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Research Article

Tribonacci and Tribonacci-Lucas Matrix Sequences with Negative Subscripts

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Abstract. In this paper, we define Tribonacci and Tribonacci-Lucas matrix sequences with negative indices and investigate their properties.

Keywords. Tribonacci numbers; Tribonacci matrix sequence; Tribonacci-Lucas matrix sequence

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1. Introduction

Fibonacci sequence is often used as a model of recursive phenomena in physics and engineering (see for example, [1]), chemistry (see for instance, [18], [31]), botany (see e.g. [19]), medicine (see e.g. [14]). As Fibonacci sequence, also Tribonacci sequence has many applications to such as coding theory (see e.g. [2]), game theory (see e.g. [8] and references therein).

By a Mathematical point of view, there are analogies properties between Horadam sequences (such as Fibonacci and Lucas sequences) and generalized Tribonacci sequences (such as Padovan and usual Tribonacci sequences). For those concerning asymptotic process if we look at the characteristic polynomial $x^2 - x - 1 = 0$ associated to Fibonacci recursive relation, we have that it has a unique (real) root of maximum modulus, that is also the limit of the ratio of two consecutive Fibonacci numbers:

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha_F,$$

where $\alpha_F:=\frac{1+\sqrt{5}}{2}$ denotes the highly celebrated Golden mean (also called Golden section or Golden ratio), (see [10, Example 4.1]).

Similarly, if we consider the characteristic polynomial $x^3-x-1=0$ associated to Padovan (a generalized Tribonacci) recursive relation, we have that it has a unique (real) root of maximum modulus, that is also the limit of the ratio of two consecutive Padovan numbers:

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=\alpha_P,$$

where $\alpha_P:=\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$ denotes the Plastic ratio (see [10, Example 4.9]) which have many applications to such as architecture, see [13]).

In our case, if we consider the characteristic polynomial $x^3-x^2-x-1=0$ associated to Tribonacci recursive relation, we have that it has a unique (real) root of maximum modulus, that is also the limit of the ratio of two consecutive Tribonacci numbers:

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\alpha_T,$$

where $\alpha_T:=\frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}$ denotes the Tribonacci ratio (see for example [10, Example 4.9] or for a basic proof see [4]). For a short introduction to these three constants, see [16].

In fact, for linear homogeneous recursive sequences with constant coefficients, Fiorenza and Vincenzi [10] find a necessary and sufficient condition for the existence of the limit of the ratio of consecutive terms. As a corollary of their results, the limit of the ratio of adjacent terms is characterized as the unique leading root of the characteristic polynomial.

On the other hand, the matrix sequences have taken so much interest for different type of numbers. For matrix sequences of generalized Horadam type numbers, see for example [7], [6], [11], [25], [26], [27], [30], [28], and for matrix sequences of generalized Tribonacci type numbers, see for instance [5], [32], [33], [23].

In this paper, for negative indices, the matrix sequences of Tribonacci and Tribonacci-Lucas numbers will be defined. Then, by giving the generating functions, the Binet formulas, and summation formulas over these new matrix sequences, we will obtain some fundamental properties on Tribonacci and Tribonacci-Lucas numbers. Also, we will present the relationship between these matrix sequences.

Now, we give some background about Tribonacci and Tribonacci-Lucas numbers. Tribonacci sequence $\{T_n\}_{n\geq 0}$ (sequence A000073 in [22]) and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ (sequence A001644 in [22]) are defined by the third-order recurrence relations

$$T_n=T_{n-1}+T_{n-2}+T_{n-3},\quad T_0=0,T_1=1,T_2=1, \tag{1.1}$$

and

$$K_n=K_{n-1}+K_{n-2}+K_{n-3},\quad K_0=3,K_1=1,K_2=3 \tag{1.2}$$

respectively. Tribonacci concept was introduced by M. Feinberg [9] in 1963. Basic properties of it is given in [3], [20], [21], [24], [29].

The sequences $\{T_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}$$

and

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}$$

for $n = 1,2,3,\ldots$, respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

We can give some relations between $\{T_n\}$ and $\{K_n\}$ as

$$K_n = 3T_{n+1} - 2T_n - T_{n-1} \tag{1.3}$$

and

$$K_n = T_n + 2T_{n-1} + 3T_{n-2} \tag{1.4}$$

and also

$$K_n = 4T_{n+1} - T_n - T_{n+2}\,. \tag{1.5}$$

Note that the last three identities hold for all integers n.

The first few Tribonacci numbers and Tribonacci Lucas numbers with positive subscript are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	$\cdots$
T_{-n}	0	0	1	−1	0	2	−3	1	4	−8	5	7	−20	$\cdots$

The first few Tribonacci numbers and Tribonacci Lucas numbers with negative subscript are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	$\cdots$
K_{-n}	3	−1	−1	5	−5	−1	11	−15	3	23	−41	21	43	$\cdots$

It is well known that for all integers n, usual Tribonacci and Tribonacci-Lucas numbers can be expressed using Binet's formulas

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.6}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n \tag{1.7}$$

respectively, where α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$. Moreover,

$$\alpha = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\beta = \frac{1 + \omega\sqrt[3]{19+3\sqrt{33}} + \omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19+3\sqrt{33}} + \omega\sqrt[3]{19-3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3),$$

is a primitive cube root of unity. Note that we have the following identities

$$\begin{aligned} \alpha+\beta+\gamma &= 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= s-1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

The generating functions for the Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} T_n x^n = \frac{x}{1-x-x^2-x^3} \quad \text{and} \quad \sum_{n=0}^{\infty} K_n x^n = \frac{3-2x-x^2}{1-x-x^2-x^3}. \tag{1.8}$$

Note that the Binet form of a sequence satisfying (1.1) and (1.2) for non-negative integers is valid for all integers n. This result of Howard and Saidak [12] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1.1 ([12]). *Let $\{w_n\}$ be a sequence such that*

$$\{w_n\} = a_1 w_{n-1} + a_2 w_{n-2} + \ldots + a_k w_{n-k}$$

for all integers n, with arbitrary initial conditions $w_0, w_1, \ldots, w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$\begin{aligned} f(x) &= x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_{k-1} x - a_k \\ &= (x-\alpha_1)^{d_1}(x-\alpha_2)^{d_2}\ldots(x-\alpha_h)^{d_h} \end{aligned} \tag{1.9}$$

with $d_1+d_2+\ldots+d_h = k$, and $\alpha_1, \alpha_2, \ldots, \alpha_k$ distinct. Then

(a) *for all n,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n, \tag{1.10}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + \ldots + A_{r_m}^{(m)} n^{r_m-1} = \sum_{u=0}^{r_m-1} A_{u+1}^{(m)} n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (1.10) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0) \neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (1.9) are distinct, as they are in our examples, then

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \ldots + A_k(\alpha_k)^n.$$

(b) *The Binet form for $\{w_n\}$ is valid for all integers n.*

In [23], Soykan introduced the following definition of Tribonacci and Tribonacci-Lucas matrix sequences and investigated their properties.

