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Research Article

Gaussian Quadrature for Two-Point Singularly Perturbed Boundary Value Problems with Exponential Fitting

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Abstract. In this paper, the Gaussian quadrature method with exponential fitting is proposed for the solution of two-point singularly perturbed boundary value problems with layer at one endpoint, dual boundary layers and internal boundary layers. The given boundary value problem is reduced into an equivalent first order differential equation with the perturbation parameter as deviating argument. Then, Gaussian two-point quadrature technique with exponential fitting is implemented to solve the first order equation with deviating parameter. The analysis of the convergence of the method is discussed. Several numerical examples are illustrated with a layer at one end, a layer at two ends and internal layers. Comparison of maximum errors in the solution of the examples with other methods is shown to justify the method.

Keywords. Singular perturbation problem; Boundary layer; Gaussian quadrature; Dual layer; Internal layer

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1. Introduction

Singularly perturbed problems have an important role in science and engineering. In several areas of applied mathematics such as quantum mechanics, fluid mechanics, chemical-reactor theory, optimal control, aerodynamics, reaction-diffusion process, geophysics and many other areas these problems have a great significance. Problems of this kind typically show solutions with boundary/interior layers; that is, the solution of the problem contains narrower domains where the solution derivatives are extremely hard. The mathematical treatment of such problems gives great computational difficulties due to the existence of layer behaviour. Various authors have developed a large number of special purpose methods for two-point singularly perturbed boundary value problems [5, 6, 8, 10, 13, 16]. Awoke and Reddy [2] proposed an exponentially fitted special second-order difference method for boundary value problems with the layer behaviour at one endpoint. A class of nonlinear singularly perturbed boundary value problems with a layer at one end is solved using a boundary value method by Attili [1]. Bawa [4] derived the fourth order finite difference method using cubic spline in compression for the solution of linear two-point singularly perturbed boundary value problem.

Hosseini and Shahraki [17] gives a new method which is a grouping of the upwind finite difference scheme and central finite difference method on a particular variable mesh for the perturbed boundary value problems. Kadalbajoo and Patidar [9] suggested a straightforward and direct method which converges uniformly to solve singular perturbation problem on a piecewise uniform mesh of Shishkin type. Mohammadi [11] developed a numerical method of order two using adaptive cubic spline functions for solving a class of two-point singular perturbation boundary value problems. Gupta and Kumar [7] introduce a scheme for the solution of perturbed problems with second order and third order by the method of multiple scales. Reddy [15] has given a numerical integration method using Simpson's formula for solving two-point singular perturbation problems with a layer on the left terminal of the domain. Aziz and Khan [3] derived a finite difference method using cubic spline in compression for the two-point singular perturbation problem.

The rest of the paper is organized as follows. The problem is described precisely and the numerical scheme is derived for the layer at left-end, right-end, dual layer and internal layer in Section 2. Convergence of the scheme is analyzed in Section 3. Numerical examples with results are provided in Section 4. Finally, some discussions and conclusion are given in the last section.

2. Description of the Method

Consider a two-point singularly perturbed boundary value problem:

$$\varepsilon x''(t)+a(t)x'(t)+b(t)x(t)=f(t), \quad p\le t\le q \tag{1}$$

with boundary conditions

$$x(p)=\alpha \text{ and } x(q)=\beta. \tag{2}$$

Here, $0<\varepsilon\ll 1$ is a small perturbation parameter, $a(t)$, $b(t)$, $f(t)$ are continuous functions in $[p,q]$ and α,β are finite constants. If $a(t)\ge P>0$ all through the domain $[p,q]$, where P is positive constant, then the layer exists in the vicinity of $t=p$. If $a(t)\le P<0$ over the domain $[p,q]$, where P is negative constant, then the layer will be in the vicinity of $t=q$.

2.1 Left-end Boundary Layer Problem

With the Taylor's expansion about the point x, we have

$$x'(t-\varepsilon) \approx x'(t) - \varepsilon x''(t)$$

implies

$$\varepsilon x''(t) = x'(t) - x'(t-\varepsilon). \tag{3}$$

In view of eq. (3), eq. (1) is replaced by the following differential equation with perturbation parameter as a small deviating argument:

$$x'(t) = x'(t-\varepsilon) - a(t)x'(t) - b(t)x(t) + f(t). \tag{4}$$

The domain $[p,q]$ is partitioned into N sub domains of mesh size $h = \frac{q-p}{N}$ so that $t_i = p + ih$, $i = 0,1,\dots,N+1$ are the mesh points. Taking integration on eq. (4) with respect to x from t_i to t_{i+1}, we get

$$\int_{t_i}^{t_{i+1}} x'(t) = \int_{t_i}^{t_{i+1}} (x'(t-\varepsilon) - a(t)x'(t) - b(t)x(t) + f(t))dt, \tag{5}$$

$$x_{i+1} - x_i = \int_{t_i}^{t_{i+1}} x'(t-\varepsilon)dt - \int_{t_i}^{t_{i+1}} a(t)x'(t)dt - \int_{t_i}^{t_{i+1}} b(t)x(t)dt + \int_{t_i}^{t_{i+1}} f(t)dt.$$

Using Gaussian two-point quadrature formula, we have

$$\int_{-1}^{1} F(t)dt = F\left(\frac{1}{\sqrt{3}}\right) + F\left(\frac{-1}{\sqrt{3}}\right).$$

For any continuous and differentiable function $F(x)$ in an arbitrary interval $[t_i, t_{i+1}]$, the Gaussian two-point quadrature formula becomes

$$\int_{t_i}^{t_{i+1}} F(t)dt = \frac{h}{2}(F(t_i+k) + F(t_{i+1}-k)), \tag{6}$$

where $k = \frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2}$. Using eq. (6) in eq. (5), we have

$$\begin{aligned} x_{i+1} - x_i = {} & x(t_{i+1}-\varepsilon) - x(t_i-\varepsilon) - a(t_{i+1})x(t_{i+1}) + a(t_i)x(t_i) \\ & + \frac{h}{2}\left[a'(t_{i+1}-k)x(t_{i+1}-k) + a'(t_i+k)x(t_i+k)\right] \\ & - \frac{h}{2}[b(t_{i+1}-k)x(t_{i+1}-k) + b(t_i+k)x(t_i+k)] + \frac{h}{2}[f(t_{i+1}-k) + f(t_i+k)]. \end{aligned} \tag{7}$$

Using the linear interpolation for $x(t_{i+1}-\varepsilon)$, $x(t_i-\varepsilon)$, $x(t_i-k)$ and $x(t_{i+1}-k)$, the eq. (7) reduces to

$$\begin{aligned} & \left\{\frac{\varepsilon}{h} + a'(t_i+k)\frac{k}{2} - b(t_i+k)\frac{k}{2}\right\} x_{i-1} \\ & + \left\{\frac{-2\varepsilon}{h} - a(t_i) - a'(t_{i+1}-k)\frac{k}{2} - a'(t_i+k)\left(\frac{h+k}{2}\right) + b(t_{i+1}-k)\frac{k}{2} + b(t_i+k)\left(\frac{h+k}{2}\right)\right\} x_i \\ & + \left\{\frac{\varepsilon}{h} - a'(t_{i+1}-k)\left(\frac{h-k}{2}\right) + a(t_{i+1}) + b(t_i+k)\left(\frac{h-k}{2}\right)\right\} x_{i+1} \\ & \quad = \frac{h}{2}\left\{f(t_{i+1}-k) + f(t_i+k)\right\}. \end{aligned}$$

Rearranging this equation, we have

$$\frac{\varepsilon}{h^2}(x_{i-1}-2x_i+x_{i+1})+\left\{\frac{a'(t_i+k)\left(\frac{k}{2}\right)-b(t_i+k)\left(\frac{k}{2}\right)}{h}\right\}x_{i-1}$$
$$+\left\{\frac{-a(t_i)-a'(t_{i+1}-k)\left(\frac{k}{2}\right)-a'(t_i+k)\left(\frac{h+k}{2}\right)+b(t_{i+1}-k)\left(\frac{k}{2}\right)+b(t_i+k)\left(\frac{h+k}{2}\right)}{h}\right\}x_i$$
$$+\left\{\frac{-a'(t_{i+1}-k)\left(\frac{h-k}{2}\right)+a(t_{i+1})+b(t_{i+1}-k)\left(\frac{h-k}{2}\right)}{h}\right\}x_{i+1}$$
$$=\left\{\frac{f(t_{i+1}+k)+f(t_i+k)}{2}\right\}. \tag{8}$$

