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Research Article

Convergence Analysis of Two Demicontractive Operators

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Abstract. In this paper, first we introduce a new iterative scheme involving demicontractive mappings in Hilbert spaces which does not require prior knowledge of operator norm and, second, by using the proposed scheme, prove some strong convergence theorems. Finally, we give some numerical examples to illustrate our main result.

Keywords. Demicontractive mappings; Common fixed point; Split common fixed problem

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1. Introduction

Let H_1, H_2 be real Hilbert space. The *Split Common Fixed Problem* (SCFPP) is the following problem:

$$\text{find } \bar{x} \in F(T) \text{ such that } A\bar{x} \in F(S), \tag{1.1}$$

where $F(S)$ and $F(T)$ stand for, respectively, the fixed point sets of $T: H_1 \to H_1$ and $S: H_2 \to H_2$, respectively.

We shall denote the solution set of the SCFPP by

$$\Gamma := \{y \in F(S) : Ay \in F(T)\} = F(S) \cap A^{-1}(F(T)). \tag{1.2}$$

We recall that $F(S)$ and $F(T)$ are nonempty, closed and convex subsets of H_1 and H_2, respectively. If $\Gamma \neq \emptyset$, then Γ is closed and convex subset of H_1.

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *Split Feasibility Problem* (SFP) is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.3}$$

where $A: H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [1, 12] and the references therein).

We observe that SCFPP is a generalization of the *Split Feasibility Problem* (SFP) and the *Convex Feasibility Problem* (CFP) (for more details, see [3]). In order to solve (1.1), Censor and Segal [3] studied, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = S(x_n + \gamma A^t(T - I)Ax_n), \quad n \geq 1, \tag{1.4}$$

where $\gamma \in \left(0, \frac{2}{\gamma}\right)$, with γ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix γ transposition). In 2011, Moudafi [9] introduced the following relaxed algorithm:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n S y_n, \quad n \geq 1, \tag{1.5}$$

where $y_n = x_n + \gamma A^*(T - I)Ax_n$, $\beta \in (0,1)$, $\alpha_n \in (0,1)$, and $\gamma \in \left(0, \frac{1}{\gamma\beta}\right)$, with γ being the spectral $\lambda\beta$ radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm (1.5) in Hilbert spaces where S and T are quasi-nonexpansive operators.

In this paper, we propose an algorithm which does not require the calculation or estimation of the operator norm, to solve the two-operator *Split Common Fixed Point Problem* (SCFPP) (1.1) when the operators S and T are demicontractive and prove strong convergence of sequence generated by our proposed algorithm. Furthermore, we give numerical example of our result to show its efficiency and implementation. Zhao and He [26], Moudafi [9], Censor and Segal [3] to the split common fixed point problem when the operators and demicontractive. Furthermore, our work improves the recent works of Moudafi [10], Tang *et al*. [17], Cholamjiak *et al*. [13], Suantai *et al*. [14–16], Vinh *et al*. [18] and Anantachai Padcharoen *et al*. [11].

2. Preliminaries

Next, we provide some definitions which will be used in the sequel.

Let $T: H \to H$ be a mapping. A point $\bar{x} \in H$ is said to be a fixed point of T provided that $T\bar{x} = \bar{x}$. In this paper, the symbols $\to$ and $\rightharpoonup$ denote by the strong convergence and the weak convergence, respectively.

The mapping $T: H \to H$ is said to be:

(1) *quasi-nonexpansive* if

$$\|Tx - Tp\| \leq \|x - p\| \tag{2.1}$$

for all $x \in H$ and $p \in F(T)$.

(2) *strictly pseudocontractive* if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2 \tag{2.2}$$

for all $x \in H$.

(3) *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2 \tag{2.3}$$

for all $x \in H$.

(4) *demicontractive* (or *k-demicontractive*) if there exists $k < 1$ such that

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2 \tag{2.4}$$

for all $x \in H$ and $p \in F(T)$.

Remark 2.1. It is clear that, in a real Hilbert space H, (2.4) is equivalent to

$$\langle x - p, x - Tx\rangle \geq \frac{1-k}{2}\|x - Tx\|^2 \tag{2.5}$$

for all $x \in H$ and $p \in F(T)$.

