to finiteness of the extreme points (extreme rays) of dual sub-problem, in the numbers of finite iteration becomes convergence. However it usually has slow convergence.

2.2 Accelerate solving the master problem and sub-problem

Inaccurate solution of master problem: Geoffrion and Graves [8] proposed that instead of finding the optimal solution of the master problem, we run several first iterations with the master problem feasible solutions(not necessarily optimal). For this reason, heuristic algorithms can be used for finding the master problem feasible solutions [16].

Inexact solution of sub-problem: it might sometimes happen due to the largeness of sub-problem, obtaining one optimal extreme point in any iterations in order to generate cut is time consuming. For this reason Zakeri *et al*. [25] have proposed the use of inexact cuts. In this method sub-problems can be solved by the interior point algorithm.

2.3 Reducing numbers of Benders decomposition iterations

Selection of more effective cuts: Magnanti and Wong [12] regarded the quality of the generated cuts, and introduced a concept entitled "Pareto optimal cuts" and afterwards the suitable strategies for simplification of generation cut process were submitted [15]. Their idea was that in the problems such as network which the sub-problem has several multiple optimal solutions that solution to be used which its corresponding cut is stronger than others and is so-called Pareto optimal. Fischetti *et al*. [5] submitted one criterion for selection of the effective feasibility cut among the possible cuts.

The addition of the primary reliable cuts to the master problem:one of the other ideas of accelerating the trend of Benders decomposition is the addition of a series of primary cuts to the master problem before beginning the common iteration of Benders decomposition. By doing this the master problem feasible region at the beginning is more restrictive and so better primary bounds are obtained for the problem. Therefore the less numbers of iteration for reaching out to the optimal solution is required. McDaniel and Devin [11] suggested that before to beginning of the iterations of Benders decomposition we do several iterations with the manner of the master problem continue and in this case to generate several numbers of primary cut and in the next iterations we consider (discrete) manner of the master problem.

The generation of multiple cuts: Saharidis *et al*. [21] generated several cuts in one iteration via using an auxiliary problem, so that the collection of these cuts covers the master problem maximum variables. Afterwards Saharidis *et al*. [19] suggested the cuts with maximum density. Furthermore, they suggested, for the problems that feasibility cuts are considerably being generated more than the optimality cuts, an extra cut in order to restrict the value of the master problem [20].

Searching for better bounds: Rei *et al*. [18] used a local searching in the Benders decomposition algorithm for finding better upper and lowerbounds in any iteration.

3. Lagrangean Relaxation Method

One of the suitable methods for solving the problems consisting of complicated constraints is the Lagrangean relaxation method. The complicated constraints are those which by omitting them from the problem set, the problem turns to have a suitable and specific structure which can be utilized. The main idea of this method includes omitting the complicated constrains and adding them to the problem objective function by using variables by the name of Lagrangean multipliers. By doing that and using duality relations, we reach to a new problem which for solving it an iterative algorithm can be used. In any iteration, a new value is considered for Langrage multipliers vector and then a suitable sub-problem which consists of only good constraints is solved. In this method find a suitable lower bound (for the problem of minimizing) becomes depending on finding the best Lagrangean multipliers (the problem of dual Lagrange). But this method does not have guaranteed convergence.

Regarding IP problem by omitting complicated constraint (3) and adding it to the objective function, Lagrangean sub-problem is written as the following:

$$\text{LSP}: \quad L_\lambda = \text{Min}\, c^T x + f^T y + \lambda(Dy - d) \tag{14}$$

$$\text{s.t.} \quad Ax + By = b \tag{15}$$

$$x \geq 0 \tag{16}$$

$$y \geq 0, \quad \text{integer} \tag{17}$$

where λ (Lagrangean multiplier) is a real, positive and adequately large enough value. Let $(\tilde{x}^k, \tilde{y}^k)$ be the optimal solution of LSP^k. We call the above model Lagrangean sub-problem. In some of problems by considering the problem structure this sub-problem can be decomposed to segregate sub-problems (the method of Lagrangean relaxation).