Inserting a fitting factor $\sigma(\rho)$ in the above scheme to control the layer behavior due the perturbation parameter ε, we get

$$\frac{\sigma\varepsilon}{h^2}(x_{i-1}-2x_i+x_{i+1})+\left\{\frac{a'(t_i+k)\left(\frac{k}{2}\right)-b(t_i+k)\left(\frac{k}{2}\right)}{h}\right\}x_{i-1}$$
$$+\left\{\frac{-a(t_i)-a'(t_{i+1}-k)\left(\frac{k}{2}\right)-a'(t_i+k)\left(\frac{h+k}{2}\right)+b(t_{i+1}-k)\left(\frac{k}{2}\right)+b(t_i+k)\left(\frac{h+k}{2}\right)}{h}\right\}x_i$$
$$+\left\{\frac{-a'(t_{i+1}-k)\left(\frac{h-k}{2}\right)+a(t_{i+1})+b(t_{i+1}-k)\left(\frac{h-k}{2}\right)}{h}\right\}x_{i+1}$$
$$=\left\{\frac{f(t_{i+1}+k)+f(t_i+k)}{2}\right\}.$$

The value of $\sigma(\rho)$ is acquired by the procedure given by Doolan et al. [6] and is given by

$$\sigma(\rho)=\frac{\rho a(0)e^{-\left(\frac{a(0)\rho}{2}\right)}}{2\sinh\left(\frac{a(o)\rho}{2}\right)}. \tag{9}$$

Eq. (8) can be rewritten in a three term recurrence relation as follows:

$$A_{iL}x_{i-1}+B_{iL}x_i+C_{iL}x_{i+1}=F_{iL};\quad i=1,2,\dots,N, \tag{10}$$

where

$$A_{iL}=\frac{\varepsilon}{h}+a'(t_i+k)\frac{k}{2}-b(t_{i+1}+k)\frac{k}{2},$$
$$B_{iL}=\frac{-2\varepsilon}{h}-a(t_i)-a'(t_{i+1}+k)\frac{k}{2}-a'(t_i+k)\left(\frac{h+k}{2}\right)+b(t_{i+1}+k)\left(\frac{k}{2}\right)+b(t_i+k)\left(\frac{h+k}{2}\right),$$
$$C_{iL}=\frac{\varepsilon}{h}-a'(t_{i+1}-k)\left(\frac{h-k}{2}\right)+a(t_{i+1})+b(t_{i+1}-k)\left(\frac{h-k}{2}\right),$$
$$F_{iL}=\frac{h}{2}[f(t_{i+1}-k)+f(t_i+k)].$$

The tri-diagonal system eq. (10) is solved efficiently by using Thomas Algorithm [2].

2.2 Right-end Boundary Layer Problem

Taylor series expansion of $x'(t+\varepsilon)$ giving

$$x'(t+\varepsilon)\approx x'(t)+\varepsilon x''(t)$$

implies

$$\varepsilon x''(t) \approx x'(t+\varepsilon) - x'(t) \tag{11}$$

and accordingly, eq. (1) is reduced to the following equation with a small deviating argument:

$$x'(t) = x'(t+\varepsilon) + a(t)x'(t) + b(t)x(t) - f(t). \tag{12}$$

Integrating eq. (12) on $[t_{i-1}, t_i]$, we get

$$\int_{t_{i-1}}^{t_i} x'(t)dt = \int_{x_{i-1}}^{x_i} x'(t+\varepsilon)dt + \int_{t_{i-1}}^{t_i} a(t)x'(t)dt + \int_{x_{i-1}}^{x_i} b(t)x(t)dt - \int_{t_{i-1}}^{t_i} f(t)dt.$$

Using Gaussian quadrature two-point formula for any continuous and differentiable function $F(t)$ in an arbitrary interval $[t_{i-1}, t_i]$, we get

$$\int_{t_{i-1}}^{t_i} F(t)dt = \frac{h}{2}(F(t_{i-1}-k) + F(t_i+k)). \tag{13}$$

Using eq. (13), from eq. (12), we get

$$\begin{aligned} x(t_i) - x(t_{i-1}) = x(t_{i-1}-\varepsilon) - x(t_i+\varepsilon) + a(t_{i-1})x(t_{i-1}) - a(t_i)x(t_i) \\ - \frac{h}{2}\left[a'(t_{i-1}-k)x(t_{i-1}-k) + a'(t_i+k)x(t_i+k)\right] \\ + \frac{h}{2}[b(t_{i-1}-k)x(t_{i-1}-k) + b(t_i+k)x(t_i+k)] - \frac{h}{2}[f(t_{i-1}-k) + f(t_i+k)]. \end{aligned} \tag{14}$$

Using linear interpolation for the terms $x(t_{i-1}-\varepsilon)$, $x(t_i+\varepsilon)$, $x(t_{i-1}-k)$ and $x(t_i+k)$, we get

$$\begin{aligned} &\left\{\frac{\varepsilon}{h} - a(t_{i-1}) - a'(t_{i-1}+k)\left(\frac{h-k}{2}\right) + b(t_{i-1}+k)\left(\frac{h-k}{2}\right)\right\}x_{i-1} \\ &+ \left\{\frac{-2\varepsilon}{h} + a(t_i) - a'(t_i-k)\left(\frac{h+k}{2}\right) - a'(t_{i-1}+k)\left(\frac{k}{2}\right) + b(t_i-k)\left(\frac{h+k}{2}\right) + b(t_{i-1}+k)\left(\frac{k}{2}\right)\right\}x_i \\ &+ \left\{\frac{\varepsilon}{h} + a'(t_i-k)\left(\frac{k}{2}\right) - b(t_i-k)\left(\frac{k}{2}\right)\right\}x_{i+1} \\ &= \frac{h}{2}\{f(t_i-k) + f(t_{i-1}+k)\}. \end{aligned} \tag{15}$$

Rearranging this equation, we have

$$\begin{aligned} &\frac{\varepsilon}{h^2}(x_{i-1} - 2x_i + x_{i+1}) + \left\{\frac{-a(t_{i+1}) - a'(t_{i-1}+k)\left(\frac{h-k}{2}\right) + b(t_{i-1}+k)\left(\frac{h-k}{2}\right)}{h}\right\}x_{i-1} \\ &+ \left\{\frac{a(t_i) - a'(t_i-k)\left(\frac{h+k}{2}\right) - a'(t_{i-1}+k)\left(\frac{k}{2}\right) + b(t_i-k)\left(\frac{h+k}{2}\right) + b(t_{i-1}+k)\left(\frac{k}{2}\right)}{h}\right\}x_i \\ &+ \left\{\frac{a'(t_i-k)\left(\frac{k}{2}\right) - b(t_i-k)\left(\frac{k}{2}\right)}{h}\right\}x_{i+1} \\ &= \left\{\frac{f(t_i-k) + f(t_{i-1}+k)}{2}\right\}. \end{aligned}$$

Introducing a fitting factor $\sigma(\rho)$ in the above scheme to control the layer behavior due the perturbation parameter ε, we get

$$\frac{\sigma\varepsilon}{h^2}(x_{i-1} - 2x_i + x_{i+1}) + \left\{\frac{-a(t_{i+1}) - a'(t_{i-1}+k)\left(\frac{h-k}{2}\right) + b(t_{i-1}+k)\left(\frac{h-k}{2}\right)}{h}\right\}x_{i-1}$$

$$+\left\{\frac{a(t_i)-a'(t_i-k)\left(\frac{h+k}{2}\right)-a'(t_{i-1}+k)\left(\frac{k}{2}\right)+b(t_i-k)\left(\frac{h+k}{2}\right)+b(t_{i-1}+k)\left(\frac{k}{2}\right)}{h}\right\}x_i$$
$$+\left\{\frac{a'(t_i-k)\left(\frac{k}{2}\right)-b(t_i-k)\left(\frac{k}{2}\right)}{h}\right\}x_{i+1}$$
$$=\left\{\frac{f(t_i-k)+f(t_{i-1}+k)}{2}\right\}.$$

The value of $\sigma(\rho)$ is obtained by using the procedure given in Doolan et al. [6] and is:

$$\sigma(\rho)=\frac{\rho a(0)e^{\left(\frac{a(n+1)\rho}{2}\right)}}{2\sinh\left(\frac{a(n+1)\rho}{2}\right)}. \tag{16}$$

From eq. (15), we can re write the tri-diagonal system:

$$A_{iR}x_{i-1}+B_{iR}x_i+C_{iR}x_{i+1}=F_{iR} \quad \text{for } 1\le i\le N-1, \tag{17}$$

where

$$A_{iR}=\left\{\frac{\varepsilon}{h}-a(t_{i-1})-a'(t_{i-1}+k)\frac{h-k}{2}+b(t_{i-1}+k)\frac{h-k}{2}\right\},$$
$$B_{iR}=\left\{\frac{-2\varepsilon}{h}+a(t_i)-a'(t_i+k)\frac{h+k}{2}-a'(t_{i-1}+k)\left(\frac{k}{2}\right)+b(t_i-k)\left(\frac{h+k}{2}\right)+b(t_{i-1}+k)\left(\frac{k}{2}\right)\right\},$$
$$C_{iR}=\left\{\frac{\varepsilon}{h}+a'(t_i-k)\left(\frac{k}{2}\right)-b(t_i-k)\left(\frac{k}{2}\right)\right\},$$
$$F_{iR}=\frac{h}{2}[f(t_i-k)+f(t_{i-1}+k)].$$

The system of eq. (17) is solved by using the Thomas algorithm [2].