Now, we give some definitions and lemmas for our main results:

Definition 2.2. A mapping $T : H \to H$ is said to be *demiclosed* at 0 if, for each sequence $\{x_n\}$ in H, the condition that the sequence $\{x_n\}$ converges weakly to x_0 and the sequence $\{Tx_n\}$ converges strongly to 0 imply $Tx_0 = 0$.

Lemma 2.3. *Let H be a real Hilbert space. Then the following results hold:*

(1) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$.

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$.

(3) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$.

(4) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$ *for all* $x, y \in H$ *and* $\alpha \in \mathbb{R}$.

Lemma 2.4. [20] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n$$

for each $n \geq 0$, where

(1) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(3) $\gamma_n \geq 0$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $a_n \to 0$ as $n \to \infty$.

3. Main Results

Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be an bounded linear operator and $A^* : H_2 \to H_1$ be a adjoint operator of A. Let $T : H_1 \to H_1$ be a k_1-demicontractive mapping such that $T - I$ is demiclosed at 0 and $C := F(T) \neq \emptyset$. Let $S : H_2 \to H_2$ be k_2-demicontractive mapping such that $S - I$ is demiclosed at 0 and $Q := F(S) \neq \emptyset$. Suppose that the problem (SCFPP) has a nonempty solution set Ω.

Algorithm 3.1.

Initialization. Given $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$.

Let $x_1 = x \in H_1$ be arbitrary.

Step 1. Set $n = 1$ and compute

$$z_n = (1-\alpha_n)x_n, \quad y_n = z_n + \rho_n A^*(S-I)Az_n,$$

where the step size ρ_n be chosen in such a way that

$$\rho_n = \left(\epsilon, \frac{(1-k_2)\|(S-I)Az_n\|^2}{\|A^*(S-I)Az_n\|^2} - \epsilon\right), \quad SAz_n \neq Az_n, \tag{3.1}$$

for small enough $\epsilon > 0$, otherwise $\rho_n = \rho$ (ρ being any nonnegative value).

Step 2. Compute

$$x_{n+1} = (1-\beta_n)z_n + \beta_n[(1-\gamma_n)y_n + \gamma_n T y_n].$$

If $y_n = z_n$ and $x_{n+1} = z_n$, then $z_n \in \Omega$.
Set $n \leftarrow n+1$ and go to *Step* 1.

Lemma 3.2. *Suppose that the problem* (SCFPP) *has a nonempty solution set Ω. Then, ρ_n defined by* (3.1) *is well-defined.*

Proof. We observe that in algorithm (3.1) the choice of the stepsize ρ_n is independent of the norm A. Furthermore, we show that ρ_n is well-defined. Now, let $\bar{x} \in \Omega$. Then $A\bar{x} = SA\bar{x}$. So

$$\begin{aligned}
\|(S-I)Az_n\|^2 &= \langle (S-I)Az_n, (S-I)Az_n\rangle \\
&= \langle (S-I)Az_n - (S-I)A\bar{x}, (S-I)Az_n\rangle \\
&= \langle SAz_n - SA\bar{x} + A\bar{x} - Az_n, (S-I)Az_n\rangle \\
&= \langle SAz_n - SA\bar{x}, (S-I)Az_n\rangle + \langle A\bar{x} - Az_n, (S-I)Az_n\rangle \\
&= \langle SAz_n - SA\bar{x}, (S-I)Az_n\rangle + \langle \bar{x} - z_n, A^*(S-I)Az_n\rangle \\
&\leq \|SAz_n - SA\bar{x}\|\|(S-I)Az_n\| + \|x - z_n\|\|A^*(S-I)Az_n\|.
\end{aligned} \tag{3.2}$$

Hence, for $SAz_n \neq Az_n$, that is, $(S-I)Az_n > 0$, we have $A^*(S-I)Az_n \neq 0$. This implies that ρ_n is well-defined. □

Lemma 3.3. *Let $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ be three sequences generated by Algorithm* 3.1 *and $\bar{x} \in \Omega$. Then the following inequality is satisfied:*

$$\|y_n - \bar{x}\|^2 \leq \|z_n - \bar{x}\|^2 - \rho_n[(1-k_2)\|(S-I)Az_n\|^2 - \rho_n\|A^*(S-I)Az_n\|^2]. \tag{3.3}$$