The best lower bound (minimizing problem)which can be obtained through Lagrangean relaxation regarding the complicated constraints $Dy = d$ is the objective function optimal value of the following model called dual Lagrangean problem:

$$L_d = \underset{\lambda}{\text{Max}}\, L_\lambda \tag{18}$$

Therefore, we will have $v(L_D) - v(IP) \leq 0$ (one lower bound for the original problem and the meaning of the sign v is the optimal value).

There are different methods for solving the following problem that one classic method is sub-gradient method which has been proposed by Held [9]. In this method Lagrangean multipliers are usually updated as well. But for the convergence we will need a suitable strategy for defining and updating the sub-gradient step size. Another method, which from the theory standpoint has better convergence properties, is the method of cutting plane which is submitted by Cheney [2]. An alternative procedure for updating the Lagrangean multiplier is based on the use of cutting plans to approximate the Lagrangean dual function. This method for convergence needs many iterations. Therefore, to gain suitable Lagrangean multipliers is a lot time consuming. For removing this problem the trust region method which is proposed by Marsten [13] is used. In this method, better updating of Lagrangean multipliers can be expected while the convergence properties are established.

For updating Lagrangean multipliers we use a combinational method which simultaneously applies three concepts of sub-gradient, cutting planes and trust region [22]. The cutting planes are reliable constraints for dual problem generated in any iteration, sub-gradient prepares a reducing direction for dual problem, while the trust region delineates the deviation value from this direction in its range. The combination of these methods while updating the Lagrangean multipliers ensure convergence.

In $k+1$ iterationby using of the following combinational dual problem, we update the Lagrangean multipliers:

$$\text{LMP}: \quad L_d^{k+1} = \text{Max}\,\eta + \frac{\delta}{2}\|\lambda - \overline{\lambda}\|_2^2 \tag{19}$$

$$\text{s.t. } \eta \le c^T \tilde{x}^k + f^T \tilde{y}^k + \lambda^k (D\tilde{y}^k - d) \tag{20}$$

$$\lambda^{k+1} = \lambda^k + \beta \frac{v(L_D^k) - v(L_\lambda^k)}{\|D\tilde{y}^k - d\|^2}(D\tilde{y}^k - d) \tag{21}$$

$$\forall\ k = 1, \cdots, k,\ \eta \in R,\ \alpha \in R,\ \lambda \in R, \tag{22}$$

$$\beta \in (-\infty, \overline{\beta}],\ \delta \in [-\overline{\delta}, \overline{\delta}],\ \ \overline{\lambda} > 0$$

where, therein $v(L_D)$ is estimated via a heuristic methods. However, instead of heuristically updating the step size, it is optimized using variable β, which is bounded by the parameter $\overline{\beta} > 0$. Note that δ variable is the deviation value form the steps of sub-gradient and the parameter $\overline{\delta} > 0$ is the maximum deviation in directions of trust region. The parameters of $\overline{\beta}$ and $\overline{\delta}$, $\overline{\lambda}$ in any iteration are updated in heuristic methods. In practice, computational testing shows that the use of fixed values is suitable selections.

Stoppage criterion, for this combinational strategy, is restricted based on Lagrange gap between the relaxed original problem and dual problem:

$$v(L_D^k) - v(L_\lambda^k) \le \varepsilon.$$

It is necessary to mention that the methods of cutting planes and trust region have features of finite convergence.

4. Dantzig-Wolfe Decomposition Method

This method of decomposition was at first introduced by Dantzig and Wolfe [7]. In this method restriction matrix structures is often in the form of angular block. In this structure, the blocks are laid in the extension of matrix diameter and they are related to each in a few rows. Let the Y set is a multidimensional bounded and integrate in the form of $Y = \{y : Dy = d,\ y \ge 0, \text{integer}\}$ for the problem (1)-(5). Regarding the representation theorem, each point from the feasible space of linear programming problem can be written in the form of convex combination from the extreme points of feasible space in addition of positive combination from the extreme rays.