2.3 Dual Boundary Layer Problems

Here, eq. (1) is considered over the domain $[p,q]$. The functions $a(t)$, $b(t)$ and $f(t)$ are assumed to be sufficiently smooth such that

$$\begin{aligned}
&a(t_m)=0,\ a'(t_m)<0, && \text{where } t_m=\frac{p+q}{2},\\
&|a(t)|\ge a_0>0 && \text{for } p<t\le q,\\
&b(t)\le b_0<0, && \text{for all } t\in[p,q],\\
&|a'(t)|\ge\frac{|a'(m)|}{2}, && \text{for all } t\in[p,q].
\end{aligned}$$

With this assumption, the turning point problem eqs. (1)-(2) possesses a unique solution having two boundary layers of at both end points $t=p$ and $t=q$.

Discretize the interval $[p,q]$ into N parts with mesh size $h=\frac{1}{N}$ so that $t_i=p+ih$ for $i=0,1,\dots,N$ are the mesh points. Denote $\frac{N}{2}=m$. Then, divide the interval $[p,q]$ into sub intervals $[t_{i-1},t_i]$ for $i=1,2,\dots,m$ and $[t_i,t_{i+1}]$ for $i=m+1,m+2,\dots,N-1$. We derive the numerical method in the two subintervals $[p,t_m]$ and $[t_m,q]$ where in $[p,t_m]$ layer will be at left end point $x=p$ whereas in $[t_m,q]$ layer exits at $t=q$.

Hence, in $[p,t_m]$ we use the finite difference scheme eq. (10) for $i = 1,2,\dots,m-1$ and in $[t_m,q]$ for $i = m+1, m+2,\dots,N-1$ the scheme eq. (17) is used to get the solution.

Now at $t = t_m$ i.e., for $i = m$, eq. (1) becomes

$$\varepsilon y''(t_m) + b(t_m)y(t_m) = f(t_m). \tag{18}$$

Since there no boundary layer at $x = x_m$, we use central finite difference scheme on eq. (18) at this point. Hence the difference equation of eq. (18) is

$$A_m x_{m-1} + B_{mi} x_m + C_m x_{m+1} = F_m \quad \text{for } \; i = m. \tag{19}$$

Here $A_m = \frac{\varepsilon}{h^2}$, $B_m = \frac{2\varepsilon}{h^2} - b_m$, $C_m = \frac{\varepsilon}{h^2}$ and $F_m = f_m$.

Now, we solve the system of eq. (10), eq. (19) and eq. (17) using Thomas algorithm.

2.4 Internal Boundary Layer

Here, we consider eq. (1) over the domain $[p,q]$. The functions $a(t), b(t)$ and $f(t)$ are taken to be sufficiently smooth, and such that eq. (1) has unique solution. The solution of eq. (1) possesses a layer or turning point behaviour depends on the coefficient $a(x)$. Under following assumptions,

$$\begin{aligned}
&a(t_m) = 0,\; a'(t_m) > 0, && \text{where } \; t_m = \frac{p+q}{2},\\
&b(t) \le b_0 < 0, && \text{for all } \; t \in [p,q],\\
&|a'(t)| \ge \frac{|a'(m)|}{2}, && \text{for all } \; t \in [p,q]
\end{aligned}$$

the given problem possesses a unique solution with interior layers at $t = t_m$.

Decompose the interval $[p,q]$ into N uniform spaced domains with mesh size $h = \frac{1}{N}$ and with mesh points $t_i = p + ih$ for $i = 0,1,\dots,N$. Let us denote $\frac{N}{2} = m$. Then, divide the interval $[p,q]$ into subintervals $[t_{i-1},t_i]$ for $i = 1,2,\dots,m$ and $[t_i,t_{i+1}]$ for $i = m+1, m+2,\dots,N-1$. For this problem, the layer is at the right endpoint of the $[p,t_m]$ and the layer is at the left endpoint of $[t_m,q]$.

Hence, in $[p,t_m]$ we use the finite difference scheme eq. (17) for $i = 1,2,\dots,m-1$ and in $[t_m,q]$ the scheme eq. (10) for $i = m+1, m+2,\dots,N-1$ is used to get the solution.

Now at $t = t_m$ i.e., for $i = m$, eq. (1) becomes

$$\varepsilon y''(t_m) + b(t_m)y(t_m) = f(t_m). \tag{20}$$

Since there is an internal layer at $x = x_m$, we take the average of the difference schemes eq. (10) and eq. (17). Hence the difference equation of eq. (20) is

$$A_m x_{m-1} + B_{mi} x_m + C_m x_{m+1} = F_m, \quad \text{for } i = m. \tag{21}$$

Here $A_m = \frac{A_{iL}+A_{iR}}{2}$, $B_m = \frac{B_{iL}+B_{iR}}{2}$, $C_m = \frac{C_{iL}+C_{iR}}{2}$ and $F_m = \frac{F_{iL}+F_{iR}}{2}$.

Now, we solve the system of eq. (10), eq. (17) and eq. (21) using the Thomas algorithm.

3. Convergence Analysis

The convergence analysis of the method described in Section 2 is considered in this section. Inserting the boundary conditions we write the system of equations in the matrix form as:

$$(D+P)x+Q+T(h)=0, \tag{22}$$

$$D=\left[\frac{\varepsilon}{h},\frac{-2\varepsilon}{h},\frac{\varepsilon}{h}\right]=\begin{bmatrix} \frac{-2\varepsilon}{h} & \frac{\varepsilon}{h} & 0 & 0 & \dots & 0 \\ \frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} & \frac{\varepsilon}{h} & 0 & \dots & 0 \\ 0 & \frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} & \frac{\varepsilon}{h} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} \end{bmatrix}$$

and

$$P=[z_i,v_i,w_i]=\begin{bmatrix} v_1 & w_1 & 0 & 0 & \dots & 0 \\ z_2 & v_2 & w_2 & 0 & \dots & 0 \\ 0 & z_3 & v_3 & w_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & z_{N-1} & v_{N-1} \end{bmatrix},$$

where

$$z_i=\frac{k}{2}\left(a'(t_i+k)-b(t_i+k)\right),$$

$$v_i=\left\{-a(t_i)-a'(t_{i+1}-k)\frac{k}{2}-a'(t_i+k)\left(\frac{h+k}{2}\right)+b(t_{i+1}-k)\frac{k}{2}+b(t_i+k)\left(\frac{h+k}{2}\right)\right\},$$

$$w_i=\left\{a(t_{i+1})-a'(t_{i+1}-k)\left(\frac{h+k}{2}\right)+b(t_{i+1}-k)\left(\frac{h-k}{2}\right)\right\},$$

$$Q=\left\{q_1+\left(\frac{\varepsilon}{h}+z_1\right)\gamma_0,q_2,q_3,\dots,q_{N-2},q_{N-1}+\left(\frac{\varepsilon}{h}+w_{N-1}\right)\gamma_1\right\},$$

where $q_i=\frac{h}{2}\{f(t_{i+1}-k)+f(t_i+k)\}$, $i=1,2,\dots,N-1$.