Definition 1.2. For any integer $n \geq 0$, the Tribonacci matrix $(\mathcal{T}_n)$ and Tribonacci-Lucas matrix $(\mathcal{K}_n)$ are defined by

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3}, \tag{1.11}$$

$$\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + \mathcal{K}_{n-3}, \tag{1.12}$$

respectively, with initial conditions

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{T}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ -1 & 4 & -1 \end{pmatrix}, \quad \mathcal{K}_1 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

$(\mathcal{T}_n)$ and $(\mathcal{K}_n)$ have some good properties which is given in next two theorems.

Theorem 1.3 ([23])**.** *For all non-negative integers m and n, we have the following identities.*

(a) $\mathcal{T}_m\mathcal{T}_n = \mathcal{T}_{m+n} = \mathcal{T}_n\mathcal{T}_m$,

(b) $\mathcal{T}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{T}_m = \mathcal{K}_{m+n}$,

(c) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = 9\mathcal{T}_{m+n+2} - 12\mathcal{T}_{m+n+1} + \mathcal{T}_{m+n} + \mathcal{T}_{m+n-1} + \mathcal{T}_{m+n-2}$,

(d) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = \mathcal{T}_{m+n} + 4\mathcal{T}_{m+n-1} + 10\mathcal{T}_{m+n-2} + 12\mathcal{T}_{m+n-3} + 9\mathcal{T}_{m+n-4}$,

(e) $\mathcal{K}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{K}_m = \mathcal{T}_{m+n} - 8\mathcal{T}_{m+n+1} + 18\mathcal{T}_{m+n+2} - 8\mathcal{T}_{m+n+3} + \mathcal{T}_{m+n+4}$.

We now give the Binet formulas for the Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 1.4 ([23])**.** *For every integer n, the Binet formulas of the Tribonacci and Tribonacci-Lucas matrix sequences are given by*

$$\mathcal{T}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \tag{1.13}$$

$$\mathcal{K}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n, \tag{1.14}$$

where

$$A_1 = \frac{\alpha\mathcal{T}_2 + \alpha(\alpha-1)\mathcal{T}_1 + \mathcal{T}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \quad B_1 = \frac{\beta\mathcal{T}_2 + \beta(\beta-1)\mathcal{T}_1 + \mathcal{T}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \quad C_1 = \frac{\gamma\mathcal{T}_2 + \gamma(\gamma-1)\mathcal{T}_1 + \mathcal{T}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}$$

$$A_2 = \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + \mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, \quad B_2 = \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + \mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, \quad C_2 = \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + \mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

Note that the Binet formulas given above hold for all integers n.

2. The Matrix Sequences of Negative Subscripts Tribonacci and Tribonacci-Lucas Numbers

The sequences $\{\mathcal{T}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{T}_{-n} = -\mathcal{T}_{-(n-1)} - \mathcal{T}_{-(n-2)} + \mathcal{T}_{-(n-3)}$$

and

$$\mathcal{K}_{-n} = -\mathcal{K}_{-(n-1)} - \mathcal{K}_{-(n-2)} + \mathcal{K}_{-(n-3)}$$

for $n = 1,2,3,\ldots$, respectively. Therefore, recurrences (1.11) and (1.12) hold for all integer. i.e. starting with $n = -1$ and working backwards, we extend $\{\mathcal{T}_n\}$ and $\{\mathcal{K}_n\}$ to negative indices. The first few Tribonacci numbers and Tribonacci Lucas numbers with negative subscript can be found as follows:

$$\mathcal{T}_{-1} = \mathcal{T}_2 - \mathcal{T}_1 - \mathcal{T}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$

$$\mathcal{T}_{-2} = \mathcal{T}_1 - \mathcal{T}_0 - \mathcal{T}_{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{pmatrix},$$

$$\mathcal{T}_{-3} = \mathcal{T}_0 - \mathcal{T}_{-1} - \mathcal{T}_{-2} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

and

$$\mathcal{K}_{-1} = \mathcal{K}_2 - \mathcal{K}_1 - \mathcal{K}_0 = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 4 & -1 \\ -1 & 0 & 5 \end{pmatrix},$$

$$\mathcal{K}_{-2} = \mathcal{K}_1 - \mathcal{K}_0 - \mathcal{K}_{-1} = \begin{pmatrix} -1 & 4 & -1 \\ -1 & 0 & 5 \\ 5 & -6 & -5 \end{pmatrix},$$

$$\mathcal{K}_{-3} = \mathcal{K}_0 - \mathcal{K}_{-1} - \mathcal{K}_{-2} = \begin{pmatrix} -1 & 0 & 5 \\ 5 & -6 & -5 \\ -5 & 10 & -1 \end{pmatrix}.$$

Actually, we can formally define $\{\mathcal{T}_n\}$ and $\{\mathcal{K}_n\}$ for negative indices as follows.

Definition 2.1. For any integer $n \geq 0$, the negative indices Tribonacci matrix $(\mathcal{T}_{-n})$ and Tribonacci-Lucas matrix $(\mathcal{K}_{-n})$ are defined by

$$\mathcal{T}_{-n} = \mathcal{T}_{-(n-3)} - \mathcal{T}_{-(n-1)} - \mathcal{T}_{-(n-2)} = \mathcal{T}_{-n+3} - \mathcal{T}_{-n+2} - \mathcal{T}_{-n+1} \tag{2.1}$$

and

$$\mathcal{K}_{-n} = \mathcal{K}_{-(n-3)} - \mathcal{K}_{-(n-1)} - \mathcal{K}_{-(n-2)} = \mathcal{K}_{-n+3} - \mathcal{K}_{-n+2} - \mathcal{K}_{-n+1} \tag{2.2}$$

respectively, with initial conditions

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{T}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad \mathcal{T}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{pmatrix} \tag{2.3}$$

and

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ -1 & 4 & -1 \end{pmatrix}, \quad \mathcal{K}_{-1} = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 4 & -1 \\ -1 & 0 & 5 \end{pmatrix}, \quad \mathcal{K}_{-2} = \begin{pmatrix} -1 & 4 & -1 \\ -1 & 0 & 5 \\ 5 & -6 & -5 \end{pmatrix}. \tag{2.4}$$

The following theorem gives the nth general terms of the Tribonacci and Tribonacci-Lucas matrix sequences with negative indices.