But if the feasible space is bounded, there are no extreme rays. Assuming E_Y is set of extreme points of bounded polyhedron Y, therefore we have:

$$Y = \left\{ \sum_{j\in E_Y} \mu_j y_j : \sum_{j\in E_Y} \mu_j = 1, \mu_j \in \{0,1\}, \ \forall\ j \in E_Y \right\} \tag{23}$$

We write Dantzig-Wolfe master problem in the following form:

$$\text{DWMP}: \quad \min \sum_{j\in E_Y} (f^T y_j)\mu_j + c^T x \tag{24}$$

$$\text{s.t.} \sum_{j\in E_Y} (B y_j)\mu_j + Ax = b \tag{25}$$

$$\sum_{j\in E_Y} \mu_j = 1, \quad \mu_j \in \{0,1\},\ j \in E_Y \tag{26}$$

$$x \geq 0 \tag{27}$$

Therefore, we have:

$$z^{DWMP} = \min \left\{ \sum_{j\in Q} (f^T y_j)\mu_j + c^T x : \sum_{j\in Q} (B y_j)\mu_j + Ax = b, \sum_{j\in Q} \mu_{j=1},\ \mu_j \in \{0,1\},\ \forall\ j \in Q \right\} \tag{28}$$

where Q is the enumerated set of integrate solutions to E_Y. The value of μ_j is 1 if integer solution y_j is chosen and zero otherwise.

In *Dantzig-Wolfe Method, We Solve the Restricted Master Problem* (DWRMP) for a bounded number of extreme points in any iteration. Then, we solve the problem of relaxation linear programming in column generation method (being binary is waived). One of the important features of this method is that in determining the optimal solution there is no need to have all variables but we consider a small sub-set of columns and put zero value for the rest variables. Then the next column (variable) by solving one optimization special sub-problem that is called the *Primal Problem of Dantzig-Wolfe* (DWPP) is added to the original problem:

$$\text{DWPP}: \quad \min f^T y - \bar{v}^T B y - \bar{z} \tag{29}$$

$$\text{s.t. } y \in Y \tag{30}$$

where $\bar{v}^T$ and $\bar{z}$ are dual solutions for the restricted master linear programming problem. It is necessary to mention that the value of the objective function of DWPP is the reduced cost of the Dantzig-Wolfe restricted master problem. If this value is non-negative then no extreme point in Y gives the results of negative reduced cost for the basic feasible solutions to the master problem; therefore, the solution of DWRMP is the optimal solution. Otherwise an entry variable of μ_{j+1} related to the primary problem solution namely y_{j+1} is added for generation of new column to the master problem. Dantzig-Wolfe method in general manner for MIP does not become convergence to the optimal solution, however if the solution μ of DWMP is an integer solution, then z^{DWMP} is a suitable upper bound for MIP.

5. New Cross Decomposition Method

The combination of Benders decomposition and Dantzig-Wolfe decomposition and Lagrangean relaxation methods leads to the new cross decomposition method in which sub-problems of each of the methods are solved in parallel and these sub-problems reinforce the master problems together in the form of ping-pong and admits suitable upper and lower bound for the optimal solution to the original problem in any iterations.

Algorithm

1. initialization

$k=0$, $K=\{0\}$, $y^{k=0}=0$, $x^{k=0}=0$, $\lambda^{k=0}=0$, $\mu^{k=0}=0$, $LB=-\infty$, $UB=\infty$, set $\varepsilon\geq 0$.

2. Benders sub-problem

For given y^k solve (BSP^k)

Store solution u^k, if $v(BSP^k)<UB$ set $UB=v\left(BSP^k\right)$ and go to step 3.

3. Dantzig-Wolfe sub-problem

For given x^k solve $(DWPP^k)$ in parallel

Store solutiony^k and go to step 4.

4. Dantzig-Wolfe master problem

For given y^k and $v(BSP^k)$ solve $DWRMP^{k+1}$

Store solution for x^{k+1} and set μ^{k+1} and update $\bar{z}$ and $\bar{v}$.

If $v\ (DWRMP^{k+1})<UB$ set $v\ (DWRMP^{k+1})=UB$ and go to step 5

Otherwise go to step 3.

5. Lagrangean sub-problem

For given λ^k solve (LSP^k) in parallel

Store $\tilde{x}^k$, $\tilde{y}^k$ and $v(LSP^k)$ and go to step 6.