The local truncation error $T(h)$ associated with the proposed scheme is

$$T(h)=-\frac{h}{2}\left(a'(t_{i+1}-k)+a'(t_i+k)\right)y_i+O\left(h^2\right)$$

for

$$k_1=\left\{\frac{\varepsilon}{h}+a'(t_i+k)\frac{k}{2}-b(t_i+k)\frac{k}{2}\right\},$$

$$k_2=\left\{\frac{-2\varepsilon}{h}-a(t_i)-a'(t_{i+1}-k)\frac{k}{2}-a'(t_i+k)\left(\frac{h+k}{2}\right)+b(t_{i+1}-k)\frac{k}{2}+b(t_i+k)\left(\frac{h+k}{2}\right)\right\},$$

$$k_3=\left\{\frac{\varepsilon}{h}+a(t_{i+1})-a'(t_{i+1}-k)\left(\frac{h-k}{2}\right)+b(t_{i+1}-k)\left(\frac{h-k}{2}\right)\right\}$$

and $X=[x_1,x_2,\dots,x_{N-1}]^T$, $T(h)=[d_1,d_2,\dots,d_{N-1}]^T$, $O=[0,0,\dots,0]^T$ are related vector of eq. (18).

Let $X=[x_1,x_2,x_3,\dots x_{N-1}]^T\cong X$ which satisfy the equation

$$(D+P)X+Q=0. \tag{23}$$

Let $l_i=x_i-X_i$, $i=1,2,\dots,N-1$ be the discretized error, so that $L=[l_1,l_2,\dots,l_{N-1}]^T=x-X$.

Subtract eqs. (22)-(23), we have the equation for error as:

$$(D+P)L = T(h). \tag{24}$$

Let $|P(t)| \le c_1$ and $|Q(t)| \le c_2$, where c_1, c_2 is a positive constant if $p_{i,j}$ be the $(i,j)^{th}$ element of P, then

$$\left|P_{i,i+1}\right| = |w_i| \le \left(a(t_{i+1}) + a'(t_{i+1}-k)\left(\frac{h+k}{2}\right) + b(t_{i+1}-k)\left(\frac{h-k}{2}\right)\right)C_1, \quad \text{where } i = 1,2,\ldots,N-2,$$

$$\left|P_{i,i-1}\right| = |Z_i| \le \frac{k}{2}\left[a'(t_i+k) - b(t_i+k)\right]C_2, \quad \text{where } i = 2,3,\ldots,N-1$$

thus for sufficiently small h, we have

$$|p_{i\,i+1}| < \varepsilon, \quad \text{for } i = 1,2,\ldots,N-2 \tag{25}$$

and

$$|P_{i\,i-1}| < \varepsilon, \quad \text{for } i = 2,3,\ldots,N-1. \tag{26}$$

Hence, $(D+P)$ is irreducible.

Let S_i be the sum of the values of the i th row of the matrix $(D+P)$, then we have

$$\bar{S}_1 = \sum_{j=1}^{N-1} M_{ij} = \left\{a(t_{i+1}) - a(t_i) + \frac{\varepsilon}{h} - a'(t_i+k)\left(\frac{h+k}{2}\right) + b(t_i+k)\left(\frac{h+k}{2}\right) + \frac{h}{2}\left(a(t_{i+1}-k) - a'(t_{i+1}-k)\right)\right\} \text{ for } i = 1,$$

$$\bar{S}_i = \sum_{j=1}^{N-1} M_{ij} = a(t_{i+1}) - a(t_i) + \frac{h}{2}\left[b(t_i+k) + b(t_{i+1}-k) - \left(a(t_i+k) + a'(t_{i+1}-k)\right)\right] \text{ for } i = N-1,$$

$$\bar{S}_{N-1} = \sum_{j=1}^{N-1} M_{N-1j} = -a(t_i) - \frac{\varepsilon}{h} - a'(t_{i+1}-k)\frac{h}{2} - a'(t_i+k)\frac{k}{2} + b(t_{i+1}-k)\frac{k}{2} + b(t_i+k)\frac{h}{2} \quad \text{for } i = 2,3,\ldots,N-2.$$

Let $C_i = \min|p(t)|$ and $C_1^* = \max|p(t)$, since $0 \le \varepsilon \le 1$ and $\varepsilon \propto O(h)$ it is possible to verify that for sufficiently small h, $(D+P)$is monotonic.

Hence, $(D+P)^{-1}$exits and $(D+P)^{-1} \ge 0$ thus eq. (24), we get

$$\|L\| \le \left\|(D+P)^{-1}\right\| \|T\|. \tag{27}$$

For sufficiently small h, we have

$\bar{s}_i > (a(t_{i+1}) - a(t_i))$ for $i = 1 \Rightarrow \bar{s}_1 > C$, where C is constant,

$\bar{s}_i > (a(t_{i+1}) - a(t_i))$ for $i = N-1 \Rightarrow \bar{s}_i > C$,

$\bar{s}_i > B_i$ for $i = 2,3,\ldots,N-2$.

Let $(D+P)^{-1}$ be the $(i,k)^{th}$ elements of $(D+P)^{-1}$ and we define

$$\left\|(D+P)^{-1}\right\| = \max \sum_{k=1}^{N-1} (D+P)^{-1}_{(i\,k)}$$

and

$$\|T(h)\| = \max|T(h)|\,. \tag{28}$$

Since $(D+P)^{-1}_{(i\,k)} \geq 0$ and

$$\sum_{k=1}^{N-1}(D+P)^{-1}_{(i\,k)}\bar{S}_K = 1 \quad \text{for } i = 1,2,\dots,N-1,$$

$$(D+P)^{-1}_{(i\,k)} \leq \frac{1}{\bar{s}_K} \leq \frac{1}{C} \quad \text{for } i = 1, \tag{29}$$

$$(D+P)^{-1}_{(iN-1)} \leq \frac{1}{\bar{s}_{N-1}} \leq \frac{1}{C} \quad \text{for } i = N-1. \tag{30}$$

Furthermore,

$$\sum_{k=2}^{N-2}(D+P)^{-1}_{(i\,k)} \leq \frac{1}{\min \bar{s}_k} \leq \frac{1}{B_i} \quad \text{for } i = 2,3,\dots,N-2. \tag{31}$$

Help of eqs. (28)-(31), from eq. (27), we have $\|L\| \leq O(h)$.

Hence, the proposed scheme is first order convergent. In similar steps, we can analyze the convergence in other cases.

4. Numerical Experiments

To describe the proposed method computationally it is implemented on two left end layer, three right end layer, two dual layer and two internal layer problems. These problems are widely discussed and numerical results are available for comparison. The maximum absolute error $E_\varepsilon^N = \max_{0\leq i\leq N}|x(t_i) - x_i|$ is calculated for each comparison by an example. Here $x(t_i)$ is the exact solution and x_i is the numerical solution. We use double mesh principle [13], $E_\varepsilon^N = \max_{0\leq i\leq N}|x_i^N - x_{2i}^{2N}|$ to find the maximum errors in the examples for which exact solution is not known. Here x_i^N is the numerical solution with N subintervals and x_{2i}^{2N} is the numerical solution with $2N$ subintervals.

Example 1. Consider the left end boundary layer problem

$$\varepsilon x''(t) + x'(t) - x(t) = 0 \quad \text{with } \; x(0) = 1 \; \text{ and } \; x(1) = 1.$$

The exact solution is

$$x(t) = \frac{[(\exp(c_2)-1)e^{c_1 t} + (1-\exp(c_1))e^{c_2 t}]}{[\exp(c_2) - \exp(c_1)]}$$

where $c_1 = (-1+\text{sqrt}(1+4\varepsilon))/(2\varepsilon)$ and $c_2 = (-1-\text{sqrt}(1+4\varepsilon))/(2\varepsilon)$.

The maximum absolute errors with comparison are shown in Table 1 for different values of ε and h. Exact and numerical solutions are shown graphically in Figure 1 for $\varepsilon = 2^{-10}$ with $h = 2^{-7}$.

