Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 541–558, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.1105

Research Article

The Dynamical System for System of Variational Inequality Problem in Hilbert Spaces

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Abstract. In this paper, we will present a dynamical system which relates to the system of variational inequality problem by starting with introducing a Wiener-Hopf equation for the system of variational inequality problem and using a such Wiener-Hopf equation for proposing a dynamical system for the system of variational inequality problem. Moreover, the existence solution of such dynamical system is considered and the stability, globally asymptotically stable and also globally exponential convergence, of the solution for the system of a such dynamical system are proved. The results in this paper improve and extend the variational inequality problems which have been appeared in literature.

Keywords. Dynamical system; The system of variational inequality; Lipschitz mapping; Lyapunov's stability theorem; Gronwall's Lemma

MSC. 37C75

Received: October 12, 2018 **Accepted:** November 13, 2018

1. Introduction

In the early 1960s, the variational inequality was studied from the mechanics problems. Later, in 1964, Stampacchia [22] introduced the variational inequality problem as follows:
Find $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in K \tag{1.1}$$

where K is a nonempty closed convex subset of a real Hilbert space H and $T : H \to H$ is a single valued mapping. Stampacchia studied the existence and uniqueness of solution of the problem and, moreover, used this problem in a field of mechanics. Variational inequality theory is powerful tools for studying problems arising in pure and applied sciences which include the work on differential equations, mechanics, control problem, equilibrium problem and transportation. Later, many authors use this problem to apply in several ways (see [7], [11], [12], [21], [24] and the references therein). From the previous, the techniques and ideas of the variational inequalities are being applied in a variety of diverse areas of sciences and are being proved to be productive and innovative. These activities have motivated to generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques (see [1], [4], [18], [20] and the references therein).

The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of researches. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solution to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving of problems. In 2001, the variational inequality is generalized in the sense of system by Verma [24], which introduced and studied a system of variational inequalities as the following problem: let $T : H \to H$ be a nonlinear mapping and K be a closed convex subset of H, to find $x^*, y^* \in K$ such that

$$\begin{aligned} &\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \text{for all } x \in K, \\ &\langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \text{for all } x \in K, \end{aligned} \tag{1.2}$$

where ρ, η are positive constants. In [24] Verma developed some iterative algorithms for approximating the solution of system of variational inequalities. Notice that, the concept of system of variational inequality is very interesting, because of a variety of equilibrium models, such as, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem, can be uniformly modelled as a system of variational inequalities.

On the other hand, a dynamical system theory is the interesting tool because this theory is used for applications in many fields, such as, economics, physics, engineering, medicine and mathematics etc. Dynamical system is the study something which is related to time, and also, in mathematics, it is the study some functions over time for using the results to apply in science and the real world problems. The fantastic of dynamical systems does not just the previous article, but also it is the relation with the variational inequality, that is, a methodology which is called projected dynamical system theory. The relation of the variational inequality theory and dynamical system theory in the same framework is that the set of stationary points of dynamical system coincides with the set of solutions of variational inequality problem. In fact, in another way, a dynamical model of competitive system is proposed in the concept of ordinary differential equation, which the right hand side of this equation is continuous, but the projected dynamical system's right hand side is projection operator and discontinuous. Then, in some papers, the projected dynamical system is used for solving some problems in linear and nonlinear in variational inequality and some related convex optimization problems (see [2, 25–27] and the references therein). For example, in 1993, Dupuis and Nagurney [5] presented the dynamical systems and variational inequalities and related of both. Later, in 1996, Nagurney and Zhang [14] proposed the projected dynamical systems and variational inequalities and also these applications. In 2000, Xia and Wang [25] have proved that the projected dynamical systems can be used effectively for designing neural network and solving variational inequalities and optimization problems. In 2002, Isac and Cojocaru [9] introduced the projection operators in a Hilbert space. In 2003, Noor [17] used the fixed point formulation to consider the dynamical system for quasi variational inequality which this dynamical system includes many previously known dynamical systems suggested by Dupuis and Nagurney [5] and Friesz *et al*. [7] as special cases. In 2005, Cojocaru *et al*. [3] considered the projected dynamical systems and evolutionary variational inequalities on any finite dimensional Hilbert space, for any closed convex set and a Lipschitz continuous operator. They also observed that if the nonlinear operator is also strictly monotone or strictly pseudomonotone, then there exists a unique stationary point for the projected dynamical system. Later, in 2010, Liu and Cao [11] and Liu and Yang [12] have developed the recurrent neural network technique for solving the extended general variational inequalities. Subsequently, projected dynamical systems can be used to study financial equilibrium problems, optimization problems, fixed point problems, complementarity problems and all those problems which can be studied in the framework of variational inequalities. From many extended researches on the concept of dynamical systems, many authors are interested and published in these articles (see [5, 6, 23, 27, 28] and the references therein).

In this paper, we show the equivalence of the solution of the system of variational inequality problem with a Wiener-Hopf equation. We introduce a dynamical system associated with the system of variational inequality problem by using a Wiener-Hopf technique and consider the existence solution of the dynamical system. Furthermore, we prove that the solution of the dynamical system is the globally asymptotically stable and also globally exponential convergence to the solution of the system of variational inequality problem.

2. Preliminaries

Now, we will recall some basic concepts and some well-known fundamental results which will be used in this work.

Throughout in this paper, let H be a real Hilbert space and K be a nonempty, closed and convex subset of H, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. We denote by $d(\cdot,K)$ for usual distance function on H to the subset K, that is, $d(u,K)=\inf_{v\in K}\|u-v\|$, for all $u\in H$.

Definition 2.1 ([20])**.** Let $u\in H$ be a point not lying in K. A point $v\in K$ is called a closest point or the projection of u onto K if $d(u,K)=\|u-v\|$. The set of all such closest points is denoted by $Proj_K(u)$, that is,

$$Proj_K(u)=\{v\in K|d(u,K)=\|u-v\|\}.$$

Notice that the projection operator $Proj_K$ is a nonexpansive mapping.

For a closed convex subset $K\subseteq H$, it is well known that for given $x\in H$ and $z\in K$ satisfy

$$\langle z-x,y-z\rangle\geq 0,\quad \text{for all}\quad y\in K \tag{2.1}$$

if and only if

$$z = Proj_K(x) \tag{2.2}$$

where $Proj_K$ is the projection of H onto K.