6. Lagrangeanmaster problem

For given $\tilde{x}^k$ and $\tilde{y}^k$ solve (LMP^{k+1}).

Store solution for Lagrangean multipliers and set λ^{k+1} and update β, δ and $\overline{\lambda}$ and go to step 7.

7. Benders master problem

For given u^k, $v\left(LSP^k\right)$ and $k\in K$ solve (BMP^{k+1}).

Store solution for y^{k+1}.

If $LB<v\left(BMP^{k+1}\right)$, set $LB=v\left(BMP^{k+1}\right)$ and go to step 8 otherwise go to step 5.

8. Check convergence (optimality)

If $UB-LB\leq\varepsilon$ stop.

Else set $k=k+1$ and include in K; go back to step 2.

Illustrative Case Study

Keeping in mind the end goal to give the proposed model, cross decomposition method has been applied under the risk of disruptions to a facility location problem with *distribution centers* (DCs). The *Capacitated Reliable Facility Location Problem* (CRFLP) design has been figured as a two-stage stochastic programming issue for incorporating the disturbances risk at distribution centers. The selection of distribution centers along with the capacity from the set of candidate locations is the first-stage decision. The demanded assignments to distribution centers in scenarios are the second-stage decisions. The model permits finding the plan choices that minimize the sum of investment cost and expected transportation costs over a finite time horizon by suspecting the dissemination methodology in the scenarios with disturbances. Snyder and Daskin [23] indicated the (CRFLP) formulation and for including the capacity design of facilities Garcia-Herroros *et al*. [6] adapted it. The (CRFLP) and (CRFLP-t) are the two versions of the model. The first one has a poor linear relaxation. For improving the linear relaxation, the second one involves a redundant collection of constraints.