Table 1. The maximum absolute errors in solution of Example 1

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Proposed method								
2^{-4}	1.242(−2)	7.607(−3)	4.169(−3)	2.168(−3)	1.105(−3)	5.575(−4)	2.800(−4)	1.403(−4)
2^{-5}	9.314(−3)	7.161(−3)	4.368(−3	2.388(−3)	1.244(−3)	6.350(−4)	3.206(−4)	1.611(−4)
2^{-6}	5.197(−3)	5.058(−3)	3.788(−3)	2.306(−3)	1.263(−3)	6.592(−4)	3.365(−4)	1.700(−4)
2^{-10}	7.484(−3)	3.689(−3)	1.697(−3)	6.770(−4)	3.571(−4)	3.318(−4)	2.463(−4)	1.499(−4)
2^{-15}	7.829(−3)	4.036(−3)	2.044(−3)	1.024(−3)	5.086(−4)	2.491(−4)	1.191(−4)	5.396(−5)
2^{-20}	7.839(−3)	4.047(−3)	2.055(−3)	1.035(−3)	5.194(−4)	2.600(−4)	1.299(−4)	6.484(−5)
Results in [18]								
2^{-4}	1.29(−2)	1.50(−2)	1.39(−2)	1.37(−2)	1.37(−2)	1.37(−2)	1.37(−2)	1.37(−2)
2^{-5}	1.00(−2)	6.80(−3)	7.70(−3)	7.20(−3)	7.10(−3)	7.00(−3)	7.00(−3	7.00(−3)
2^{-6}	1.50(−2)	5.60(−3)	3.50(−3)	3.90(−3)	3.70(−3)	3.60(−3)	3.60(−3)	3.60(−3)
2^{-10}	2.00(−2)	1.06(−2)	5.30(−3)	2.50(−3)	1.10(−3)	3.84(−4)	2.23(−4)	2.49(−4)
2^{-15}	2.03(−2)	1.09(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.05(−4)	3.47(−4)	1.68(−4)
2^{-20}	2.04(−2)	1.09(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.16(−4)	3.58(−4)	1.79(−4)

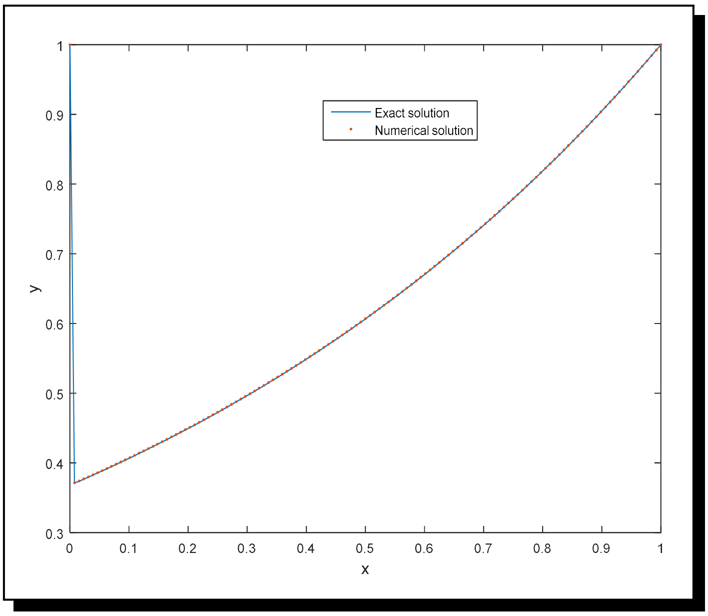

Figure 1. Graphical representation of the solution in Example 1 with $\varepsilon = 2^{-10}$ with $h = 2^{-7}$

Example 2. Consider the left end boundary layer with variable coefficient

$$\varepsilon x''(t) + (1+t)^2 x'(t) + 2(1+t)x(t) = f(t); \quad t \in [0,1]$$

where $f(t) = e^{-t/2}\frac{(1+t)(3-t)+\frac{\varepsilon}{2}}{2}$, with $x(0) = 0$ and $x(1) = e^{-0.5} - e^{-7/3\varepsilon}$.

The exact solution is $x(t) = e^{-\frac{t}{2}} - e^{\frac{-t(t^2+3t+3)}{3\varepsilon}}$.

The maximum errors with comparison are given in Table 2 for different values of ε and h. Figure 2 depicts the exact and numerical solution with $\varepsilon = 2^{-10}$ with $h = 2^{-7}$.

Table 2. The maximum absolute errors in solution of Example 2

ε \ h	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Proposed method					
2^{-4}	$2.5057e-03$	$1.3584e-03$	$7.0752e-04$	$3.6109e-04$	$1.8241e-04$
2^{-8}	$1.2295e-03$	$8.8492e-04$	$5.6997e-04$	$3.0859e-04$	$1.6099e-04$
10^{-4}	$2.0032e-05$	$1.9548e-05$	$1.9424e-05$	$1.9178e-05$	$1.8696e-05$
10^{-5}	$1.9611e-06$	$1.9548e-06$	$1.9424e-06$	$1.9178e-06$	$1.8696e-06$
10^{-6}	$1.9611e-07$	$1.9548e-07$	$1.9424e-07$	$1.9178e-07$	$1.8697e-07$
Results in [9]					
2^{-4}	$0.29e-02$	$0.73e-03$	$0.18e-03$	$0.46e-04$	$0.11e-04$
2^{-8}	$0.39e-01$	$0.12e-01$	$0.38e-02$	$0.12e-02$	$0.37e-03$
10^{-4}	$0.38e-01$	$0.11e-01$	$0.37e-02$	$0.12e-02$	$0.36e-03$
10^{-5}	$0.38e-01$	$0.11e-01$	$0.37e-02$	$0.12e-02$	$0.36e-03$
10^{-6}	$0.38e-01$	$0.11e-01$	$0.37e-02$	$0.12e-02$	$0.36e-03$

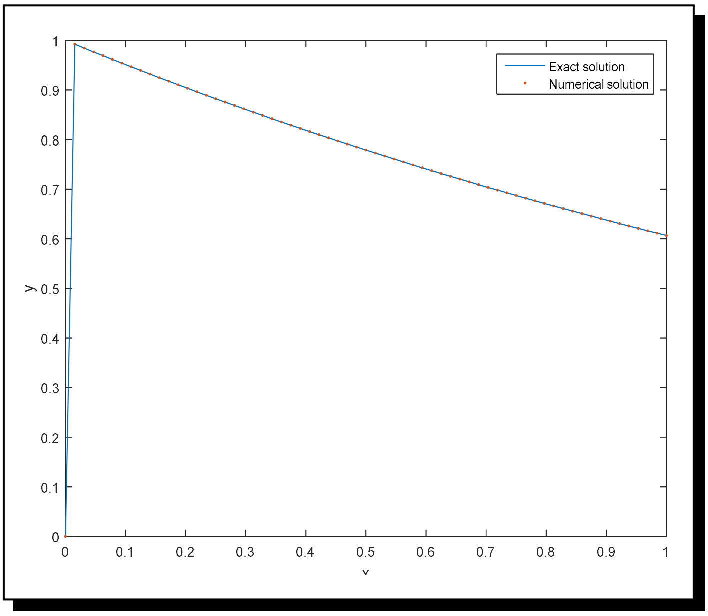

Figure 2. Graphical representation of the solution in Example 2 with $\varepsilon = 2^{-10}$ with $h = 2^{-7}$

Example 3. Consider the right end boundary layer problem

$$\varepsilon x''(t) - x'(t) - (1+\varepsilon)x(t) = 0; \quad t \in [0,1]$$

$$\text{with } x(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon) \text{ and } x(0) = 1 + \frac{1}{\varepsilon}.$$

$x(t) = \exp((1+\varepsilon)(t-1)/\varepsilon) + \exp(-t)$ is the exact solution of the problem. Table 3 represents the maximum absolute errors in the solution for different values of ε and h. Graphical representation of the solution is given in Figure 3.

Table 3. The maximum absolute errors of Example 3

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Proposed method								
2^{-4}	1.313(−2)	8.102(−3)	4.420(−3)	2.300(−3)	1.115(−3)	5.105(−4)	2.963(−4)	1.487(−4)
2^{-5}	9.482(−3)	7.403(−3)	4.504(−3)	2.468(−3)	1.248(−3)	6.557(−4)	3.302(−4)	1.662(−4)
2^{-6}	6.967(−3)	5.147(−3)	3.855(−3)	2.344(−3)	1.236(−3)	6.692(−4)	3.418(−4)	1.727(−4)
2^{-10}	1.201(−2)	6.037(−3)	2.891(−3)	1.280(−3)	4.654(−4)	3.321(−4)	2.461(−4)	1.500(−4)
2^{-15}	1.234(−2)	6.376(−3)	3.236(−3)	1.622(−3)	8.109(−4)	4.006(−4)	1.949(−4)	9.190(−5)
2^{-20}	1.235(−2)	6.387(−3)	3.244(−3)	1.630(−3)	8.214(−4)	4.115(−4)	1.299(−4)	6.484(−5)
Results in [18]								
2^{-4}	2.54(−2)	2.63(−2)	2.39(−2)	2.33(−2)	2.32(−2)	2.32(−2)	2.32(−2)	2.32(−2)
2^{-5}	1.03(−2)	1.29(−2)	1.31(−2)	1.19(−2)	1.16(−2)	1.16(−2)	1.15(−2)	1.15(−2)
2^{-6}	1.54(−2)	5.70(−3)	6.50(−3)	6.50(−3)	5.90(−3)	5.80(−3)	5.80(−3)	5.80(−3)
2^{-10}	2.02(−2)	1.06(−2)	5.30(−3)	2.50(−3	1.10(−3)	3.84(−4)	4.10(−4)	4.05(−4)
2^{-15}	2.05(−2)	1.10(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.05(−4)	3.47(−4)	1.68(−4)
2^{-20}	2.05(−2)	1.10(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.16(−4)	3.58(−4)	1.79(−4)