Next, we will recall the Gateaux directional derivative of $Proj_K$ as follows.

Definition 2.2 ([9])**.** For $x \in K$, the set $T_K(x) = \overline{\bigcup_{h>0} \frac{1}{h}(K-x)}$ is called the tangent cone to K at the point x. The cone $T_K(x)$ is a closed convex cone. The normal cone to the set K at the point x is the polar cone of $T_K(x)$ and it is given by

$$N_K(x) := \{p \in H | \langle p, x - x' \rangle \geq 0, \text{ for all } x' \in K\}.$$

The normal cone is also a closed, convex cone.

Proposition 2.3 ([9])**.** *For any $x \in K$ and any element $v \in H$ the limit*

$$\prod_K(x,v) := \lim_{\delta \to 0^+} \frac{Proj_K(x+\delta v) - x}{\delta}$$

exists and $\prod_K(x,v) = Proj_{T_K(x)}(v)$.

Definition 2.4 ([9])**.** Let H be a Hilbert space of arbitrary dimension and $K \subseteq H$ be a non-empty, closed and convex subset. Let $F : K \to K$ be a single-valued mapping. Then the ordinary differential equation

$$\frac{dx(t)}{dt} = \prod_K(x(t), -F(x(t))) \tag{2.3}$$

is called the projected differential equation associated with F and K.

Now, we will introduce some relations between the projected differential equation and variational inequality problem.

Theorem 2.5 ([9])**.** *Let H be a Hilbert space and $K \subseteq H$ be a nonempty, closed and convex subset. Let $F : K \to H$ be a single-valued mapping. Then the solution to the variational inequality problem* (1.1) *coincide with the critical points of the projected differential equation* (2.3)*.*

If the initial value problem as follow:

$$\frac{dx(t)}{dt} = \prod_K(x(t), -F(x(t))), \quad x(0) = x_0 \in K. \tag{2.4}$$

Then an absolutely continuous function x is said to be a solution to (2.4) if $x : [0,T) \subseteq \mathbb{R} \to H$ such that $x(t) \in K$ for all $t \in [0,T)$ and $\frac{dx(t)}{dt} = \prod_K(x(t), -F(x(t)))$ for almost all $t \in [0,T)$.

Definition 2.6 ([9])**.** A mapping $\phi : K \times \mathbb{R}^+ \to K$ is called a projected dynamical system if ϕ solves the initial value problem, that is,

$$\dot{\phi}_x(t) = \prod_K(\phi_x(t), -F(\phi_x(t))), \quad \phi_x(0) = x \tag{2.5}$$

where $\phi_x(\cdot) := x(\cdot)$.

The following definitions and lemma are important to prove our results in the concepts of stability in dynamical system.

Observe the general dynamical system

$$\frac{dx}{dt} = f(x) \tag{2.6}$$

for $x \in H$ and f is a continuous function from H into H.

Definition 2.7 ([8])**.** (a) A point x^* is an equilibrium point for (2.6) if $f(x^*) = 0$;

(b) An equilibrium point x^* of (2.6) is stable if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x_0 \in B(x^*,\delta)$, the solution $x(t)$ of the dynamical system with $x(0) = x_0$ exists and is contained in $B(x^*,\varepsilon)$ for all $t > 0$, where $B(x^*,r)$ denotes the open ball with center x^* and radius r;

(c) A stable equilibrium point x^* of (2.6) is asymptotically stable if there exist $\delta > 0$ such that, for every solution $x(t)$ with $x(0) \in B(x^*,\delta)$, one has

$$\lim_{t\to\infty} x(t) = x^*.$$

Definition 2.8 ([14])**.** Let $x(t) = \phi_x(t)$ in (2.5). For any $x^* \in K$, let V be a real continuous function defined on a neighborhood $N(x^*)$ of x^*, and differentiable everywhere on $N(x^*)$ except possibly at x^*. V is called a Lyapunov function at x^*, if satisfies:

(i) $V(x^*) = 0$ and $V(x) > 0$, for all $x \neq x^*$,

(ii) $\dot{V}(x) \le 0$ for all $x \neq x^*$ where

$$\dot{V}(x) = \frac{d}{dt}V(x(t))|_{t=0}. \tag{2.7}$$

Notice that, the equilibrium x, which satisfies Definition 2.8(ii), is stable in the sense of Lyapunov.

Definition 2.9 ([19])**.** A dynamical system is said to be globally convergent to the solution set K of (2.6) if, irrespective of initial point, the trajectory of dynamical system satisfies

$$\lim_{t\to\infty} d(x(t),K) = 0. \tag{2.8}$$

If the set K has a unique point x^*, then (2.8) satisfies $\lim_{t\to\infty} x(t) = x^*$. If the dynamical system is still stable at x^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at x^* .

Definition 2.10 ([19])**.** The dynamical system is said to be globally exponentially stable with degree ω at x^* if, irrespective of the initial point, the trajectory of the dynamical system $x(t)$ satisfies

$$\|x(t) - x^*\| \le c_0\|x(t_0) - x^*\| \exp^{-\omega(t-t_0)}$$

for all $t \ge t_0$, where c_0 and ω are positive constants independent of initial point.

Notice that, if it is a globally exponentially stability then it is a globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 2.11 ([13])**.** *Let $\widehat{u}$ and $\widehat{v}$ be real valued nonnegative continuous functions with domain $\{t|t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for all $t \geq t_0$,*

$$\widehat{u}(t) \leq \alpha(t) + \int_{t_0}^{t} \widehat{u}(s)\widehat{v}(s)ds,$$

then,

$$\widehat{u}(t) \leq \alpha(t)\exp^{\int_{t_0}^{t} \widehat{v}(s)ds}.$$

Next, we will introduce the type of mapping which is used for solving our results.

Definition 2.12 ([20])**.** A mapping $T : H \to H$ is said to be Lipschitz continuous mapping with a constant β, if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\| \tag{2.9}$$

for all $x, y \in H$. It is well known that if $\beta = 1$, then the mapping T is said to be nonexpansive mapping.