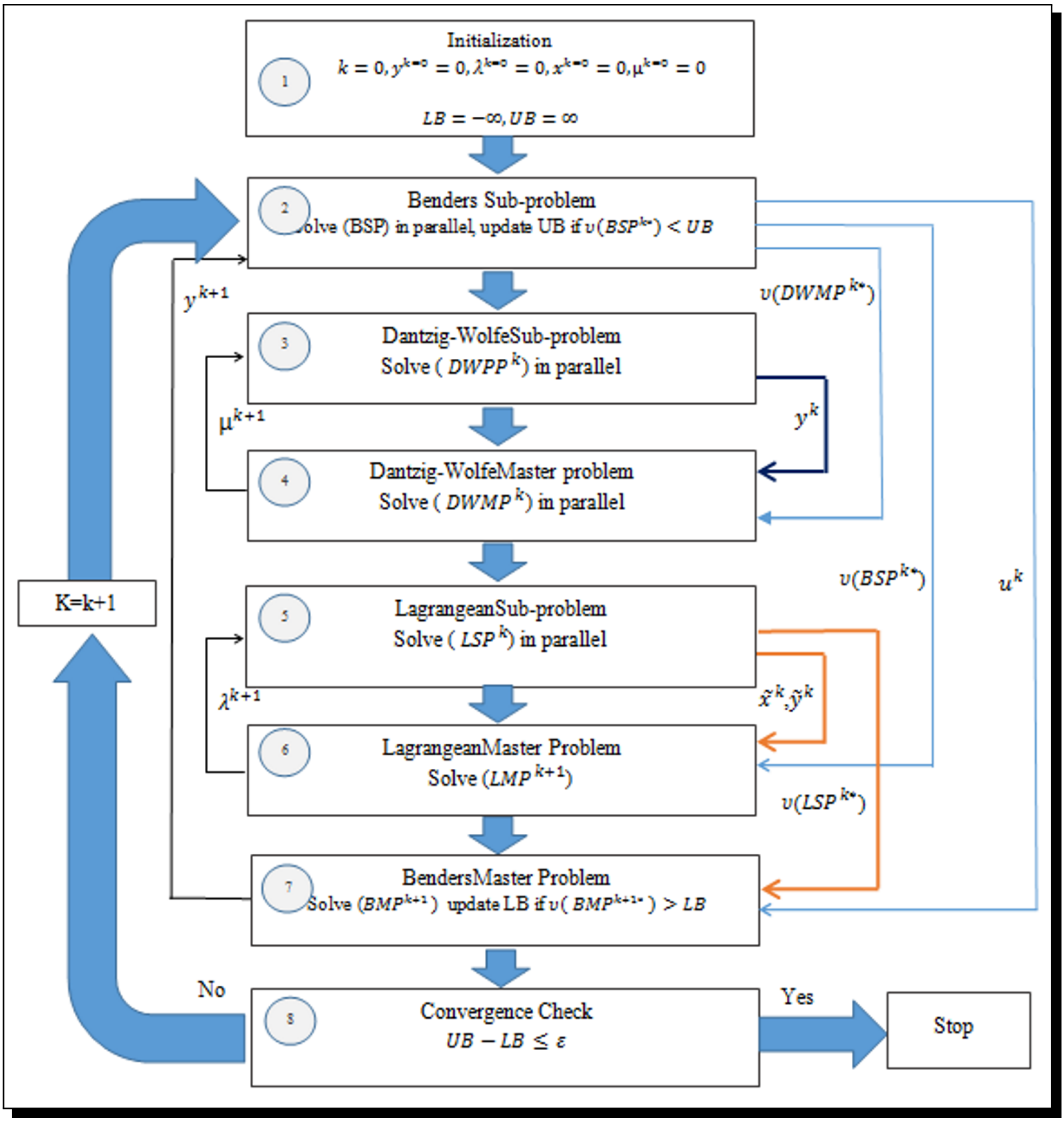

Figure 1. Cross Decomposition Algorithm

6. Implementation

In GAMS 24.1.2 on the 8 processors of an Intel i7- 2600 (3.40 GHZ) machine with 8 GB RAM, the full-space model, the classical multi-cut Benders decomposition and cross decomposition methods are achieved.

7. Model

The formulation of *Capacitated Reliable Facility Location Problem* (CRFLP) indicates:

$$\text{CRFLP}: \quad \text{Min} \sum_{j}^{|J|-1} (F_j x_j + V_j c_j) + \sum_{s\in S}\sum_{j\in J}\sum_{i\in I} \tau_s A_{j,i} D_i y_{s,j,i} \tag{31}$$

$$\text{s.t. } c_j - c^{\max} x_j \le 0 \quad \forall\ j < |j| \tag{32}$$

$$\sum_{i\in i} D_i y_{s,j,i} - T_{s,j} c_j \le 0 \quad \forall\ s \in S,\ j \in J \tag{33}$$

$$\sum_{j\in j} y_{s,j,i} = 1 \quad \forall\ s \in S,\ i \in I \tag{34}$$

$$x_j \in \{0,1\},\ 0 \le c_j \le c^{\max} \quad \forall\ j \in J \tag{35}$$

$$0 \le y_{s,j,i} \le 1 \quad \forall\ s \in S,\ i \in I,\ j \in J \tag{36}$$

The objective function (31) minimizes the total of investment cost at distribution centers, the expected cost of transportation from distribution centers. Based on the place selection, constraints (32) bound the capacity of storage of DCs. The inventory availability show that the constraints (33) in every scenario for the customer assignments are limited based on the binary parameter $T_{s,j}$that in the scenarios represent the form of distributions ($T_{s,j} = 0$). Constraints (34) ensure demand assignments for all scenarios. Finally, the domain of the variables is presented in constraint (35) and (36).

For strengthening the formulation of MILP, a redundant set of constraint (39) specifically avoid demand assignment to distribution centers that are not chosen and can be added to the model:

$$\text{CRFLP-}t: \quad \text{Min} \sum_{j}^{|J|-1} (F_j x_j + V_j c_j) + \sum_{s\in S}\sum_{j\in J}\sum_{i\in I} \tau_s A_{j,i} D_i y_{s,j,i} \tag{37}$$