Theorem 2.2. *For any integer $n \geq 0$, we have the following formulas of the matrix sequences:*

$$\mathcal{T}_{-n} = \begin{pmatrix} T_{-n+1} & T_{-n}+T_{-n-1} & T_{-n} \\ T_{-n} & T_{-n-1}+T_{-n-2} & T_{-n-1} \\ T_{-n-1} & T_{-n-2}+T_{-n-3} & T_{-n-2} \end{pmatrix}, \tag{2.5}$$

$$\mathcal{K}_{-n} = \begin{pmatrix} K_{-n+1} & K_{-n}+K_{-n-1} & K_{-n} \\ K_{-n} & K_{-n-1}+K_{-n-2} & K_{-n-1} \\ K_{-n-1} & K_{-n-2}+K_{-n-3} & K_{-n-2} \end{pmatrix}. \tag{2.6}$$

Proof. We prove (2.5) by strong mathematical induction on n. (2.6) can be proved similarly. If $n = 0$ then since $T_1 = 1$, $T_0 = T_{-1} = 0$, $T_{-2} = 1$, $T_{-3} = -1$, $T_{-4} = 0$, we have

$$\mathcal{T}_0 = \begin{pmatrix} T_1 & T_0+T_{-1} & T_0 \\ T_0 & T_{-1}+T_{-2} & T_{-1} \\ T_{-1} & T_{-2}+T_{-3} & T_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true and

$$\mathcal{T}_{-1} = \begin{pmatrix} T_0 & T_{-1}+T_{-2} & T_{-1} \\ T_{-1} & T_{-2}+T_{-3} & T_{-2} \\ T_{-2} & T_{-3}+T_{-4} & T_{-3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, we have

$$\begin{aligned}
\mathcal{T}_{-(k+1)} &= \mathcal{T}_{-(k+1)+3} - \mathcal{T}_{-(k+1)+2} - \mathcal{T}_{-(k+1)+1} = \mathcal{T}_{-(k-2)} - \mathcal{T}_{-(k-1)} - \mathcal{T}_{-k} \\
&= \mathcal{T}_{2-k} - \mathcal{T}_{1-k} - \mathcal{T}_{-k} \\
&= \begin{pmatrix} T_{3-k} & T_{1-k}+T_{2-k} & T_{2-k} \\ T_{2-k} & T_{-k}+T_{1-k} & T_{1-k} \\ T_{1-k} & T_{-k}+T_{-k-1} & T_{-k} \end{pmatrix} - \begin{pmatrix} T_{2-k} & T_{-k}+T_{1-k} & T_{1-k} \\ T_{1-k} & T_{-k}+T_{-k-1} & T_{-k} \\ T_{-k} & T_{-k-1}+T_{-k-2} & T_{-k-1} \end{pmatrix} \\
&\quad - \begin{pmatrix} T_{1-k} & T_{-k}+T_{-k-1} & T_{-k} \\ T_{-k} & T_{-k-1}+T_{-k-2} & T_{-k-1} \\ T_{-k-1} & T_{-k-2}+T_{-k-3} & T_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} T_{-k} & T_{-k-1}+T_{-k-2} & T_{-k-1} \\ T_{-k-1} & T_{-k-2}+T_{-k-3} & T_{-k-2} \\ T_{-k-2} & T_{-k-3}+T_{-k-4} & T_{-k-3} \end{pmatrix} \\
&= \begin{pmatrix} T_{-(k+1)+1} & T_{-(k+1)}+T_{-(k+1)-1} & T_{-(k+1)} \\ T_{-(k+1)} & T_{-(k+1)-1}+T_{-(k+1)-2} & T_{-(k+1)-1} \\ T_{-(k+1)-1} & T_{-(k+1)-2}+T_{-(k+1)-3} & T_{-(k+1)-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on n, this proves (2.5). □

The Binet formulas for the matrix sequences of Tribonacci and Tribonacci-Lucas numbers are given in [23], see Theorem 1.4 above. For the completeness of the paper, we now give the Binet formula for the Tribonacci and Tribonacci-Lucas matrix sequences with negative indices.

Theorem 2.3. *For every non-negative integer n, the Binet formulas of the Tribonacci and Tribonacci-Lucas matrix sequences are given by*

$$\mathcal{T}_{-n} = A_3\alpha^{-n} + B_3\beta^{-n} + C_3\gamma^{-n}, \tag{2.7}$$

$$\mathcal{K}_{-n} = A_4\alpha^{-n} + B_4\beta^{-n} + C_4\gamma^{-n}, \tag{2.8}$$

where

$$A_3 = \frac{\alpha\mathcal{T}_{-2}+(\alpha-1)\alpha^2\mathcal{T}_{-1}+\alpha^2\mathcal{T}_0}{(\alpha-\gamma)(\alpha-\beta)}, B_3 = \frac{\beta\mathcal{T}_{-2}+(\beta-1)\beta^2\mathcal{T}_{-1}+\beta^2\mathcal{T}_0}{(\beta-\gamma)(\beta-\alpha)}, C_3 = \frac{\gamma\mathcal{T}_{-2}+(\gamma-1)\gamma^2\mathcal{T}_{-1}+\gamma^2\mathcal{T}_0}{(\gamma-\beta)(\gamma-\alpha)},$$

$$A_4 = \frac{\alpha\mathcal{K}_{-2}+(\alpha-1)\alpha^2\mathcal{K}_{-1}+\alpha^2\mathcal{K}_0}{(\alpha-\gamma)(\alpha-\beta)}, B_4 = \frac{\beta\mathcal{K}_{-2}+(\beta-1)\beta^2\mathcal{K}_{-1}+\beta^2\mathcal{K}_0}{(\beta-\gamma)(\beta-\alpha)}, C_4 = \frac{\gamma\mathcal{K}_{-2}+(\gamma-1)\gamma^2\mathcal{K}_{-1}+\gamma^2\mathcal{K}_0}{(\gamma-\beta)(\gamma-\alpha)}.$$

Proof. Note that the proof is based on the recurrence relations (2.3) and (2.4) in Definition 2.1.

We prove (2.7). By the assumption, the characteristic equation of (2.1) is $x^3+x^2+x-1=0$ and the roots of it are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. So it's general solution is given by

$$\mathcal{T}_{-n} = A_3\alpha^{-n} + B_3\beta^{-n} + C_3\gamma^{-n}.$$

Using initial condition which is given in Definition 2.1, and also applying lineer algebra operations, we obtain the matrices A_3, B_3, C_3 as desired. This gives the formula for $\mathcal{T}_{-n}$.

Similarly, we have the formula (2.8). □

In fact, again by Theorem 1.1, Theorem 2.3 is true for all integers n. If we compare Theorem 1.4 and Theorem 2.3 we obtain

$$A_1 = A_3,\ B_1 = B_3,\ C_1 = C_3, \tag{2.9}$$

and

$$A_2 = A_4,\ B_2 = B_4,\ C_2 = C_4, \tag{2.10}$$

i.e.,

$$\alpha\mathcal{T}_{-2}+(\alpha-1)\alpha^2\mathcal{T}_{-1}+\alpha^2\mathcal{T}_0 = \frac{\alpha\mathcal{T}_2+\alpha(\alpha-1)\mathcal{T}_1+\mathcal{T}_0}{\alpha},$$

$$\beta\mathcal{T}_{-2}+(\beta-1)\beta^2\mathcal{T}_{-1}+\beta^2\mathcal{T}_0 = \frac{\beta\mathcal{T}_2+\beta(\beta-1)\mathcal{T}_1+\mathcal{T}_0}{\beta},$$

$$\gamma\mathcal{T}_{-2}+(\gamma-1)\gamma^2\mathcal{T}_{-1}+\gamma^2\mathcal{T}_0 = \frac{\gamma\mathcal{T}_2+\gamma(\gamma-1)\mathcal{T}_1+\mathcal{T}_0}{\gamma},$$

$$\alpha\mathcal{K}_{-2}+(\alpha-1)\alpha^2\mathcal{K}_{-1}+\alpha^2\mathcal{K}_0 = \frac{\alpha\mathcal{K}_2+\alpha(\alpha-1)\mathcal{K}_1+\mathcal{K}_0}{\alpha},$$

$$\beta\mathcal{K}_{-2}+(\beta-1)\beta^2\mathcal{K}_{-1}+\beta^2\mathcal{K}_0 = \frac{\beta\mathcal{K}_2+\beta(\beta-1)\mathcal{K}_1+\mathcal{K}_0}{\beta},$$

$$\gamma\mathcal{K}_{-2}+(\gamma-1)\gamma^2\mathcal{K}_{-1}+\gamma^2\mathcal{K}_0 = \frac{\gamma\mathcal{K}_2+\gamma(\gamma-1)\mathcal{K}_1+\mathcal{K}_0}{\gamma}.$$