3. Main Results

In this work, we are interested to study the system of variational inequality problems (1.2). In 2004, Kim and Kim [10] introduced a new system of generalized nonlinear mixed variational inequalities, which the problem (1.2) is a special case in Hilbert spaces and proved the existence and uniqueness of the such solution of the such problem. Furthermore, if $x^* = y^*$ and $\rho = \eta = 1$, then (1.2) reduces to the Stampacchia's variational inequality problem (1.1).

In [24], Verma presented the following lemma.

Lemma 3.1. *$x^*, y^* \in K$ are the solutions of the problem* (1.2) *if and only if*

$$\begin{aligned} x^* &= Proj_K(y^* - \rho T(y^*)), \\ y^* &= Proj_K(x^* - \eta T(x^*)), \end{aligned} \tag{3.1}$$

for some positive constants $\rho, \eta > 0$.

Now, we will present the Wiener-Hopf equation, which is equivalent to the system of variational inequality problem (1.2). Let $T : H \to H$ be a nonlinear mapping and ρ, η be positive constants, we consider to find $u^*, v^*, x^*, y^* \in H$ such that

$$\begin{aligned} Q_K(v^*) + \rho TProj_K(u^*) &= y^* - x^*, \\ Q_K(u^*) + \eta TProj_K(v^*) &= x^* - y^*, \end{aligned} \tag{3.2}$$

where $Q_K = I - Proj_K$ with I is an identity operator.

Remark 3.2. (1) If $x^* = y^*, u^* = v^*$ and $\rho = \eta = 1$, then (3.2) reduces to the following problem: to find $u^* \in H$ such that

$$(Q_K + TProj_K)(u^*) = 0, \tag{3.3}$$

which is the Wiener-Hopf associate with the Stampacchia's variational inequality problem (1.1).

(2) If $x^* = y^*$, $u^* = v^*$ and $\rho = \eta$, then (3.2) reduces to the Wiener-Hopf equation of variational

inequality which was studied by Noor [15].

The following lemma, we show the equivalence of the solutions of the problem (1.2) and the solutions of (3.2).

Lemma 3.3. *Let $T : H \to H$ be a Lipschitz continuous mapping. The system of variational inequality problem* (1.2) *has solutions $x^*, y^* \in K$ if and only if the Wiener-Hopf equation* (3.2) *has solutions $u^*, v^*, x^*, y^* \in H$, where satisfy*

$$\begin{aligned} x^* &= Proj_K(v^*), \\ y^* &= Proj_K(u^*), \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} u^* &= x^* - \eta T(x^*), \\ v^* &= y^* - \rho T(y^*). \end{aligned} \tag{3.5}$$

Proof. ($\Rightarrow$) Assume that x^*, y^* are solutions of the system of the variational inequality problem (1.2). That is, $x^*, y^* \in K$ such that

$$\begin{aligned} &\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \text{for all } x \in K, \\ &\langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \text{for all } x \in K. \end{aligned}$$

We will show that $u^*, v^*, x^*, y^* \in H$, which satisfy (3.4) and (3.5), are solutions of the problem (3.2). We see that

$$\begin{aligned} &\langle x^* - (y^* - \rho T(y^*)), x - x^* \rangle \geq 0, \quad \text{for all } x \in K, \\ &\langle y^* - (x^* - \eta T(x^*)), x - y^* \rangle \geq 0, \quad \text{for all } x \in K. \end{aligned}$$

By using the property of the convexity, we obtain that

$$\begin{aligned} x^* &= Proj_K(y^* - \rho T(y^*)) \\ y^* &= Proj_K(x^* - \eta T(x^*)). \end{aligned}$$

Since $Q_K = I - Proj_K$ and (3.4), we have

$$\begin{aligned} Q_K(y^* - \rho T(y^*)) &= (I - Proj_K)(y^* - \rho T(y^*)) \\ &= y^* - \rho T(y^*) - Proj_K(y^* - \rho T(y^*)) \\ &= y^* - \rho T(y^*) - x^*. \end{aligned}$$

This implies that

$$Q_K(y^* - \rho T(y^*)) + \rho T Proj_K(x^* - \eta T(x^*)) + x^* - y^* = 0.$$

By (3.5), we have

$$Q_K(v^*) + \rho T Proj_K(u^*) = y^* - x^*.$$

In the same way, we obtain that

$$Q_K(x^* - \eta T(x^*)) + \eta T Proj_K(y^* - \rho T(y^*)) + y^* - x^* = 0.$$

Thus,

$$Q_K(u^*) + \eta T Proj_K(v^*) = x^* - y^*.$$

We conclude that u^*, v^*, x^*, y^* are solutions of the problem (3.2).

($\Leftarrow$) Assume that u^*, v^*, x^*, y^* are the solutions of the problem (3.2) and satisfy (3.4), (3.5). Replacing (3.5) in (3.4), this implies that

$$x^* = Proj_K(y^* - \rho T(y^*)),$$
$$y^* = Proj_K(x^* - \eta T(x^*)).$$

By Lemma 3.1, we obtain that x^*, y^* are the solutions of the problem (1.2). This completes the proof. □

Notice that, if we are replacing (3.2) with (3.5), then,

$$Q_K(y^* - \rho T(y^*)) + \rho T Proj_K(x^* - \eta T(x^*)) = y^* - x^*,$$
$$Q_K(x^* - \eta T(x^*)) + \eta T Proj_K(y^* - \eta T(y^*)) = x^* - y^*.$$

Since $Q_K = I - Proj_K$, we have

$$(I - Proj_K)(y^* - \rho T(y^*)) + \rho T Proj_K(x^* - \eta T(x^*)) = y^* - x^*,$$
$$(I - Proj_K)(x^* - \eta T(x^*)) + \eta T Proj_K(y^* - \eta T(y^*)) = x^* - y^*.$$

This implies that

$$x^* - \rho T(y^*) - Proj_K(y^* - \rho T(y^*)) + \rho T Proj_K(x^* - \eta T(x^*)) = 0,$$
$$y^* - \eta T(x^*) - Proj_K(x^* - \eta T(x^*)) + \eta T Proj_K(y^* - \rho T(y^*)) = 0. \tag{3.6}$$

This observation suggest the following system of dynamical system associated with the problem (1.2):

$$\frac{dx}{dt} = \lambda\{Proj_K(y - \rho T(y)) - \rho T Proj_K(x - \eta T(x)) + \rho T(y) - x\},$$
$$\frac{dy}{dt} = \gamma\{Proj_K(x - \eta T(x)) - \eta T Proj_K(y - \rho T(y)) + \eta T(x) - y\}. \tag{3.7}$$

which $x(t_0) = x_0$ and $y(t_0) = y_0$ in K and λ, γ are positive constants with a positive real number t_0.