$$\text{s.t. } (32)-(36) \tag{38}$$

$$y_{s,j,i} - T_{s,j} x_j \le 0 \quad s \in S,\ i \in I,\ j \in J \tag{39}$$

8. Data

Daskin's [3] data has been employed for this case study. The 49 United States cities which are the demand sites and the feasible centers of distribution are considered as the original problem. In 1990 this demand for the population of state was considered to be proportional for each commodity. With investment costs based on the real-state market, the main formulation contains uncapacitated DCs. For a rate of \$0.0001 for each type of product the variable costs along with the capacities of DC have been added. Between facilities, the cost of transportations to the great-circle distance are proportional. The researcher selected the facilities of subsets of 10, 11, and 12 as the places of the candidate for DCs problem of various instances by considering a large number of potential scenarios (2^{49}). The scenarios classified into a group where all the demands are penalized through the small feasibility of them with more than five simultaneous disruptions. The value of originality which was employed by Synder and Daskin [23] that indicates $q = 0.05$ is the potential failure of all DCs.

9. Computational results

We compare four methods Full-space model, Benders decomposition, new and old cross decomposition (cross decomposition of Mitra *et al.*) methods for 3 instance of the two problem formulations (CRFLP) and (CRFLP-t).

In Table 1 and 3, we report the problem sizes of the model for (CRFLP) and (CRFLP-t).

Table 1. Size of Full-space model for (CRFLP) in terms of constraint and variables

DCs (N)	Scenarios	Constraints	Variables	Binary Var
10	639	38,992	345,084	10
11	1025	63,564	603,751	11
12	1587	99,996	1,012,534	12

Given that Table 2, it is seen that owing to the existence of poor LP relaxation. Full-space models in this case is hardly solved. For example, in the largest instance with 12 distribution centers, time of 722 minutes is elapsed to solve the model but it is possible to find the optimality in all instances. In this case, to solve the model with the help of Benders method, an increase in the number of DCs results in failure. For example, a time of 2781 minutes (about 2 days) in an instance for 10 DCs and also a time of 4000 minutes in an instance of 11 and 12 DCs is elapsed to solve the model and also for the large instance with 11 and 12 candidate DCs in algorithm, optimality gap with 12.2% and 18.8% is created after 130 and 239 iteration respectively. In old crossdecomposition method, any of three instances with the number of 21, 37 and 36 iterations in turn reach optimality. Instance of 10 and 12 DCs with runtime of 48% and 34% in comparison with Full-space model is also solved respectively. In new cross decomposition method, any of three instances with the number of iterations 21, 32 and 21 reach optimality respectively, but an instance of 12 DCs with runtime of 50% in solved in comparison to the Full-space model.

Table 2. Computational results for (CRFLP)

DCs (N)		Full-space	Benders	Old Cross	New Cross
10	Objective ($)	1,003,707.231	1,003,707.231	1,003,707.231	1,003,707.231
	LP relaxation ($)	520,311.87	-	-	-
	Optimality gap (%)	0	0	0	0
	Iterations (#)	-	440	21	27
	Runtime (min)	33	2781	17	28
11	Objective ($)	1,003,632.26	1,007,279.28	1,003,632.26	1,003,632.26
	LP relaxation ($)	495,055.02	-	-	-
	Optimality gap (%)	0	12.2	0	0
	Iterations (#)	-	239	37	32
	Runtime (min)	167	4000	182	172
12	Objective ($)	1,004,855.83	1,028,650.66	1,004,855.83	1,004,855.83
	LP relaxation ($)	479,563.62	-	-	-
	Optimality gap (%)	0	18.8	0	0
	Iterations (#)	-	130	36	21
	Runtime (min)	722	4000	474	364

As one can see in Table 3, the tightening constraint (39) increases the problem size for the model.

Table 3. Size of Full-space model for (CRFLP-t) in terms of constraint and variables

DCs (N)	Scenarios	Constraints	Variables	Binary Var
10	639	383,413	345,084	10
11	1025	666,234	603,751	11
12	1587	1,110,915	1,012,534	12

It is considered in Table 4, owning to the existence of a tightening constraint (39) that improves the LP relaxation, in terms of time of solution of (CRFLP-t) for Full-space model in comparison to the time of solution of (CRFLP) for the Full-space model increases. But the number iterations and the times of solution for Benders method highly decreases. This is because the number of iterations and times of solution for old cross decomposition method decreases and even this number of iterations and times of solution for new cross decomposition methods will become less again in a way that is seen with an instances of 12 DCs in Benders method with 65 iterations and old cross decomposition method and also with 24 iterations in new cross decomposition with 19 iteration reaches optimality. Also we find answer in term of time in comparison to old cross decomposition methods with Benders decomposition with runtime the less than 63% and in comparison to new cross decomposition methods with Benders decomposition with a less 83% runtime.