The well known Binet formulas (for positive and negative indices) for Tribonacci and Tribonacci-Lucas numbers are given in (1.6) and (1.7) respectively. But, for negative indices, we will obtain these functions in terms of Tribonacci and Tribonacci-Lucas matrix sequences as a consequence of Theorems 2.2 and 2.3. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 2.4. *For every non-negative integers n, the Binet's formulas for Tribonacci and Tribonacci-Lucas numbers are given as*

$$
\begin{aligned}
T_{-n} &= \frac{\alpha^{-n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{-n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{-n+1}}{(\gamma-\beta)(\gamma-\alpha)},\\
K_{-n} &= \alpha^{-n} + \beta^{-n} + \gamma^{-n}.
\end{aligned}
$$

Proof. From Theorem 2.3, we have

$$
\begin{aligned}
\mathcal{T}_{-n} = {} & \frac{\alpha^{-n}}{(\alpha-\gamma)(\alpha-\beta)} \begin{pmatrix} \alpha^2 & \alpha^2(\alpha-1) & \alpha \\ \alpha & \alpha^2-\alpha & \alpha^2(\alpha-1)-\alpha \\ \alpha^2(\alpha-1)-\alpha & 2\alpha-\alpha^2(\alpha-1) & \alpha^2-\alpha^2(\alpha-1) \end{pmatrix} \\
& + \frac{\beta^{-n}}{(\beta-\gamma)(\beta-\alpha)} \begin{pmatrix} \beta^2 & \beta^2(\beta-1) & \beta \\ \beta & \beta^2-\beta & \beta^2(\beta-1)-\beta \\ \beta^2(\beta-1)-\beta & 2\beta-\beta^2(\beta-1) & \beta^2-\beta^2(\beta-1) \end{pmatrix} \\
& + \frac{\gamma^{-n}}{(\gamma-\beta)(\gamma-\alpha)} \begin{pmatrix} \gamma^2 & \gamma^2(\gamma-1) & \gamma \\ \gamma & \gamma^2-\gamma & \gamma^2(\gamma-1)-\gamma \\ \gamma^2(\gamma-1)-\gamma & 2\gamma-\gamma^2(\gamma-1) & \gamma^2-\gamma^2(\gamma-1) \end{pmatrix}.
\end{aligned}
$$

By Theorem 2.2, we know that

$$
\mathcal{T}_{-n} = \begin{pmatrix} T_{-n+1} & T_{-n}+T_{-n-1} & T_{-n} \\ T_{-n} & T_{-n-1}+T_{-n-2} & T_{-n-1} \\ T_{-n-1} & T_{-n-2}+T_{-n-3} & T_{-n-2} \end{pmatrix}.
$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$
T_{-n} = \frac{\alpha^{-n}\alpha}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{-n}\beta}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{-n}\gamma}{(\gamma-\beta)(\gamma-\alpha)} = \frac{\alpha^{-n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{-n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{-n+1}}{(\gamma-\beta)(\gamma-\alpha)}.
$$

Tribonacci-Lucas case can be proved similarly.

Now, we present summation formulas for Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 2.5. *For $m > j \geq 0$, we have*

$$
\sum_{i=0}^{n-1} \mathcal{T}_{-mi-j} = \frac{\mathcal{T}_{-mn+m-j} + \mathcal{T}_{-mn-m-j} + (1-K_{-m})\mathcal{T}_{-mn-j}}{K_{-m}-K_m} - \frac{\mathcal{T}_{-m-j} + \mathcal{T}_{-j+m} + (1-K_{-m})\mathcal{T}_{-j}}{K_{-m}-K_m} \tag{2.11}
$$

and

$$
\sum_{i=0}^{n-1} \mathcal{K}_{-mi-j} = \frac{\mathcal{K}_{-mn+m-j} + \mathcal{K}_{-mn-m-j} + (1-K_{-m})\mathcal{K}_{-mn-j}}{K_{-m}-K_m} - \frac{\mathcal{K}_{-m-j} + \mathcal{K}_{-j+m} + (1-K_{-m})\mathcal{K}_{-j}}{K_{-m}-K_m}. \tag{2.12}
$$

Proof. Note that

$$
\sum_{i=0}^{n-1} \mathcal{T}_{-mi-j} = \sum_{i=0}^{n-1} (A_3\alpha^{-mi-j} + B_3\beta^{-mi-j} + C_3\gamma^{-mi-j})
$$

$$= A_3\alpha^{-j}\left(\frac{\alpha^{-mn}-1}{\alpha^{-m}-1}\right)+B_3\beta^{-j}\left(\frac{\beta^{-mn}-1}{\beta^{-m}-1}\right)+C_3\gamma^{-j}\left(\frac{\gamma^{-mn}-1}{\gamma^{-m}-1}\right)$$

and

$$\begin{aligned}\sum_{i=0}^{n-1}\mathcal{K}_{-mi-j} &= \sum_{i=0}^{n-1}(A_2\alpha^{-mi-j}+B_2\beta^{-mi-j}+C_2\gamma^{-mi-j})\\ &= A_2\alpha^{-j}\left(\frac{\alpha^{-mn}-1}{\alpha^{-m}-1}\right)+B_2\beta^{-j}\left(\frac{\beta^{-mn}-1}{\beta^{-m}-1}\right)+C_2\gamma^{-j}\left(\frac{\gamma^{-mn}-1}{\gamma^{-m}-1}\right).\end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (2.11) and (2.12) as required. □

As in Corollary 2.4, in the proof of next corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

Corollary 2.6. *For $m > j > 0$, we have*

$$\sum_{i=0}^{n-1}T_{-mi-j}=\frac{T_{-mn+m-j}+T_{-mn-m-j}+(1-K_{-m})T_{-mn-j}}{K_{-m}-K_m}-\frac{T_{-m-j}+T_{-j+m}+(1-K_{-m})T_{-j}}{K_{-m}-K_m} \tag{2.13}$$

and

$$\sum_{i=0}^{n-1}K_{-mi-j}=\frac{K_{-mn+m-j}+K_{-mn-m-j}+(1-K_{-m})K_{-mn-j}}{K_{-m}-K_m}-\frac{K_{-m+j}+K_{-j+m}+(1-K_{-m})K_{-j}}{K_{-m}-K_m}. \tag{2.14}$$

Note that using the above Corollary we obtain the following well known formulas (taking $m=1$, $j=0$):

$$\sum_{i=0}^{n-1}T_{-i}=\frac{T_{-n+1}+T_{-n-1}+2T_{-n}-1}{-2},\qquad \sum_{i=0}^{n-1}K_{-i}=\frac{K_{-n+1}+K_{-n-1}+2K_{-n}-6}{-2}.$$

We now give generating functions of $\mathcal{T}$ and $\mathcal{K}$ for negative indices.