Remark 3.4. (1) If $x = y$, $\lambda = \gamma$ and $\rho = \eta$, then the problem (3.7) reduces to the following equation:

$$\frac{dx}{dt} = \lambda\{Proj_K(x - \rho T(x)) - \rho T Proj_K(x - \rho T(x)) + \rho T(x) - x\}, \tag{3.8}$$

which $x(t_0) = x_0$ in K and λ is a positive constant with a positive real number t_0. Then, the dynamical system (3.8) is the dynamical system of variational inequality problem which was studied by Noor [16].

(2) If $x = y$, $\lambda = \gamma$, $\rho = \eta$ and x is a solution of the problem of variational inequality problem (1.1), then the problem (3.7) reduces to the following equation:

$$\frac{dx}{dt} = \lambda\{Proj_K(x - \rho T(x)) - x\}, \tag{3.9}$$

which $x(t_0) = x_0$ in K and λ is a positive constant with a positive real number t_0. The dynamical system (3.9) is the dynamical system of variational inequality problem which was studied by Ha *et al.* [8].

(3) If $x = y$, $\lambda = \gamma$, $\rho = \eta = 1$ and x is a solution of the problem of variational inequality problem (1.1), then the problem (3.7) reduces to the following equation:
$$\frac{dx}{dt} = \lambda\{Proj_K(x - T(x)) - x\}, \tag{3.10}$$
which $x(t_0) = x_0$ in K and λ is a positive constant with a positive real number t_0. The dynamical system (3.10) is the dynamical system of Stampacchia's variational inequality problem.

Theorem 3.5. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with constant β. Then, for each $x_0, y_0 \in H$, there exist the unique continuous solutions, $x(t), y(t)$, of the system of dynamical system* (3.7) *with $x(t_0) = x_0$ and $y(t_0) = y_0$ over $[t_0, \infty)$.*

Proof. Now, we let λ, γ are positive constants and define the mapping $F : H \times H \to H \times H$ by $F(x, y) = (f(x), g(y))$ where
$$f(x) = \lambda\{Proj_K(y - \rho T(y)) - \rho T Proj_K(x - \eta T(x)) + \rho T(y) - x\},$$
$$g(y) = \gamma\{Proj_K(x - \eta T(x)) - \eta T Proj_K(y - \rho T(y)) + \eta T(x) - y\},$$
for all $x, y \in H$. Next, let us define a norm $\|\cdot\|^+$ on $H \times H$ by
$$\|(x, y)\|^+ = \|x\| + \|y\| \tag{3.11}$$
for all $x, y \in H$. It is well known that $(H \times H, \|\cdot\|^+)$ is a Hilbert space. We will show that F is a Lipschitz continuous mapping. Let $(x_1, y_1), (x_2, y_2) \in H \times H$. We have
$$\begin{aligned}
&\|F(x_1, y_1) - F(x_2, y_2)\|^+ \\
&= \|(f(x_1), g(y_1)) - (f(x_2), g(y_2))\|^+ \\
&= \|(f(x_1) - f(x_2), g(y_1) - g(y_2))\|^+ \\
&= \|f(x_1) - f(x_2)\| + \|g(y_1) - g(y_2)\| \\
&= \|\lambda\{Proj_K(y_1 - \rho T(y_1)) - \rho T Proj_K(x_1 - \eta T(x_1)) + \rho T(y_1) - x_1\} \\
&\quad - \lambda\{Proj_K(y_2 - \rho T(y_2)) - \rho T Proj_K(x_2 - \eta T(x_2)) + \rho T(y_2) - x_2\}\| \\
&\quad + \|\gamma\{Proj_K(x_1 - \eta T(x_1)) - \eta T Proj_K(y_1 - \rho T(y_1)) + \eta T(x_1) - y_1\} \\
&\quad - \gamma\{Proj_K(x_2 - \eta T(x_2)) - \eta T Proj_K(y_2 - \rho T(y_2)) + \eta T(x_2) - y_2\}\| \\
&= \lambda\|Proj_K(y_1 - \rho T(y_1)) - \rho T Proj_K(x_1 - \eta T(x_1)) + \rho T(y_1) - x_1 \\
&\quad - Proj_K(y_2 - \rho T(y_2)) + \rho T Proj_K(x_2 - \eta T(x_2)) - \rho T(y_2) + x_2\| \\
&\quad + \gamma\|Proj_K(x_1 - \eta T(x_1)) - \eta T Proj_K(y_1 - \rho T(y_1)) + \eta T(x_1) - y_1 \\
&\quad - Proj_K(x_2 - \eta T(x_2)) + \eta T Proj_K(y_2 - \rho T(y_2)) - \eta T(x_2) + y_2\| \\
&\le \lambda\{\|Proj_K(y_1 - \rho T(y_1)) - Proj_K(y_2 - \rho T(y_2))\| + \rho\|T Proj_K(x_1 - \eta T(x_1)) - T Proj_K(x_2 - \eta T(x_2))\| \\
&\quad + \rho\|T(y_1) - T(y_2)\| + \|x_1 - x_2\|\} + \gamma\{\|Proj_K(x_1 - \eta T(x_1)) - Proj_K(x_2 - \eta T(x_2))\| \\
&\quad + \eta\|T Proj_K(y_1 - \rho T(y_1)) - T Proj_K(y_2 - \rho T(y_2))\| + \eta\|T(x_2) - T(x_1)\| + \|y_1 - y_2\|\} \\
&\le \lambda\{\|y_1 - \rho T(y_1) - y_2 + \rho T(y_2)\| + \rho\beta\|x_1 - \eta T(x_1) - x_2 + \eta T(x_2)\| + \rho\beta\|y_1 - y_2\| \\
&\quad + \|x_1 - x_2\|\} + \gamma\{\|x_1 - \eta T(x_1) - x_2 + \eta T(x_2)\| + \eta\beta\|y_1 - \rho T(y_1) - y_2 + \rho T(y_2)\| \\
&\quad + \eta\beta\|x_1 - x_2\| + \|y_1 - y_2\|\}
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda\{\|y_1-y_2\|+\rho\beta\|y_1-y_2\|+\rho\beta\{\|x_1-x_2\|+\eta\beta\|x_2-x_1\|\}+\rho\beta\|y_1-y_2\|+\|x_1-x_2\|\}\\
&\quad +\gamma\{\|x_1-x_2\|+\eta\beta\|x_1-x_2\|+\eta\beta\{\|y_1-y_2\|+\rho\beta\|y_2-y_1\|\}+\eta\beta\|x_1-x_2\|+\|y_1-y_2\|\}\\
&\le \lambda\{(1+2\rho\beta)\|y_1-y_2\|+(1+\rho\beta+\rho\eta\beta^2)\|x_1-x_2\|\}+\gamma\{(1+2\eta\beta)\|x_1-x_2\|+(1+\eta\beta+\rho\eta\beta^2)\|y_1-y_2\|\}\\
&\le \lambda\{(1+2\Phi\beta)\|y_1-y_2\|+(1+\Phi\beta+\Phi^2\beta^2)\|x_1-x_2\|\}+\gamma\{(1+2\Phi\beta)\|x_1-x_2\|+(1+\Phi\beta+\Phi^2\beta^2)\|y_1-y_2\|\}\\
&\le \Delta\{(1+2\Phi\beta)\|y_1-y_2\|+(1+\Phi\beta+\Phi^2\beta^2)\|x_1-x_2\|\}+\Delta\{(1+2\Phi\beta)\|x_1-x_2\|+(1+\Phi\beta+\Phi^2\beta^2)\|y_1-y_2\|\}\\
&= \Delta(1+2\Phi\beta)\{\|y_1-y_2\|+\|x_1-x_2\|\}+\Delta(1+\Phi\beta+\Phi^2\beta^2)\{\|x_1-x_2\|+\|y_1-y_2\|\}\\
&\le 2\Delta(1+2\Phi\beta+\Phi^2\beta^2)\{\|x_1-x_2\|+\|y_1-y_2\|\}\\
&= 2\Delta(1+\Phi\beta)^2\{\|x_1-x_2\|+\|y_1-y_2\|\}\\
&= 2\Delta(1+\Phi\beta)^2\|(x_1-x_2,y_1-y_2)\|^+,
\end{aligned}
$$