Table 4. Computational results for (CRFLP-t)

DCs (N)		Full-space	Benders	Old Cross	New Cross
10	Objective ($)	1,003,707.231	1,003,707.231	1,003,707.231	1,003,707.231
	LP relaxation ($)	1,000,314.99	-	-	-
	Optimality gap (%)	0	0	0	0
	Iterations (#)	-	13	12	11
	Runtime (min)	61	2.5	6.5	4.5
11	Objective ($)	1,003,707.23	1,003,632.26	1,003,632.26	1,003,632.26
	LP relaxation ($)	995,531.18	-	-	-
	Optimality gap (%)	0	0	0	0
	Iterations (#)	-	38	20	16
	Runtime (min)	328	42	44	43
12	Objective ($)	1,004,855.83	1,004,855.83	1,004,855.83	1,004,855.83
	LP relaxation ($)	996,777.36	-	-	-
	Optimality gap (%)	0	0	0	0
	Iterations (#)	-	65	24	19
	Runtime (min)	2,015	435	158	96

10. Conclusion

This investigation suggests a new framework for a cross decomposition method that is different from the current cross decomposition methods. This method utilizes Dantzig-Wolfe decomposition, Benders decomposition and Lagrangean relaxation methods simultaneously and alternatively which therein a trail of upper bound is bounded by Bender sub-problem and Dantzig-Wolfe bounded master problem and also a trail of lower bounds by Lagrangean master problem, a combination of three concepts (cutting planes, sub-gradient and trust region) and *Benders Master Problem* (BMP) is obtained. The numerical results obtained from applying this method for capacitated reliable facility location problem shows while this method becomes convergence to the original problem optimal solution, reaching to the solution speeds up too.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] J. F. Benders, Partitioning procedures for solving mixed-variables programming problems, *Numerische Mathematik* **4**(1) (1962), 238 – 252, DOI: 10.1007/BF01386316.

[2] E. W. Cheney and A. A. Goldstein, Newton's method for convex programming and Tchebycheff approximation, *Numerische Mathematik* **1**(1) (1959), 253 – 268, DOI: 10.1007/BF01386389.

[3] M. S. Daskin, *Network and Discrete Location: Models, Algorithms, and Applications*, John Wiley & Sons, Inc, Chap Appendix H. Longitudes, Latitudes, Demands, and Fixed Cost for SORTCAP.GRT: A 49-Node Problem Defined on the Continental United States (1995).

[4] E. Ogbe and X. Li, A new cross decomposition method for stochastic mixed-integer linear programming, *European Journal of Operational Research* **256**(2) (2017), 487 – 499, DOI: 10.1016/j.ejor.2016.08.005.

[5] M. Fischetti, D. Salvagnin and A. Zanette, A note on the selection of Benders' cuts, *Mathematical Programming* **124**(1-2) (2010), 175 – 182, DOI: 10.1007/s10107-010-0365-7.

[6] P. Garcia-Herreros, J. Wassick and I. E. Grossmann, Design of Resilient Supply Chains with Risk of Facility Disruptions, *Industrial & Engineering Chemistry Research* **53** (44) (2014), 17240 – 17251, DOI: 10.1021/ie5004174.

[7] G. B. Dantzig and P. Wolfe, Decomposition principle for linear programs, *Operations Research* **8**(1) (1960), 101 – 111, DOI: 10.1287/opre.8.1.101.

[8] A. M. Geofrion and G. W. Graves, Multicommodity distribution system design by Benders ecomposition, *Management Science* **20**(5) (1974), 822 – 844, DOI: 10.1287/mnsc.20.5.822.