Theorem 2.7. *For negative indices, the generating function for the Tribonacci and Tribonacci-Lucas matrix sequences are given as*

$$\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n=\frac{1}{1+x+x^2-x^3}\begin{pmatrix}x^2+x+1 & x^2+x & x^2\\ x^2 & x+1 & x\\ x & x^2-x & 1\end{pmatrix}$$

and

$$\sum_{n=0}^{\infty}\mathcal{K}_{-n}x^n=\frac{1}{1+x+x^2-x^3}\begin{pmatrix}ccc3x^2+4x+1 & 4x^2+2 & x^2+2x+3\\ x^2+2x+3 & 2x^2+2x-2 & 3x^2-2x-1\\ 3x^2-2x-1 & -2x^2+4x+4 & -x^2+4x-1\end{pmatrix},$$

respectively.

Proof. Then, using Definition 2.1, and adding $xg(x)$ and $x^2g(x)$ to $g(x)$ and also substracting $x^3g(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}(1+x+x^2-x^3)g(x) &= \sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n+x\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n+x^2\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n-x^3\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n\\ &= \sum_{n=0}^{\infty}\mathcal{T}_{-n}x^n+\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^{n+1}+\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^{n+2}-\sum_{n=0}^{\infty}\mathcal{T}_{-n}x^{n+3}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathcal{T}_{-n}x^n + \sum_{n=1}^{\infty} \mathcal{T}_{-n+1}x^n + \sum_{n=2}^{\infty} \mathcal{T}_{-n+2}x^n - \sum_{n=3}^{\infty} \mathcal{T}_{-n+3}x^n \\
&= (\mathcal{T}_0 + \mathcal{T}_{-1}x + \mathcal{T}_{-2}x^2) + (\mathcal{T}_0 x + \mathcal{T}_{-1}x^2) + \mathcal{T}_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\mathcal{T}_{-n} + \mathcal{T}_{-n+1} + \mathcal{T}_{-n+2} - \mathcal{T}_{-n+3})x^n \\
&= (\mathcal{T}_0 + \mathcal{T}_{-1}x + \mathcal{T}_{-2}x^2) + (\mathcal{T}_0 x + \mathcal{T}_{-1}x^2) + \mathcal{T}_0 x^2 \\
&= \mathcal{T}_0 + (\mathcal{T}_0 + \mathcal{T}_{-1})x + (\mathcal{T}_0 + \mathcal{T}_{-1} + \mathcal{T}_{-2})x^2 .
\end{aligned}
$$

Rearranging above equation, we get

$$g(x) = \frac{\mathcal{T}_0 + (\mathcal{T}_0 + \mathcal{T}_{-1})x + (\mathcal{T}_0 + \mathcal{T}_{-1} + \mathcal{T}_{-2})x^2}{1 + x + x^2 - x^3}$$

which equals the $\sum\limits_{n=0}^{\infty} \mathcal{T}_n x^n$ in the theorem.

Tribonacci-Lucas case can be proved similarly. □

Now, we will obtain generating functions for Tribonacci and Tribonacci-Lucas numbers in terms of Tribonacci and Tribonacci-Lucas matrix sequences with negative indices as a consequence of Theorem 2.7. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.7. Thus we have the following corollary.

Corollary 2.8. *The generating functions for the Tribonacci sequence $\{T_{-n}\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_{-n}\}_{n\geq 0}$ are given as*

$$\sum_{n=0}^{\infty} T_{-n}x^n = \frac{x^2}{1+x+x^2-x^3} \quad \textit{and} \quad \sum_{n=0}^{\infty} K_{-n}x^n = \frac{x^2+2x+3}{1+x+x^2-x^3},$$

respectively.

Note that using above Corollary we can obtain Binet formulas for T_{-n} and K_{-n} again.

3. Relation Between Tribonacci and Tribonacci-Lucas Matrix Sequences With Negative Indices

The following theorem shows that there always exist interrelation between Tribonacci and Tribonacci-Lucas matrix sequences with negative indices.

Theorem 3.1. *For the matrix sequences $\{\mathcal{T}_{-n}\}_{n\geq 0}$ and $\{\mathcal{K}_{-n}\}_{n\geq 0}$, we have the following identities.*

(a) $\mathcal{K}_{-n} = 3\mathcal{T}_{-n+1} - 2\mathcal{T}_{-n} - \mathcal{T}_{-n-1}$,

(b) $\mathcal{K}_{-n} = \mathcal{T}_{-n} + 2\mathcal{T}_{-n-1} + 3\mathcal{T}_{-n-2}$,

(c) $\mathcal{K}_{-n} = -\mathcal{T}_{-n+2} + 4\mathcal{T}_{-n+1} - \mathcal{T}_{-n}$,

(d) $\mathcal{T}_{-n} = \frac{1}{22}(5\mathcal{K}_{-n+2} - 3\mathcal{K}_{-n+1} - 4\mathcal{K}_{-n})$.

Proof. From (1.3), (1.4) and (1.5), (a), (b) (c) follow. It is easy to show that $K_{-n} = -T_{-n+2} + 4T_{-n+1} - T_{-n}$ and $22T_{-n} = 5K_{-n+2} - 3K_{-n+1} - 4K_{-n}$, so now (d) and (e) follow. □

Lemma 3.2. *For all non-negative integers m and n, we have the following identities.*

(a) $\mathcal{K}_0\mathcal{T}_{-n}=\mathcal{T}_{-n}\mathcal{K}_0=\mathcal{K}_{-n}$,

(b) $\mathcal{T}_0\mathcal{K}_{-n}=\mathcal{K}_{-n}\mathcal{T}_0=\mathcal{K}_{-n}$.

Proof. Identities can be established easily. Note that to show (a) we need to use all the relations (1.3), (1.4) and (1.5). □

Next corollary gives another relation between the numbers T_{-n} and K_{-n} and also the matrices $\mathcal{T}_{-n}$ and $\mathcal{K}_{-n}$.

Corollary 3.3. *We have the following identities.*

(a) $T_{-n}=\frac{1}{22}(K_{-n}+5K_{-n-1}+2K_{-n+1})$,

(b) $\mathcal{T}_{-n}=\frac{1}{22}(\mathcal{K}_{-n}+5\mathcal{K}_{-n-1}+2\mathcal{K}_{-n+1})$.