where $\Delta=\max\{\lambda,\gamma\}$ and $\Phi=\max\{\rho,\eta\}$. Then, F is a Lipschitz continuous on $\|\cdot\|^+$. Hence, for each $(x_0,y_0)\in H\times H$, there exists a unique continuous solution $(x(t),y(t))$ of the system of dynamical system of (3.7), defined in a initial $t_0\le t<\Gamma$ with the initial conditions $x(t_0)=x_0$ and $y(t_0)=y_0$.

Let $[t_0,\Gamma)$ be its maximal interval of existence, we now show that $\Gamma=\infty$. Under the assumptions made of T, it is well known that the problem (1.2) has unique solution, $x^*,y^*\in K$ which

$$
\begin{aligned}
x^* &= Proj_K(y^*-\rho T(y^*))\\
y^* &= Proj_K(x^*-\eta T(x^*)),
\end{aligned}
$$

for some ρ,η are positive constants. Let $x,y\in H$. We have

$$
\begin{aligned}
&\|F(x,y)\|^+\\
&= \|(f(x),g(y))\|^+\\
&= \|f(x)\|+\|g(y)\|\\
&= \|\lambda\{Proj_K(y-\rho T(y))-\rho T Proj_K(x-\eta T(x))+\rho T(y)-x\}\|\\
&\quad +\|\gamma\{Proj_K(x-\eta T(x))-\eta T Proj_K(y-\rho T(y))+\eta T(x)-y\}\|\\
&\le \lambda\{\|Proj_K(y-\rho T(y))-x\|+\|\rho T(y)-\rho T Proj_K(x-\eta T(x))\|\}\\
&\quad +\gamma\{\|Proj_K(x-\eta T(x))-y\|+\|\eta T(x)-\eta T Proj_K(y-\rho T(y))\|\}\\
&\le \lambda\|Proj_K(y-\rho T(y))-x\|+\lambda\rho\beta\|Proj_K(x-\eta T(x))-y\|\\
&\quad +\gamma\|Proj_K(x-\eta T(x))-y\|+\gamma\eta\beta\|Proj_K(y-\rho T(y))-x\|\\
&= (\lambda+\gamma\eta\beta)\|Proj_K(y-\rho T(y))-x\|+(\gamma+\lambda\rho\beta)\|Proj_K(x-\eta T(x))-y\|\\
&\le (\Delta+\Delta\Phi\beta)\{\|Proj_K(y-\rho T(y))-x\|+\|Proj_K(x-\eta T(x))-y\|\}\\
&\le (\Delta+\Delta\Phi\beta)\{\|Proj_K(y-\rho T(y))-Proj_K(y^*-\rho T(y^*))\|+\|Proj_K(y^*-\rho T(y^*))-x^*\|+\|x^*-x\|\\
&\quad +\|Proj_K(x-\eta T(x))-Proj_K(x^*-\eta T(x^*))\|+\|Proj_K(x^*-\eta T(x^*))-y^*\|+\|y^*-y\|\}\\
&= (\Delta+\Delta\Phi\beta)\{\|x^*-x\|+\|y^*-y\|+\|Proj_K(y-\rho T(y))-Proj_K(y^*-\rho T(y^*))\|\\
&\quad +\|Proj_K(x-\eta T(x))-Proj_K(x^*-\eta T(x^*))\|\}\\
&\le (\Delta+\Delta\Phi\beta)\{\|x^*-x\|+\|y^*-y\|+\|y-\rho T(y)-y^*+\rho T(y^*)\|+\|x-\eta T(x)-x^*+\eta T(x^*)\|\}
\end{aligned}
$$