Proof. Let $\bar{x} \in \Omega$. From (3.1) and Lemma 2.3(1), we have

$$\begin{aligned}\|y_n - \bar{x}\|^2 &= \|z_n - \bar{x} + \rho_n A^*(S-I)Az_n\|^2 \\ &\le \|z_n - \bar{x}\|^2 + 2\rho_n \langle z_n - \bar{x}, A^*(S-I)Az_n\rangle + \rho_n^2 \|A^*(S-I)Az_n\|^2,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\rho_n^2 \|A^*(S-I)Az_n\|^2 &= \rho_n^2 \langle A^*(S-I)Az_n, A^*(S-I)Az_n\rangle \\ &\le \rho_n^2 \langle AA^*(S-I)Az_n, (S-I)Az_n\rangle \\ &\le \rho_n^2 \|A\|^2 \|(S-I)Az_n\|^2.\end{aligned} \tag{3.5}$$

Since S is a demicontractive mapping and $A\bar{x} \in Q = F(S)$, we have

$$\begin{aligned}\langle z_n - \bar{x}, A^*(S-I)Az_n\rangle &= \langle A(z_n - \bar{x}), (S-I)Az_n\rangle \\ &= \langle A(z_n - \bar{x}) + (S-I)Az_n - (S-I)Az_n, (S-I)Az_n\rangle \\ &= \langle SAz_n - A\bar{x}, (S-I)Az_n\rangle - \|(S-I)Az_n\|^2 \\ &= \frac{1}{2}\left(\|SAz_n - A\bar{x}\|^2 + \|(S-I)Az_n\|^2 - \|Az_n - A\bar{x}\|^2\right) - \|(S-I)Az_n\|^2 \\ &\le \frac{1}{2}\left(\|Az_n - A\bar{x}\|^2 + k_2\|(S-I)Az_n\|^2\right) \\ &\quad + \frac{1}{2}\left(\|(S-I)Az_n\|^2 - \|Az_n - A\bar{x}\|^2\right) - \|(S-I)Az_n\|^2 \\ &= \frac{k_2 - 1}{2}\|(S-I)Az_n\|^2.\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), it follows that

$$\|y_n - \bar{x}\|^2 \le \|z_n - \bar{x}\|^2 - \rho_n[(1-k_2)\|(S-I)Az_n\|^2 - \rho_n\|A^*(S-I)Az_n\|^2]. \qquad \square$$

Lemma 3.4. *Let $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ be three sequences generated by Algorithm* 3.1 *and $\bar{x} \in \Omega$. Then the following inequality is satisfied:*

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \|z_n - \bar{x}\|^2 - \beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n - y_n\|^2 \\ &\quad - \beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2 - \rho_n\|A^*(S-I)Az_n\|^2].\end{aligned} \tag{3.7}$$

Proof. By using the convexity of $\|\cdot\|^2$ and Lemma 2.3(4), we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|(1-\beta_n)(z_n - \bar{x}) + \beta_n[(1-\gamma_n)y_n + \gamma_n Ty_n - \bar{x}\|^2] \\ &\le (1-\beta)\|z_n - \bar{x}\|^2 + \beta_n[\|(1-\gamma_n)y_n + \gamma_n Ty_n - \bar{x}\|]^2 \\ &= (1-\beta)\|z_n - \bar{x}\|^2 + \beta_n[\|(1-\gamma_n)(y_n - \bar{x}) + \gamma_n(Ty_n - \bar{x})\|^2] \\ &= (1-\beta)\|z_n - \bar{x}\|^2 + \beta_n[(1-\gamma_n)\|y_n - \bar{x}\|^2 \\ &\quad + \gamma_n\|Ty_n - T\bar{x}\|^2 - \gamma_n(1-\gamma_n)\|Ty_n - y_n\|^2] \\ &\le (1-\beta)\|z_n - \bar{x}\|^2 + \beta_n[(1-\gamma_n)\|y_n - \bar{x}\|^2 \\ &\quad + \gamma_n(\|y_n - \bar{x}\|^2 + k_1\|y_n - Ty_n\|^2) - \gamma_n(1-\gamma_n)\|Ty_n - y_n\|^2] \\ &= (1-\beta)\|z_n - \bar{x}\|^2 + \beta_n\|y_n - \bar{x}\|^2 - \beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n - y_n\|^2.\end{aligned} \tag{3.8}$$