Proof. From Lemma 3.2(a), we know that $\mathcal{K}_0\mathcal{T}_{-n}=\mathcal{K}_{-n}$. To show (a), use Theorem 2.2 for the matrix $\mathcal{T}_{-n}$ and calculate the matrix operation $\mathcal{K}_0^{-1}\mathcal{K}_{-n}$ and then compare the 2nd row and 1st column entries with the matrices $\mathcal{T}_{-n}$ and $\mathcal{K}_0^{-1}\mathcal{K}_{-n}$. Now (b) follows from (a). □

The following theorem shows that there exist relation between the positive indices and negative indices for Tribonacci matrix sequences.

Theorem 3.4. *For $n\geq 0$, we have the following identity:*

$$\mathcal{T}_{-n}=(\mathcal{T}_n)^{-1}. \tag{3.1}$$

Proof. We prove by mathematical induction. If $n=0$ then we have

$$\mathcal{T}_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}^{-1}=(\mathcal{T}_0)^{-1}$$

which is true and

$$\mathcal{T}_{-1}=\begin{pmatrix}0&1&0\\0&0&1\\1&-1&-1\end{pmatrix}=\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\end{pmatrix}^{-1}=(\mathcal{T}_1)^{-1}$$

which is true. Assume that the equality holds for $n\leq k$. For $n=k+1$, by using Theorem 1.3, we obtain

$$\begin{aligned}(\mathcal{T}_{k+1})^{-1}&=(\mathcal{T}_k\mathcal{T}_1)^{-1}=(\mathcal{T}_1)^{-1}(\mathcal{T}_k)^{-1}=\mathcal{T}_{-1}\mathcal{T}_{-k}\\
&=\begin{pmatrix}0&1&0\\0&0&1\\1&-1&-1\end{pmatrix}\begin{pmatrix}T_{-k+1}&T_{-k}+T_{-k-1}&T_{-k}\\T_{-k}&T_{-k-1}+T_{-k-2}&T_{-k-1}\\T_{-k-1}&T_{-k-2}+T_{-k-3}&T_{-k-2}\end{pmatrix}\\
&=\begin{pmatrix}T_{-k}&T_{-k-1}+T_{-k-2}&T_{-k-1}\\T_{-k-1}&T_{-k-2}+T_{-k-3}&T_{-k-2}\\T_{1-k}-T_{-k-1}-T_{-k}&T_{-k}-2T_{-k-2}-T_{-k-3}&T_{-k}-T_{-k-1}-T_{-k-2}\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} T_{-k} & T_{-k-1}+T_{-k-2} & T_{-k-1} \\ T_{-k-1} & T_{-k-2}+T_{-k-3} & T_{-k-2} \\ T_{-k-2} & T_{-k-3}+T_{-k-4} & T_{-k-3} \end{pmatrix} \\
&= \begin{pmatrix} T_{-(k+1)+1} & T_{-(k+1)}+T_{-(k+1)-1} & T_{-(k+1)} \\ T_{-(k+1)} & T_{-(k+1)-1}+T_{-(k+1)-2} & T_{-(k+1)-1} \\ T_{-(k+1)-1} & T_{-(k+1)-2}+T_{-(k+1)-3} & T_{-(k+1)-2} \end{pmatrix} \\
&= \mathcal{T}_{-(k+1)}.
\end{aligned}
$$

Thus, by induction on n, this proves (3.1). □

In the following Theorem we will use the next lemma.

Lemma 3.5 ([23]). *Let A_1, B_1, C_1; A_2, B_2, C_2 as in Theorem 1.4. Then the following relations hold:*

$$
\begin{aligned}
&A_1^2 = A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\
&A_1B_1 = B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0).
\end{aligned}
$$

As the following theorem shows $\mathcal{T}_{-m}$ and $\mathcal{K}_{-m}$ has nice properties.

Theorem 3.6. *For $n, m \geq 0$, we have the following identities:*

(a) $\mathcal{T}_{-m}\mathcal{T}_{-n} = \mathcal{T}_{-m-n}$

(b) $\mathcal{T}_{m}\mathcal{T}_{-n} = \mathcal{T}_{m-n}$

(c) $\mathcal{T}_{-m}\mathcal{K}_{-n} = \mathcal{K}_{-n}\mathcal{T}_{-m} = \mathcal{K}_{-m-n}$

(d) $\mathcal{T}_{m}\mathcal{K}_{-n} = \mathcal{K}_{m}\mathcal{T}_{-n} = \mathcal{K}_{m-n}$

(e) $\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_{0}\mathcal{K}_{-m-n}$

(f) $\mathcal{K}_{m}\mathcal{K}_{-n} = \mathcal{K}_{0}\mathcal{K}_{m-n}$

(g) $\sum\limits_{i=0}^{n-1} \mathcal{T}_{-mi-j} = (\mathcal{T}_{-mn+m-j} - \mathcal{T}_{m-j})(\mathcal{T}_0 - \mathcal{T}_m)^{-1}$.

Proof. (a) Using Theorems 3.4 and 1.3, we obtain

$$
\mathcal{T}_{-m}\mathcal{T}_{-n} = (\mathcal{T}_m)^{-1}(\mathcal{T}_n)^{-1} = (\mathcal{T}_n\mathcal{T}_m)^{-1} = (\mathcal{T}_{m+n})^{-1} = \mathcal{T}_{-m-n}.
$$

(b) Using (2.9), Lemma 3.5 and Theorem 1.3 we obtain

$$
\begin{aligned}
\mathcal{T}_m\mathcal{T}_{-n} &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^{-m})(A_3\alpha^{-n} + B_3\beta^n + C_3\gamma^{-n}) \\
&= A_1^2\alpha^{m-n} + B_1^2\beta^{m-n} + C_1^2\gamma^{m-n} + A_1B_1\alpha^m\beta^{-n} + B_1A_1\alpha^{-n}\beta^m \\
&\quad + A_1C_1\alpha^m\gamma^{-n} + C_1A_1\alpha^{-n}\gamma^m + B_1C_1\beta^m\gamma^{-n} + C_1B_1\beta^{-n}\gamma^m \\
&= A_1\alpha^{m-n} + B_1\beta^{m-n} + C_1\gamma^{m-n} \\
&= \mathcal{T}_{m-n}.
\end{aligned}
$$

(c) By Lemma 3.2(a) we have

$$
\mathcal{T}_{-m}\mathcal{K}_{-n} = \mathcal{T}_{-m}\mathcal{T}_{-n}\mathcal{K}_0.
$$

By (a) and again by Lemma 3.2(a) we obtain

$$\mathcal{T}_{-m}\mathcal{K}_{-n} = \mathcal{T}_{-m-n}\mathcal{K}_0 = \mathcal{K}_{-m-n}.$$

The other equality can be obtained similarly.

(d) By Lemma 3.2(a) we have

$$\mathcal{T}_m\mathcal{K}_{-n} = \mathcal{T}_m\mathcal{T}_{-n}\mathcal{K}_0.$$

By (b) and again by Lemma 3.2(a) we obtain

$$\mathcal{T}_m\mathcal{K}_{-n} = \mathcal{T}_{m-n}\mathcal{K}_0 = \mathcal{K}_{m-n}.$$

The other equality can be obtained similarly.