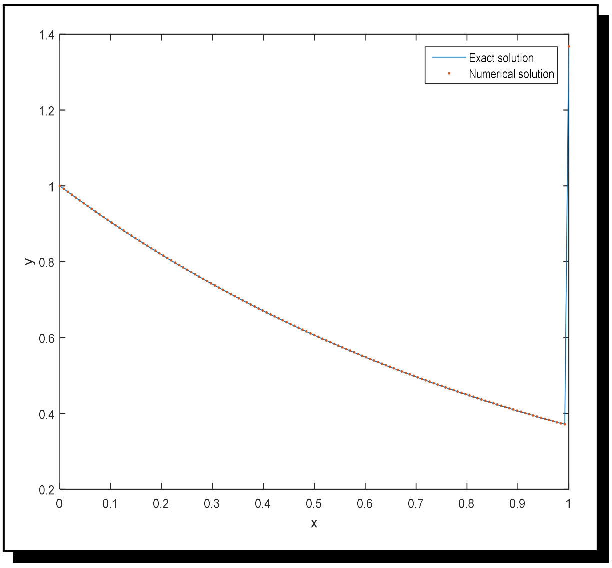

Figure 3. Graphical representation of the solution in Example 3 with $\varepsilon = 10^{-8}$ with $h = 2^{-7}$

Example 4. Consider the right end boundary layer problem

$$-\varepsilon x''(t) + x'(t) = e^t \quad \text{with} \quad x(0) = x(1) = 0.$$

The exact solution is

$$x(t) = \frac{1}{1-\varepsilon}\left[e^t - \frac{1 - e^{1-\left(\frac{1}{\varepsilon}\right)} + \left[e^t - 1\right] e^{(t-1)/\varepsilon}}{1 - e^{-1/\varepsilon}}\right].$$

Table 4 depicts the maximum absolute errors in the solution for different values of ε and h. The layer behavior in the solution is shown graphically in Figure 4.

Table 4. The maximum absolute errors in solution of Example 4

ε \ N	64	128	256	512	1024
Proposed method					
10^{-5}	1.6762(−5)	1.6971(−5)	1.7077(−5)	1.7130(−5)	1.7156(−5)
10^{-6}	1.6762(−6)	1.6971(−6)	1.7077(−6)	1.7130(−6)	1.7156(−6)
10^{-7}	1.6764(−7)	1.6971(−7)	1.7077(−7)	1.7130(−7)	1.7156(−7)
10^{-8}	1.6785(−8)	1.6973(−8)	1.7077(−8)	1.7130(−8)	1.7156(−8)
Results in [11]					
10^{-5}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-6}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-7}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)
10^{-8}	4.09(−1)	3.41(−1)	1.96(−1)	1.68(−2)	1.64(−3)

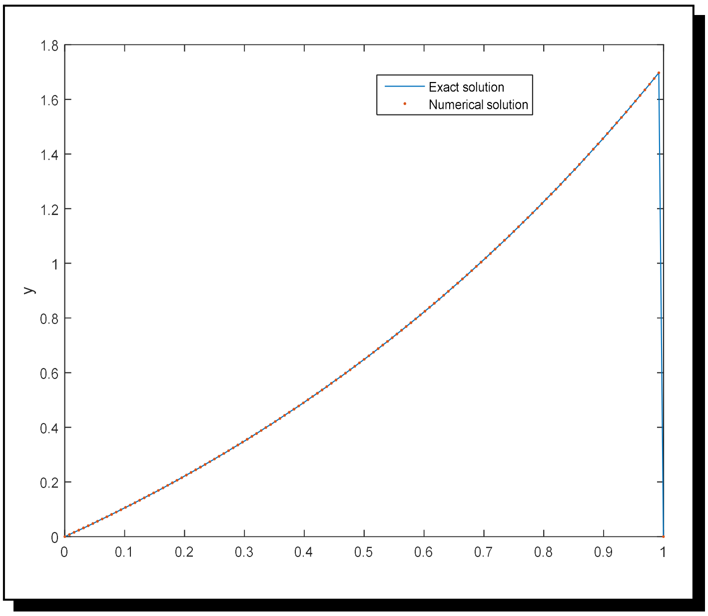

Figure 4. Graphical representation of the solution in Example 4 with $\varepsilon = 10^{-8}$ with $h = 2^{-7}$

Example 5. Consider the right end boundary layer problem

$$-\varepsilon x''(t) + \frac{1}{t+1}x'(t) + \frac{1}{t+2}x(t) = f(t); \quad x(0) = 0, x(1) = e+2,$$

$$f(t) = \left(-\varepsilon + (t+1)^{-1} + (t+2)^{-1}\right)e^t + \frac{1}{t+2}2^{\frac{-1}{\varepsilon}}(t+1)^{1+\frac{1}{\varepsilon}},$$

$x(t) = e^t + 1/2^{\frac{1}{\varepsilon}}(t+1)^{-\left(1+\frac{1}{\varepsilon}\right)}$ is the exact solution of the problem. The maximum errors in the solution are given in Table 5 for different values of h and ε. Numerical and exact solutions are illustrated graphically in Figure 5.

Table 5. The maximum absolute errors in solution of Example 5

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Proposed method					
2^{-6}	5.0408(−3)	2.6969(−3)	1.3932(−3)	7.0790(−3)	3.5678(−3)
2^{-7}	5.4870(−3)	3.0766(−3)	1.6232(−3)	8.3298(−3)	4.2186(−3)
2^{-8}	6.7148(−3)	3.0961(−3)	1.7151(−3)	9.0004(−4)	4.6068(−4)
2^{-9}	1.0473(−2)	3.3027(−3)	1.6521(−3)	9.1039(−4)	4.7668(−4)
Results in [11] for $A1=A4=1/3$ and $A2=A3=1/6$					
2^{-6}	1.50(−2)	3.68(−3)	9.17(−4)	2.29(−4)	5.72(−5)
2^{-7}	6.75(−2)	1.54(−2)	3.77(−3)	9.37(−4)	2.34(−4)
2^{-8}	2.66(−1)	6.83(−2)	1.55(−2)	3.81(−3)	9.48(−4)
2^{-9}	6.92(−1)	2.68(−1)	6.87(−2)	1.56(−2)	3.83(−3)
Results in [11] for $A1=A4=4/9$ and $A2=A3=1/18$					
2^{-6}	1.57(−2)	3.86(−3)	9.62(−4)	2.39(−4)	6.00(−5)
2^{-7}	6.91(−2)	1.56(−2)	3.85(−3)	9.59(−4)	2.39(−4)
2^{-8}	2.69(−1)	6.90(−2)	1.57(−2)	3.85(−3)	9.58(−4)
2^{-9}	6.97(−1)	2.70(−1)	6.90(−2)	1.57(−2)	3.85(−3)

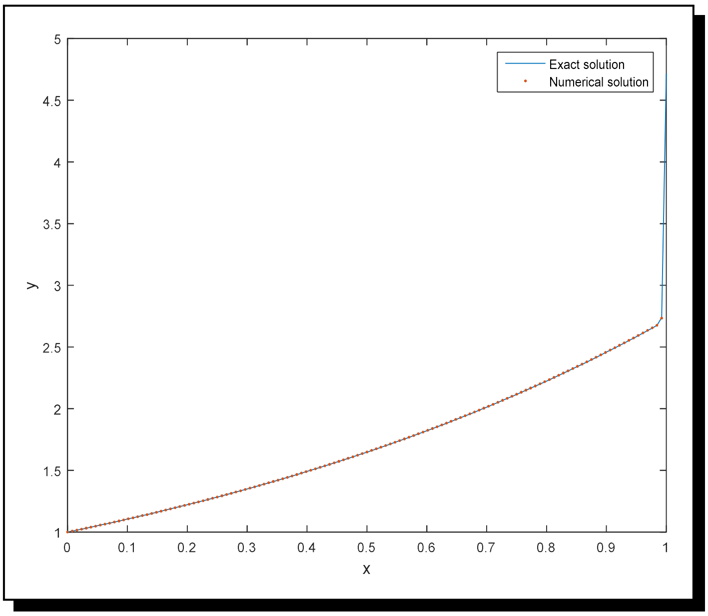

Figure 5. Graphical representation of the solution in Example 5 with $\varepsilon=10^{-8}$ with $h=2^{-7}$

Example 6. Consider the dual boundary layers problem

$$\varepsilon x''(t)-2(2t-1)x'(t)-4x(t)=0, \quad t\in[0,1]$$

with $x(0)=1$, $x(1)=1$. This problem exhibits dual layers at $t=0$ and $t=1$.