$$
\begin{aligned}
&\le (\Delta+\Delta\Phi\beta)\{2\|x^*-x\|+2\|y^*-y\|+\rho\beta\|y^*-y\|+\eta\beta\|x^*-x\|\}\\
&\le (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\|x^*-x\|+\|y^*-y\|\}\\
&\le (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\|x^*\|+\|x\|+\|y^*\|+\|y\|\}\\
&= (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\|x^*\|+\|y^*\|\}+(\Delta+\Delta\Phi\beta)(2+\Phi\beta)\{\|x\|+\|y\|\}\\
&= (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x^*,y^*)\|^+ +(\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x,y)\|^+
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|(x(t),y(t))\|^+ &\le \|(x(t_0),y(t_0))\|^+ + \int_{t_0}^{t}\|F(x(s),y(s))\|^+ ds\\
&= \|(x(t_0),y(t_0))\|^+ + \int_{t_0}^{t}(\Delta+\Phi\Delta\beta)(2+\Phi\beta)\|(x(s),y(s))\|^+ ds\\
&\quad + \int_{t_0}^{t}(\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x^*,y^*)\|^+ ds\\
&= \|(x(t_0),y(t_0))\|^+ + (\Delta+\Phi\Delta\beta)(2+\Phi\beta)\int_{t_0}^{t}\|(x(s),y(s))\|^+ ds\\
&\quad + (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x^*,y^*)\|^+\int_{t_0}^{t} ds\\
&= \|(x(t_0),y(t_0))\|^+ + (\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x^*,y^*)\|^+(t-t_0)\\
&\quad + (\Delta+\Phi\Delta\beta)(2+\Phi\beta)\int_{t_0}^{t}\|(x(s),y(s))\|^+ ds\\
&= \|(x(t_0),y(t_0))\|^+ + k_1(t-t_0)+k_2\int_{t_0}^{t}\|(x(s),y(s))\|^+ ds
\end{aligned}
$$

where $k_1=(\Delta+\Delta\Phi\beta)(2+\Phi\beta)\|(x^*,y^*)\|^+$ and $k_2=(\Delta+\Delta\Phi\beta)(2+\Phi\beta)$. By Gronwall's Lemma, we obtain that

$$\|(x(t),y(t))\|^+ \le \{\|(x(t_0),y(t_0))\|^+ + k_1(t-t_0)\}e^{k_2(t-t_0)},$$

where $t\in[t_0,\Gamma)$. Therefore, the solution $(x(t),y(t))$ is bounded on $[t_0,\Gamma)$, if Γ is finite. We conclude that $\Gamma=\infty$. This completes the proof. □

Theorem 3.6. *Let $T:H\to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with a constant β. Assume that*

$$2\Phi\beta(1+2\Phi\beta)<\frac{\omega}{\Delta} \tag{3.12}$$

where $\Phi=\max\{\rho,\eta\}$, $\Delta=\max\{\gamma,\lambda\}$ and $\omega=\min\{\gamma,\lambda\}$ with ρ,η satisfy the problem (1.2) *and γ, λ satisfy the problem* (3.7)*. Then, the system of dynamical system* (3.7) *is a globally exponentially stable, and also, globally asymptotically stable to the solution of* (1.2)*.*

Proof. Assume that T is a Lipschitz continuous mapping with a constant β. By Theorem 3.5, we obtain that the system of dynamical system (3.7) has unique continuous solutions $x(t),y(t)$ over $[t_0,T)$ for any fixed $(x_0,y_0)\in K\times K$.

Let $(x_0(t),y_0(t))=(x(t,t_0;x_0),y(t,t_0;y_0))$ be the solution of the initial value problem (3.7) and $x^*(t),y^*(t)\in K$ be solutions of the problem (1.2). We define the Lyapunov function $L:H\times H\to\mathbb{R}$

by

$$L(x,y)=\frac{1}{2}\left(\|(x,y)-(x^*,y^*)\|^+\right)^2$$

for all $x,y\in H$. Since,

$$\begin{aligned}
\frac{\partial L}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\|(x,y)-(x^*,y^*)\|^+\right)^2\right)\\
&=\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\|x-x^*\|+\|y-y^*\|\right)^2\right)\\
&=\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\|x-x^*\|^2+2\|x-x^*\|\|y-y^*\|+\|y-y^*\|^2\right)\right)\\
&=\frac{\partial}{\partial x}\left(\frac{1}{2}\|x-x^*\|^2\right)+\frac{\partial}{\partial x}\left(\|x-x^*\|\|y-y^*\|\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}\|y-y^*\|^2\right)\\
&=(x-x^*)+\frac{\|y-y^*\|(x-x^*)}{\|x-x^*\|}\\
&=\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)(x-x^*)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{1}{2}\left(\|(x,y)-(x^*,y^*)\|^+\right)^2\right)\\
&=\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)(y-y^*),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{dL}{dt}&=\frac{\partial L}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial L}{\partial y}\cdot\frac{dy}{dt}\\
&=\left\langle\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)(x-x^*),\frac{dx}{dt}\right\rangle+\left\langle\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)(y-y^*),\frac{dy}{dt}\right\rangle\\
&=\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\left\langle x-x^*,\frac{dx}{dt}\right\rangle+\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\left\langle y-y^*,\frac{dy}{dt}\right\rangle\\
&=\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*,\lambda\{Proj_K(y-\rho T(y))-\rho TProj_K(x-\eta T(x))+\rho T(y)-x\}\rangle\\
&\quad+\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*,\gamma\{Proj_K(x-\eta T(x))-\eta TProj_K(y-\rho T(y))+\eta T(x)-y\}\rangle\\
&=\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*,Proj_K(y-\rho T(y))-\rho TProj_K(x-\eta T(x))+\rho T(y)-x\rangle\\
&\quad+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*,Proj_K(x-\eta T(x))-\eta TProj_K(y-\rho T(y))+\eta T(x)-y\rangle\\
&=-\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*,x-x^*+x^*-Proj_K(y-\rho T(y))+\rho TProj_K(x-\eta T(x))-\rho T(y)\rangle\\
&\quad-\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*,y-y^*+y^*-Proj_K(x-\eta T(x))+\eta TProj_K(y-\rho T(y))-\eta T(x)\rangle\\
&=-\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*,x-x^*\rangle\\
&\quad-\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*,x^*-Proj_K(y-\rho T(y))+\rho TProj_K(x-\eta T(x))-\rho T(y)\rangle
\end{aligned}$$