[9] M. Held and R. M. Karp, The traveling salesman problem and minimum spanning trees: Part II, *Mathprogramming* 16–25 (1971), DOI: 10.1007/BF01584070.

[10] K. Holmberg, On the convergence of cross decomposition, *Mathematical Programming* **47**(2) (1990), 269 – 296, DOI: 10.1007/BF01580863.

[11] D. MacDaneil and M. Devine, A modified bendes partitioning algorithm for mixed integer programming, *Management Science* **24**(3) (1977), 312 – 319, DOI: 10.1287/mnsc.24.3.312.

[12] T. L. Magnanti and R. T. Wong, Accelerating Benders decomposition: Algorithmic enhancement and model selection criteria, *Operations Research* **29**(3) (1981), 464 – 484, DOI: 10.1287/opre.29.3.464.

[13] R. E. Marsten, W. W. Hogan and J. W. Blankenship, The box step method for large-scale optimization, *Operations Research* **23**(3) (1975), 389 – 405, DOI: 10.1287/opre.23.3.389.

[14] S. Mitra, P. Garcia-Herreros and I. E. Grossmann, A Novel Cross-decomposition Multi-cut Scheme for Two-Stage Stochastic Programming, in 24th European Symposium on Computer Aided Process Engineering: Part A and B, June 20 2014, Vol. 33, p. 241, DOI: 10.1016/B978-0-444-63456-6.50041-7.

[15] N. Papadakos, Practical enhancements to the magnanti-wong method, *Operations Research Letters* **36**(4) (2008), 444 – 449, DOI: 10.1016/j.orl.2008.01.005.

[16] C. A. Poojari and J. E. Beasley, Improving Benders decomposition using a genetic algorithm, *European Journal of Operational Research* **199**(1) (2009), 89 – 97, DOI: 10.1016/j.ejor.2008.10.033.

[17] R. Rahmaniani, T. G. Crainic, M. Gendreau and W. Rei, The Benders decomposition algorithm: A literature review, *European Journal of Operational Research* **259**(3) (2017), 801 – 817, DOI: 10.1016/j.ejor.2016.12.005.

[18] W. Rei, J.-F. Cordeau, M. Gendreau and P. Soriano, Accelerating Benders decomposition by local branching, *INFORMS Journal on Computing* **21**(2) (2009), 333 – 345, DOI: 10.1287/ijoc.1080.0296.

[19] G. K. Saharidis and M. G. Ierapetritou, Speed-up Benders decomposition using maximum density cut (MDC) generation, *Annals of Operations Research* **210**(1) (2013), 101 – 123, DOI: 10.1007/s10479-012-1237-8.

[20] G. K. Saharidis and M. G. Ierapetritou, Improving Benders decomposition using maximum feasible subsystem (MFS) cut generation strategy, *Computers & Chemical Engineering* **34**(8) (2010), 1237 – 1245, DOI: 10.1016/j.compchemeng.2009.10.002.

[21] G. K. Saharidis, M. Minoux and M. C. Ierapetritou, Accelerating Benders method using covering cut bundle generation, *International Transactions in Operational Research* **17**(2) (2010), 221 – 237, DOI: 10.1111/j.1475-3995.2009.00706.x.

[22] S. Mouret, I. E. Grossmann and P. Pestiaux, Anew Lagrangean decomposition approach applied to the integration of refinery planning and crude-oil scheduling, *Computers and Chemical Engineering* **35** (2011), 2750 – 2766, DOI: 10.1016/j.compchemeng.2011.03.026.

[23] L. V. Snyder and M. S. Daskin, Reliability models for facility location: the expected failure cost case, *Transportation Science* **39**(3) (2005), 400 – 416, DOI: 10.1287/trsc.1040.0107.

[24] T.J. Van Roy, Cross decomposition for mixed integer programming, *Mathematical Programming* **25**(1) (1983), 46 – 63, DOI: 10.1007/BF02591718.

[25] G. Zakeri, A. B. Philpott, D. M. Ryan, Inexact cuts in Benders decomposition, *SIAM Journal on Optimization* **10**(3) (2000), 643 – 657, DOI: 10.1137/S1052623497318700.