The exact solution is given by $x(t)=\exp(-2t(1-t)/\varepsilon)$.

Table 6 shows the maximum absolute errors for different values of ε and h. Figure 6 depicts the graphical representation of the exact and numerical solution.

Table 6. Maximum errors in Example 6

ε \ N	16	32	64	128	256	512	1024
Proposed method							
10^{-5}	3.5871(−4)	9.3494(−5)	2.3928(−5)	6.0560(−6)	1.5235(−6)	3.8209(−7)	9.5674(−8)
10^{-6}	3.5871(−4)	9.3494(−5)	2.3928(−5)	6.0560(−6)	1.5235(−6)	3.8209(−7)	9.5674(−8)
10^{-7}	3.5871(−4)	9.3494(−5)	2.3928(−5)	6.0560(−6)	1.5235(−6)	3.8209(−7)	9.5674(−8)
10^{-8}	3.5871(−4)	9.3494(−5)	2.3928(−5)	6.0560(−6)	1.5235(−6)	3.8209(−7)	9.5674(−8)
10^{-9}	3.5871(−4)	9.3494(−5)	2.3928(−5)	6.0560(−6)	1.5235(−6)	3.8209(−7)	9.5674(−8)
Results in Natesan et al. [16]							
10^{-5}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-6}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-7}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-8}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-9}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097

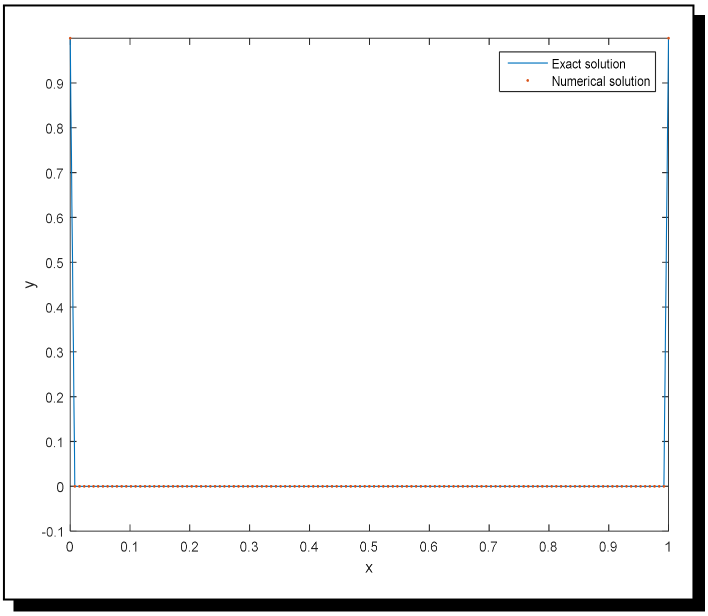

Figure 6. Graphical representation of the solution in Example 6 with $\varepsilon = 10^{-5}$ with $h = 2^{-7}$

Example 7. Consider the dual boundary layers problem

$$\varepsilon x'' - tx' - x = 0, \quad -1 \le t \le 1$$

with $x(-1) = 1$ and $x(1) = 2$. For this problem we have two boundary layers one at $t = -1$ and another at $t = 1$.

The maximum absolute errors are posed in Table 7 for different values of ε and h. Figure 7 shows graphical representation of the numerical solution.

Table 7. Maximum absolute errors in Example 7

ε \ N	16	32	64	128	256	512	1024
Proposed method							
10^{-5}	6.2746(−4)	1.6501(−4)	4.2393(−5)	1.0748(−5)	2.7062(−6)	6.7898(−7)	1.7005(−7)
10^{-6}	6.2746(−4)	1.6501(−4)	4.2393(−5)	1.0748(−5)	2.7062(−6)	6.7898(−7)	1.7005(−7)
10^{-7}	6.2746(−4)	1.6501(−4)	4.2393(−5)	1.0748(−5)	2.7062(−6)	6.7898(−7)	1.7005(−7)
10^{-8}	6.2746(−4)	1.6501(−4)	4.2393(−5)	1.0748(−5)	2.7062(−6)	6.7898(−7)	1.7005(−7)
10^{-9}	6.2746(−4)	1.6501(−4)	4.2393(−5)	1.0748(−5)	2.7062(−6)	6.7898(−7)	1.7005(−7)
Results in [14]							
10^{-5}	3.1298(−1)	1.3929(−1)	6.6063(−2)	3.2279(−2)	1.6036(−2)	8.0725(−3)	4.1337(−3)
10^{-6}	3.1265(−1)	1.3905(−1)	6.5855(−2)	3.2084(−2)	1.5846(−2)	7.8828(−3)	3.9390(−3)
10^{-7}	3.1262(−1)	1.3903(−1)	6.5834(−2)	3.2065(−2)	1.5828(−2)	7.8844(−3)	3.9207(−3)
10^{-8}	3.1262(−1)	1.3902(−1)	6.5832(−2)	3.2063(−2)	1.5826(−2)	7.8626(−3)	3.9189(−3)
10^{-9}	3.1262(−1)	1.3902(−1)	6.5832(−2)	3.2063(−2)	1.5826(−2)	7.8624(−3)	3.9187(−3)

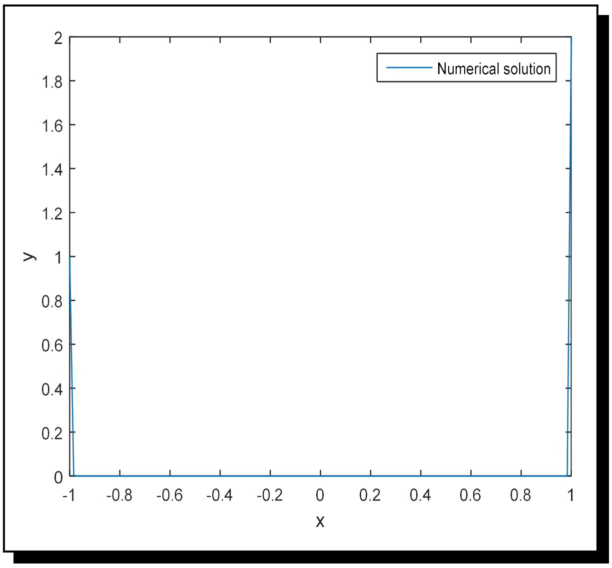

Figure 7. Graphical representation of the solution in Example 7 with $\varepsilon = 10^{-5}$ with $h = 2^{-7}$

Example 8. Consider the internal boundary layers problem

$$\varepsilon x''(t) + 2(2t-1)x'(t) - 4x(t) = 0; \quad t \in (0,1)$$

with $x(0) = 1$ and $x(1) = 1$.

The exact solution of this problem is

$$x(t) = \frac{-e^{\frac{1}{2\varepsilon}-\frac{(1-2t)^2}{2\varepsilon}}\left(2e^{\frac{(1-2t)^2}{2\varepsilon}}\sqrt{2\pi}x\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2\varepsilon}}\right) - e^{\frac{(1-2t)^2}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1-2t}{\sqrt{2\varepsilon}}\right) - 2\varepsilon\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 2\sqrt{\varepsilon}}.$$

This problem has an internal layer at $t = \frac{1}{2}$.

Table 8 shows the maximum absolute errors for different values of ε and h. The numerical and exact solutions are plotted graphically in Figure 8.