$$
\begin{aligned}
&-\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*, y-y^*\rangle \\
&-\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*, y^*-Proj_K(x-\eta T(x))+\eta TProj_K(y-\rho T(y))-\eta T(x)\rangle \\
=&-\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|^2 \\
&+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\langle x-x^*, Proj_K(y-\rho T(y))-\rho TProj_K(x-\eta T(x))+\rho T(y)-x^*\rangle \\
&-\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|^2 \\
&+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\langle y-y^*, Proj_K(x-\eta T(x))-\eta TProj_K(y-\rho T(y))+\eta T(x)-y^*\rangle \\
\le&-\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|\|Proj_K(y-\rho T(y))-\rho TProj_K(x-\eta T(x))+\rho T(y)-x^*\| \\
&-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|\|Proj_K(x-\eta T(x))-\eta TProj_K(y-\rho T(y))+\eta T(x)-y^*\| \\
\le&-\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|\left(\|Proj_K(y-\rho T(y))-x^*\|+\|\rho T(y)-\rho TProj_K(x-\eta T(x))\|\right) \\
&-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|\left(\|Proj_K(x-\eta T(x))-y^*\|+\|\eta T(x)-\eta TProj_K(y-\rho T(y))\|\right) \\
\le&-\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|(\|Proj_K(y-\rho T(y))-Proj_K(y^*-\rho T(y^*))\|+\rho\beta\|y-Proj_K(x-\eta T(x))\|) \\
&-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|\left(\|Proj_K(x-\eta T(x))-Proj_K(x^*-\eta T(x^*))\|+\eta\beta\|x-Proj_K(y-\rho T(y))\|\right) \\
\le&-\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|(\|y-\rho T(y)-y^*+\rho T(y^*)\|+\rho\beta\|y-Proj_K(x-\eta T(x))\|) \\
&-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|\left(\|x-\eta T(x)-x^*+\eta T(x^*)\|+\eta\beta\|x-Proj_K(y-\rho T(y))\|\right) \\
\le&-\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\|+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|(\|y-y^*\|+\|\rho T(y^*)-\rho T(y)\| \\
&+\rho\beta\{\|y-y^*\|+\|y^*-Proj_K(x^*-\eta T(x^*))\|+\|Proj_K(x^*-\eta T(x^*))-Proj_K(x-\eta T(x))\|\}) \\
&-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\|+\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|(\|x-x^*\|+\|\eta T(x^*)-\eta T(x)\|
\end{aligned}
$$

$$
\begin{aligned}
&+\eta\beta\{\|x-x^*\|+\|x^*-Proj_K(y^*-\rho T(y^*))\|+\|Proj_K(y^*-\rho T(y^*))-Proj_K(y-\rho T(y))\|\}) \\
&\le -\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\|+\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|(\|y-y^*\|+\rho\beta\|y^*-y\| \\
&\quad +\rho\beta\{\|y-y^*\|+\|x^*-x\|+\eta\beta\|x-x^*\|\})-\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&\quad +\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|(\|x-x^*\|+\eta\beta\|x^*-x\|+\eta\beta\{\|x-x^*\|+\|y^*-y\|+\rho\beta\|y-y^*\|\}) \\
&\le -\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&\quad +\lambda\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|((1+2\rho\beta)\|y-y^*\|+(\rho\beta+\rho\eta\beta^2)\|x^*-x\|) \\
&\quad -\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&\quad +\gamma\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|((1+2\eta\beta)\|x-x^*\|+(\eta\beta+\rho\eta\beta^2)\|y^*-y\|) \\
&\le -\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\| \\
&\quad +\lambda(1+2\rho\beta)\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|\|y-y^*\|+\lambda(\rho\beta+\rho\eta\beta^2)\left(1+\frac{\|y-y^*\|}{\|x-x^*\|}\right)\|x-x^*\|^2 \\
&\quad -\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\| \\
&\quad +\gamma(1+2\eta\beta)\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|\|x-x^*\|+\gamma(\eta\beta+\rho\eta\beta^2)\left(1+\frac{\|x-x^*\|}{\|y-y^*\|}\right)\|y-y^*\|^2 \\
&\le -\lambda\|x-x^*\|^2-\lambda\|y-y^*\|\|x-x^*\|+\lambda(1+2\rho\beta)\|x-x^*\|\|y-y^*\|+\lambda(1+2\rho\beta)\|y-y^*\|^2 \\
&\quad +\lambda(\rho\beta+\rho\eta\beta^2)\|x-x^*\|^2+\lambda(\rho\beta+\rho\eta\beta^2)\|y-y^*\|\|x-x^*\| \\
&\quad -\gamma\|y-y^*\|^2-\gamma\|x-x^*\|\|y-y^*\|+\gamma(1+2\eta\beta)\|y-y^*\|\|x-x^*\|+\gamma(1+2\eta\beta)\|x-x^*\|^2 \\
&\quad +\gamma(\eta\beta+\rho\eta\beta^2)\|y-y^*\|^2+\gamma(\eta\beta+\rho\eta\beta^2)\|x-x^*\|\|y-y^*\| \\
&\le (-\lambda+\gamma(1+2\eta\beta)+\lambda(\rho\beta+\rho\eta\beta^2))\|x-x^*\|^2+(-\gamma+\lambda(1+2\rho\beta)+\gamma(\eta\beta+\rho\eta\beta^2))\|y-y^*\|^2 \\
&\quad (-\lambda+\lambda(1+2\rho\beta)+\lambda(\rho\beta+\rho\eta\beta^2)-\gamma+\gamma(1+2\eta\beta)+\gamma(\eta\beta+\rho\eta\beta^2))\|x-x^*\|\|y-y^*\| \\
&\le (-\lambda+\gamma(1+2\Phi\beta)+\lambda(\Phi\beta+\Phi^2\beta^2))\|x-x\|^2+(-\gamma+\lambda(1+2\Phi\beta)+\gamma(\Phi\beta+\Phi^2\beta^2))\|y-y^*\|^2 \\
&\quad (-\lambda+\lambda(1+2\Phi\beta)+\lambda(\Phi\beta+\Phi^2\beta^2)-\gamma+\gamma(1+2\Phi\beta)+\gamma(\Phi\beta+\Phi^2\beta^2))\|x-x^*\|\|y-y^*\| \\
&= (-\lambda+\gamma(1+2\Phi\beta)+\lambda(\Phi\beta+\Phi^2\beta^2))\|x-x^*\|^2+(-\lambda+\lambda(1+2\Phi\beta)+\lambda(\Phi\beta+\Phi^2\beta^2))\|x-x^*\|\|y-y^*\| \\
&\quad +(-\gamma+\lambda(1+2\Phi\beta)+\gamma(\Phi\beta+\Phi^2\beta^2))\|y-y^*\|^2+(-\gamma+\gamma(1+2\Phi\beta)+\gamma(\Phi\beta+\Phi^2\beta^2))\|x-x^*\|\|y-y^*\| \\
&\le (-\lambda+(\Delta+\lambda\Phi\beta)(1+2\Phi\beta))\|x-x^*\|(\|x-x^*\|+\|y-y^*\|) \\
&\quad +(-\gamma+(\Delta+\gamma\Phi\beta)(1+2\Phi\beta))\|y-y^*\|(\|y-y^*\|+\|x-x^*\|) \\
&\le (-\lambda+2\Delta\Phi\beta(1+2\Phi\beta))\|x-x^*\|(\|x-x^*\|+\|y-y^*\|) \\
&\quad +(-\gamma+2\Delta\Phi\beta(1+2\Phi\beta))\|y-y^*\|(\|y-y^*\|+\|x-x^*\|) \\
&\le (-\lambda+2\Delta\Phi\beta(1+2\Phi\beta))(\|x-x^*\|+\|y-y^*\|)+(-\gamma+2\Delta\Phi\beta(1+2\Phi\beta))(\|y-y^*\|+\|x-x^*\|) \\
&\le 2(-\omega+2\Delta\Phi\beta(1+2\Phi\beta))(\|y-y^*\|+\|x-x^*\|) \\
&= 2\psi(\|y-y^*\|+\|x-x^*\|) \\
&= 2\psi\|(x,y)-(x^*,y^*)\|^+,
\end{aligned}
$$