(e) By Lemma 3.2(a) we have

$$\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_0\mathcal{T}_{-m}\mathcal{K}_{-n}.$$

By (c) we obtain

$$\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_0\mathcal{K}_{-m-n}.$$

The other equality can be obtained similarly.

(f) By Theorem 1.3 we can write

$$\mathcal{K}_m\mathcal{K}_{-n} = \mathcal{K}_0\mathcal{T}_m\mathcal{K}_{-n}.$$

From (d), we obtain

$$\mathcal{K}_m\mathcal{K}_{-n} = \mathcal{K}_0\mathcal{K}_{m-n}.$$

(g) We use (b). Since

$$\left(\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j}\right)\mathcal{T}_m = \sum_{i=0}^{n-1}\mathcal{T}_{-mi-j+m} = \mathcal{T}_{m-j} + \left(\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j}\right) - \mathcal{T}_{-m(n-1)-j},$$

we obtain

$$\mathcal{T}_{-m(n-1)-j} - \mathcal{T}_{m-j} = \left(\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j}\right) - \left(\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j}\right)\mathcal{T}_m = \left(\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j}\right)(\mathcal{T}_0 - \mathcal{T}_m)$$

and so

$$\sum_{i=0}^{n-1}\mathcal{T}_{-mi-j} = (\mathcal{T}_{-mn+m-j} - \mathcal{T}_{m-j})(\mathcal{T}_0 - \mathcal{T}_m)^{-1}.$$

□

Theorem 3.7. *For all non-negative integers m and n, we have the following identities.*

(a) $\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_{-n}\mathcal{K}_{-m} = 9\mathcal{T}_{-m-n+2} - 12\mathcal{T}_{-m-n+1} - 2\mathcal{T}_{-m-n} + 4\mathcal{T}_{-m-n-1} + \mathcal{T}_{-m-n-2}$,

(b) $\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_{-n}\mathcal{K}_{-m} = \mathcal{T}_{-m-n} + 4\mathcal{T}_{-m-n-1} + 10\mathcal{T}_{-m-n-2} + 12\mathcal{T}_{-m-n-3} + 9\mathcal{T}_{-m-n-4}$,

(c) $\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_{-n}\mathcal{K}_{-m} = \mathcal{T}_{-m-n} - 8\mathcal{T}_{-m-n+1} + 18\mathcal{T}_{-m-n+2} - 8\mathcal{T}_{-m-n+3} + \mathcal{T}_{-m-n+4}$.

Proof. (a) Using Theorem 3.6 (a) and Theorem 3.1 (a) we obtain

$$\begin{aligned}\mathcal{K}_{-m}\mathcal{K}_{-n} &= (3\mathcal{T}_{-m+1} - 2\mathcal{T}_{-m} - \mathcal{T}_{-m-1})(3\mathcal{T}_{-n+1} - 2\mathcal{T}_{-n} - \mathcal{T}_{-n-1})\\ &= 2\mathcal{T}_{-n}\mathcal{T}_{-m-1} - 6\mathcal{T}_{-n}\mathcal{T}_{-m+1} + 2\mathcal{T}_{-m}\mathcal{T}_{-n-1} - 6\mathcal{T}_{-m}\mathcal{T}_{-n+1} + 4\mathcal{T}_{-m}\mathcal{T}_{-n}\\ &\quad + \mathcal{T}_{-m-1}\mathcal{T}_{-n-1} - 3\mathcal{T}_{-m-1}\mathcal{T}_{-n+1} - 3\mathcal{T}_{-m+1}\mathcal{T}_{-n-1} + 9\mathcal{T}_{-m+1}\mathcal{T}_{-n+1}\end{aligned}$$

$$\begin{aligned}&= 2\mathcal{T}_{-m-n-1} - 6\mathcal{T}_{-m-n+1} + 2\mathcal{T}_{-m-n-1} - 6\mathcal{T}_{-m-n+1} + 4\mathcal{T}_{-m-n} + \mathcal{T}_{-m-n-2}\\ &\quad - 3\mathcal{T}_{-m-n} - 3\mathcal{T}_{-m+n} + 9\mathcal{T}_{-m-n+2}\\ &= 9\mathcal{T}_{-m-n+2} - 12\mathcal{T}_{-m-n+1} - 2\mathcal{T}_{-m-n} + 4\mathcal{T}_{-m-n-1} + \mathcal{T}_{-m-n-2}.\end{aligned}$$

It can be shown similarly that

$$\mathcal{K}_{-n}\mathcal{K}_{-m} = 9\mathcal{T}_{-m-n+2} - 12\mathcal{T}_{-m-n+1} - 2\mathcal{T}_{-m-n} + 4\mathcal{T}_{-m-n-1} + \mathcal{T}_{-m-n-2}.$$

The remaining of identities can be proved by considering again (a) and Theorem 3.1. □

Comparing matrix entries and using Theorem 2.2 we have next two result.

Corollary 3.8. *For Tribonacci and Tribonacci-Lucas numbers, we have the following identities:*

(a) $T_{-m-n} = T_{-m}T_{-n+1} + T_{-n}(T_{-m-1} + T_{-m-2}) + T_{-m-1}T_{-n-1}$

(b) $K_{-m-n} = T_{-m}K_{-n+1} + K_{-n}(T_{-m-1} + T_{-m-2}) + K_{-n-1}T_{-m-1}$

(c) $K_{-m}K_{-n+1} + K_{-n}(K_{-m-1} + K_{-m-2}) + K_{-m-1}K_{-n-1} = 9T_{-m-n+2} - 12T_{-m-n+1} - 2T_{-m-n} + 4T_{-m-n-1} + T_{-m-n-2}$

(d) $K_{-m}K_{-n+1} + K_{-n}(K_{-m-1} + K_{-m-2}) + K_{-m-1}K_{-n-1} = T_{-m-n} + 4T_{-m-n-1} + 10T_{-m-n-2} + 12T_{-m-n-3} + 9T_{-m-n-4}$

(e) $K_{-m}K_{-n+1} + K_{-n}(K_{-m-1} + K_{-m-2}) + K_{-m-1}K_{-n-1} = T_{-m-n} - 8T_{-m-n+1} + 18T_{-m-n+2} - 8T_{-m-n+3} + T_{-m-n+4}$

The following theorem shows that there exist relation between the positive indices and negative indices for Tribonacci-Lucas matrix sequences.