Table 8. Maximum absolute errors in Example 8

ε \ N	32	64	128	256	512	1024
Proposed method						
2^{-5}	5.9701(−3)	3.3654(−3)	1.7391(−3)	8.7449(−4)	4.3697(−4)	2.1822(−4)
2^{-6}	5.3525(−3)	3.2322(−3)	1.7219(−3)	8.7336(−4)	4.3719(−4)	2.1834(−4)
2^{-7}	1.1177(−2)	2.9851(−3)	1.6827(−3)	8.6953(−4)	4.3725(−4)	2.1848(−4)
2^{-8}	2.5867(−2)	2.6763(−3)	1.6161(−3)	8.6093(−4)	4.3668(−4)	2.1860(−4)
2^{-9}	4.7842(−2)	5.5886(−3)	1.4925(−3)	8.4134(−4)	4.3477(−4)	2.1862(−4)
2^{-10}	7.5829(−2)	1.2934e − 02	1.3381(−3)	8.0805(−4)	4.3046(−4)	2.1834(−4)
Results in [14]						
2^{-5}	5.2198(−2)	8.3601(−2)	1.0368(−1)	1.1631(−1)	1.2439(−1)	1.2967(−1)
2^{-6}	2.2246(−2)	4.9178(−2)	6.7036(−2)	7.8288(−2)	8.5410(−2)	9.0014(−2)
2^{-7}	1.2867(−2)	2.6099(−2)	4.1801(−2)	5.1840(−2)	5.8156(−2)	6.2194(−2)
2^{-8}	2.8508(−2)	1.1123(−2)	2.4589(−2)	3.3518(−2)	3.9144(−2)	4.2705(−2)
2^{-9}	3.9171(−2)	6.4337(−3)	1.3050(−2)	2.0900(−2)	2.5920(−2)	2.9078(−2)
2^{-10}	4.6439(−2)	1.4254(−2)	5.5616(−3)	1.2295(−2)	1.6759(−2)	1.9572(−2)

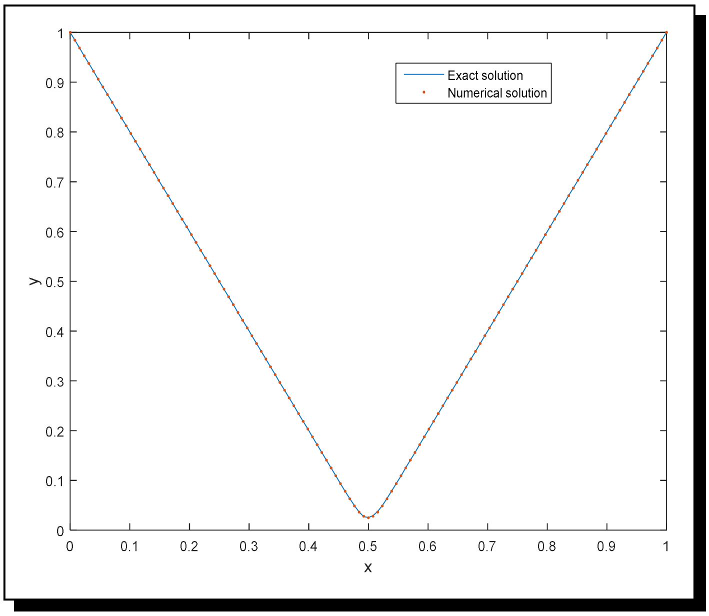

Figure 8. Graphical representation of the solution in Example 8 with $\varepsilon = 2^{-10}$ with $h = 2^{-7}$

Example 9. Consider the internal boundary layers problem

$$\varepsilon x'' + tx' - x = 0, \quad -1 \le t \le 1$$

with $x(-1) = 1$ and $x(1) = 2$.The exact solution is

$$x(t) = \frac{2e^{\frac{-1}{2\varepsilon}}\sqrt{\varepsilon}\left(t + 3e^{-\frac{t^2-1}{2\varepsilon}}\right) + \left(\sqrt{2\pi}x\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 3\sqrt{2\pi}\,\mathrm{erf}\left(\frac{t}{\sqrt{2\varepsilon}}\right)\right)}{2\sqrt{2\pi}\,\mathrm{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 4e^{\frac{-1}{2\varepsilon}}\sqrt{\varepsilon}}.$$

For this problem, we have internal layers at $t = 0$.

Table 9 represents the maximum absolute errors with comparisons for different values of ε and h. Figure 9 gives the graphical representation of the numerical result and exact solution of the problem.

Table 9. Maximum absolute errors in Example 9

ε \ N	32	64	128	256	512	1024
Proposed method						
2^{-5}	1.0203(−2)	5.5017(−3)	2.7453(−3)	1.3449(−3)	6.6344(−4)	3.2928(−4)
2^{-6}	9.1104(−3)	5.3503(−3)	2.7665(−3)	1.3581(−3)	6.6733(−4)	3.3030(−4)
2^{-7}	7.9110(−3)	5.1014(−3)	2.7508(−3)	1.3726(−3)	6.7246(−4)	3.3172(−4)
2^{-8}	2.2330(−2)	4.5552(−3)	2.6751(−3)	1.3832(−3)	6.7905(−4)	3.3366(−4)
2^{-9}	4.6794(−2)	3.9555(−3)	2.5507(−3)	1.3754(−3)	6.8632(−4)	3.3623(−4)
2^{-10}	7.7601(−2)	1.1165(−2)	2.2776(−3)	1.3376(−3)	6.9162(−4)	3.3953(−4)
Results in [14]						
2^{-5}	1.0111(−1)	1.3778(−1)	1.6169(−1)	1.7746(−1)	1.8802(−1)	1.9520(−1)
2^{-6}	5.3820(−2)	8.5843(−2)	1.0677(−1)	1.2048(−1)	1.2958(−1)	1.3573(−1)
2^{-7}	2.2863(−2)	5.0556(−2)	6.8889(−2)	8.0847(−2)	8.8730(−2)	9.4011(−2)
2^{-8}	5.6850(−3)	2.6910(−2)	4.2921(−2)	5.3386(−2)	6.0238(−2)	6.4792(−2)
2^{-9}	2.0288(−2)	1.1431(−2)	2.5278(−2)	3.4445(−2)	4.0424(−2)	4.4365(−2)
2^{-10}	2.9876(−2)	2.8425(−3)	1.3455(−2)	2.1461(−2)	2.6693(−2)	3.0119(−2)

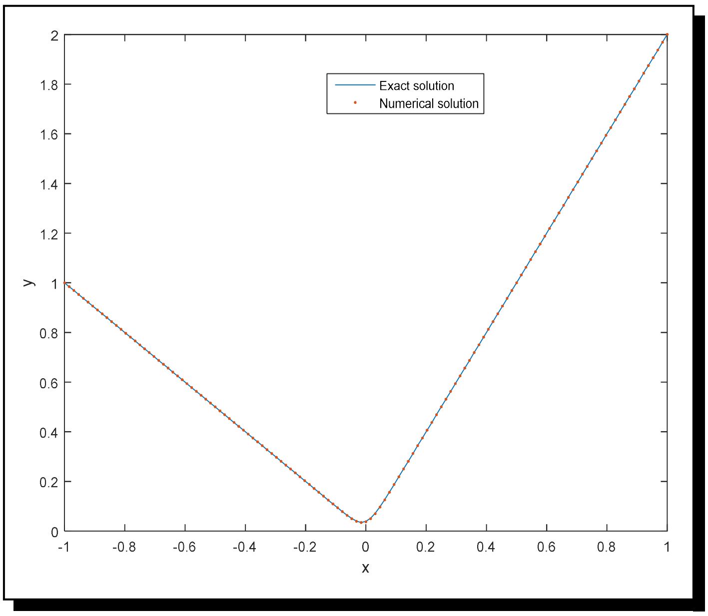

Figure 9. Graphical representation of the solution in Example 9 with $\varepsilon = 2^{-10}$ with $h = 2^{-7}$

5. Discussions and Conclusion

In this paper, Gaussian quadrature with exponential fitting is implemented for the solution of two-point singularly perturbed boundary value problems with layer at one endpoint, two endpoints and interior point of the domain. When ε, the perturbation parameter is small, then to control the layer behavior, a fitting parameter is inserted in the Gaussian two point quadrature formula.

The method is analyzed clearly for one end boundary layer problem, dual boundary layer and internal boundary layer problem. Analysis of convergence of the method is discussed. Numerical computations for the several examples with layer behavior are shown by comparison to justify the proposed scheme. We noticed that the method produces good results for $\varepsilon < h$ also. Graphical representation of the numerical solution and exact solution of the examples is presented in figures. We took note that the numerical method meets the exact solution very well. The proposed method is easy to implement with less computational work.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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