where $\Phi=\max\{\rho,\eta\}$, $\Delta=\max\{\lambda,\gamma\}$, $\omega=\min\{\lambda,\gamma\}$ and $\psi=-\omega+2\Delta\Phi\beta(1+2\Phi\beta)$. Since Φ satisfies

(3.12), we have $\psi < 0$. We see that

$$\begin{aligned}
\|(x(t),y(t))-(x^*,y^*)\|^+ &\le \|(x_0,y_0)-(x^*,y^*)\|^+ + \int_{t_0}^{t} \|L(x(s),y(s))\| ds \\
&\le \|(x_0,y_0)-(x^*,y^*)\|^+ + 2\psi \int_{t_0}^{t} \|(x(s),y(s))-(x^*,y^*)\|^+ ds \\
&\le \|(x_0,y_0)-(x^*,y^*)\|^+ e^{2\psi(t-t_0)}.
\end{aligned}$$

Since $2\psi < 0$, this implies that the system of dynamical system (3.7) is a globally exponentially stable with degree -2ψ at (x^*,y^*). Moreover, we see that $L(x,y)$ is a global Lyapunov function for the problem (3.7) and the problem (3.7) is stable in the sense of Lyapunov. Thus, the dynamical system is also globally asymptotically stable. We conclude that the solutions of the system of dynamical system (3.7) converge to unique solutions of the system of variational inequality problem (1.2). This completes the proof. □

In the dynamical system (3.7), if $x = y$, $\lambda = \gamma$ and $\rho = \eta$, then we obtain as the following results which were studied by Noor in [16].

Corollary 3.7. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with constant β. Then, for each $x_0 \in H$, there exists the unique continuous solution $x(t)$ of the dynamical system* (3.8) *with $x(t_0) = x_0$ over $[t_0,\infty)$.*

Corollary 3.8. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with a constant β and $\rho \in \left(0, \frac{-1+\sqrt{5}}{4\beta}\right)$. Then, the dynamical system* (3.8) *is a globally exponentially stable, and also, globally asymptotically stable to the solution of* (1.1).

Moreover, if $x = y$, $\lambda = \gamma$, $\rho = \eta$, and x is a solution of the problem (1.1) we obtain as the following results which were studied by Ha *et al.* in [8].

Corollary 3.9. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with constant β. Then, for each $x_0 \in H$, there exists the unique continuous solution $x(t)$ of the dynamical system* (3.9) *with $x(t_0) = x_0$ over $[t_0,\infty)$.*

Corollary 3.10. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with a constant $\rho \in \left(0, \frac{-1+\sqrt{5}}{4\beta}\right)$. Then, the dynamical system* (3.9) *is a globally exponentially stable, and also, globally asymptotically stable to the solution of* (1.1).

Remark 3.11. In our work, we assume a condition on ρ, then we obtain the same result of Noor in [16] and Ha *et al.* in [8].

Furthermore, if $x = y$, $\lambda = \gamma$, $\rho = \eta = 1$, and x is a solution of the problem (1.1), then we obtain as the following results which are the dynamical system of Stampacchia's variational inequality.

Corollary 3.12. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with constant β. Then, for each $x_0 \in H$, there exists the unique continuous solution $x(t)$ of the dynamical system* (3.10) *with $x(t_0) = x_0$ over $[t_0,\infty)$.*

Corollary 3.13. *Let $T : H \to H$ be a strongly monotone mapping and a Lipschitz continuous mapping with a constant $\beta \in \left(0, \frac{-1+\sqrt{5}}{4}\right)$. Then, the dynamical system* (3.10) *is a globally exponentially stable, and also, globally asymptotically stable to the solution of* (1.1).

4. Conclusion

In this work, we study the system of variational inequality problem and introduce the dynamical system which associates the system of variational inequality problem by using the Wiener-Hopf technique. We show that the solution of system of variational inequality problem is equivalent to the solution of the dynamical system which associates the problem. Furthermore, we can prove the existence solution of the dynamical system and show that the solution is stable and globally converges to the solution of the system of variational inequality problem. In finally, we show that our results extend the dynamical system of classical variational inequality problem in literature. We desire that the results which presented here will be useful and valuable for researchers who study the branch of variational inequality and related applications.

Acknowledgement

This work is supported by the Thailand research fund (Project No. MRG5980099) and Pibulsongkram Rajabhat University.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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