Theorem 3.9. *For all non-negative integers n, we have the following identity:*

$$\mathcal{K}_{-n} = (\mathcal{K}_0)^{1-n}(\mathcal{K}_{-1})^n.$$

Proof. Taking $(n-1)$ for m and 1 for n in $\mathcal{K}_{-m}\mathcal{K}_{-n} = \mathcal{K}_0\mathcal{K}_{-m-n}$ which is given in Theorem 3.6 (e), we obtain that

$$\mathcal{K}_0\mathcal{K}_{-n} = \mathcal{K}_{-n+1}\mathcal{K}_{-1}. \tag{3.2}$$

If we multiply both side of the equation (3.2) with $\mathcal{K}_0$ we have the relation

$$\begin{aligned}\mathcal{K}_0\mathcal{K}_0\mathcal{K}_{-n} &= \mathcal{K}_0\mathcal{K}_{-n+1}\mathcal{K}_{-1}\\ &= \mathcal{K}_{-n+2}\mathcal{K}_{-1}\mathcal{K}_{-1}.\end{aligned}$$

Repeating this process we then obtain

$$\mathcal{K}_0^{n-1}\mathcal{K}_{-n} = \mathcal{K}_{-1}^n.$$

Thus, it follows that

$$\mathcal{K}_{-n} = \mathcal{K}_0^{1-n}\mathcal{K}_{-1}^n.$$

This completes the proof. □

Note that using Theorem 3.6(d) in Theorem 3.9, we obtain

$$\mathcal{K}_{-n} = (\mathcal{K}_n\mathcal{T}_{-n})^{1-n}\mathcal{K}_{-1}^n = \mathcal{T}_{-n}^{1-n}\mathcal{K}_n^{1-n}\mathcal{K}_{-1}^n$$

and then by Theorem 3.4 we get

$$\mathcal{K}_{-n} = (\mathcal{T}_n\mathcal{K}_n^{-1}\mathcal{K}_{-1})^{n-1}\mathcal{K}_{-1}.$$

The next two theorems provide us the convenience to obtain the powers of Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 3.10. *For non-negatif integers m,n and r with $n \geq r$, the following identities hold:*

(a) $(\mathcal{T}_{-n})^m = \mathcal{T}_{-mn}$,

(b) $(\mathcal{T}_{-n-1})^m = (\mathcal{T}_{-1})^m\mathcal{T}_{-mn}$,

(c) $\mathcal{T}_{-n-r}\mathcal{T}_{-n+r} = (\mathcal{T}_{-n})^2 = (\mathcal{T}_{-2})^n$.

Proof. (a) By Theorem 3.4 we have

$$(\mathcal{T}_{-n})^m = ((\mathcal{T}_n)^{-1})^m = ((\mathcal{T}_n)^m)^{-1}.$$

Using Theorem 1.3, and also again Theorem 3.4 we obtain

$$(\mathcal{T}_{-n})^m = (\mathcal{T}_{mn})^{-1} = \mathcal{T}_{-mn}.$$

(b) Using Theorem 3.6(a) and method used in above (a) we can write

$$(\mathcal{T}_{-n-1})^m = \mathcal{T}_{m(-n-1)} = \mathcal{T}_{-m}\mathcal{T}_{-mn} = \mathcal{T}_{-1}\mathcal{T}_{-m+1}\mathcal{T}_{-mn}.$$

Similarly, we obtain $\mathcal{T}_{-m+1} = \mathcal{T}_{-1}\mathcal{T}_{-m+2}$. Continuing to this iterative process, we then obtain

$$(\mathcal{T}_{-n-1})^m = \underbrace{\mathcal{T}_{-1}\mathcal{T}_{-1}\dots\mathcal{T}_{-1}}_{m\,\text{times}}\mathcal{T}_{-mn} = (\mathcal{T}_{-1})^m\mathcal{T}_{-mn}.$$

(c) The proof is similar to (b). □

We have analogues results for the matrix sequence $\mathcal{K}_{-n}$.

Theorem 3.11. *For non-negatif integers m, n and r with $n \geq r$, the following identities hold:*

(a) $\mathcal{K}_{-n-r}\mathcal{K}_{-n+r} = (\mathcal{K}_{-n})^2$,

(b) $(\mathcal{K}_{-n})^m = \mathcal{K}_0^m\mathcal{T}_{-mn} = \mathcal{K}_0^{m-1}\mathcal{K}_{-mn}$.

Proof. (a) Applying Theorem 3.6(e), we find

$$\mathcal{K}_{-n-r}\mathcal{K}_{-n+r} = \mathcal{K}_0\mathcal{K}_{-2n} = \mathcal{K}_0\mathcal{K}_{-n-n} = \mathcal{K}_{-n}\mathcal{K}_{-n} = (\mathcal{K}_{-n})^2.$$

(b) By Lemma 3.2(a), we see that

$$(\mathcal{K}_{-n})^m = (\mathcal{K}_0\mathcal{T}_{-n})^m = (\mathcal{K}_0)^m(\mathcal{T}_{-n})^m.$$

Using Theorem 3.6(a), we obtain

$$(\mathcal{K}_{-n})^m = \mathcal{K}_0^m\mathcal{T}_{-mn}.$$

Thus, we can write $(\mathcal{K}_{-n})^m = \mathcal{K}_0^{m-1}\mathcal{K}_0\mathcal{T}_{-mn}$ and by Lemma 3.2(a) we get $(\mathcal{K}_{-n})^m = \mathcal{K}_0^{m-1}\mathcal{K}_{-mn}$. This completes the proof. □

4. On the Theorem of Rabinowitz and Bruckman

We now present a remarkable theorem of Rabinowitz and Bruckman [15].

Theorem 4.1 (Rabinowitz, 1994 [17], solution: Bruckman, 1995 [15])**.** *Assume that H_n satisfies a second-order linear recurrence with constant coefficients. Let $\{a_i\}$ and $\{b_i\}$, $i = 1,2,\dots,r$ be integer constants and let $f(x_0,x_1,x_2,\dots,x_r)$ be a polynomial with integer coefficients. If the expression*

$$f((-1)^n,H_{a_1n+b_1},H_{a_2n+b_2},\dots,H_{a_rn+b_r})$$

vanishes for all integers $n > N$ then the expression vanishes for all integers n.

(*As a special case, if an identity involving Fibonacci and Lucas numbers is true for all positive subscripts, then it must be true for all negative subscripts as well.*)

Proof. For details, see Bruckman [15]. □

It follows from above Theorem that if an identity involving generalized Fibonacci numbers (Horadam numbers) is true for all positive subscripts, then it is true for all non-positive subscripts as well.

After seeing the results of this paper, we can propose the following conjecture.

Conjecture 4.2. *Assume that H_n satisfies an order-k linear homogeneous recurrences with constant coefficients. Let $\{a_i\}$ and $\{b_i\}$, $i = 1,2,\dots,r$ be integer constants and let $f(x_0,x_1,x_2,\dots,x_r)$ be a polynomial with integer coefficients. If the expression*

$$f(H_{a_1n+b_1},H_{a_2n+b_2},\dots,H_{a_rn+b_r})$$

vanishes for all integers $n > N$ then the expression vanishes for all integers n.

It seems that the proof of the Theorem 4.1 can be applied to prove the above conjecture. It follows that if Conjecture 4.2 is true then if an identity involving generalized Tribonacci numbers as a third-order linear recurrence with constant coefficients (such as Tribonacci and Tribonacci-Lucas numbers) is true for all positive subscripts, then it is true for all non-positive subscripts as well.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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