By Lemma 3.3, we have

$$\|x_{n+1} - \bar{x}\|^2 \le (1-\beta)\|z_n - \bar{x}\|^2 - \beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n - y_n\|^2$$

$$
\begin{aligned}
&+\beta_n[\|z_n-\bar{x}\|^2-\rho_n((1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2)] \\
&=\|z_n-\bar{x}\|^2-\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2 \\
&\quad-\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2]. \qquad (3.9)
\end{aligned}
$$

□

Theorem 3.5. *Let $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm* 3.1 *converges strongly to an element $\bar{x}$ of Ω, where $\bar{x}$ is the minimum-norm solution of the problem* (SCFPP)*, for each $n\geq 1$, the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(2) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n\leq 1$;

(3) $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n\leq 1$;

(4) $1-k_1-\gamma_n\geq\epsilon$ *for some $\epsilon>0$ small enough.*

Proof. From Lemma 3.4, we have

$$\|x_{n+1}-\bar{x}\|\leq\|z_n-\bar{x}\|.$$

Therefore, we have

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|&\leq\|z_n-\bar{x}\| \\
&\leq(1-\alpha_n)\|x_n-\bar{x}\|+\alpha\|\bar{x}\| \\
&\leq\max\{\|x_n-\bar{x}\|,\|\bar{x}\|\}.
\end{aligned}
$$

By induction, we have

$$\|x_n-\bar{x}\|\leq\max\{\|x_1-\bar{x}\|,\|\bar{x}\|\}.$$

Thus $\{x_n-\bar{x}\}$ is bounded and so $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ are bounded.

Next, we discuss two cases to establish the strong convergence.

Case I. Suppose that $\{\|x_{n+1}-\bar{x}\|\}$ is monotonically decreasing sequence. Then $\{\|x_n-\bar{x}\|\}$ is convergent and, as $n\to\infty$,

$$\|x_{n+1}-\bar{x}\|^2-\|x_n-\bar{x}\|^2\to 0. \qquad (3.10)$$

From Lemma 3.4, we have

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2&\leq\|z_n-\bar{x}\|^2-\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2 \\
&\quad-\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2] \\
&=\|(1-\alpha_n)x_n-\bar{x}\|^2-\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2 \\
&\quad-\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2] \\
&=\|x_n-\bar{x}-\alpha_nx_n\|^2-\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2 \\
&\quad-\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2] \\
&\leq\|x_n-\bar{x}\|^2+\alpha_n(\alpha_n\|x_n\|^2-(1-\alpha_n)\langle x_n-x,x_n\rangle) \\
&\quad-\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2
\end{aligned}
$$

$$-\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2]. \tag{3.11}$$

Since $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ are bounded, there exists $\mathscr{M}>0$ such that

$$\alpha_n\|x_n\|^2-(1-\alpha_n)\langle x_n-\bar{x},x_n\rangle<\mathscr{M}$$

for all $n\geq 1$. Thus we have

$$\begin{aligned}&\|x_n-\bar{x}\|^2-\|x_{n+1}-\bar{x}\|^2+\beta_n\gamma_n(1-k_1-\gamma_n)\|Ty_n-y_n\|^2\\&\quad+\beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2]\leq\alpha_n\mathscr{M}.\end{aligned} \tag{3.12}$$

From this together with (3.12) and $\alpha_n\to 0$ as $n\to\infty$, it follows that

$$\|y_n-Ty_n\|\to 0 \text{ and } \beta_n\rho_n[(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2]\to 0. \tag{3.13}$$

It follows from the condition on ρ_n that

$$\rho_n<\frac{(1-k_2)\|(S-I)Az_n\|^2}{\|A^*(S-I)Az_n\|^2}-\epsilon. \tag{3.14}$$

Also, we have

$$\rho_n\|A^*(S-I)Az_n\|^2<(1-k_2)\|(S-I)Az_n\|^2-\epsilon\|A^*(S-I)Az_n\|^2$$

and hence we have

$$\epsilon\|A^*(S-I)Az_n\|^2<(1-k_2)\|(S-I)Az_n\|^2-\rho_n\|A^*(S-I)Az_n\|^2\to 0 \tag{3.15}$$

as $n\to\infty$, which shows that

$$\|A^*(S-I)Az_n\|^2\to 0 \tag{3.16}$$

as $n\to\infty$ and so

$$\|y_n-z_n\|\to 0 \tag{3.17}$$

as $n\to\infty$. Furthermore, we obtain from Lemma 3.3 that

$$\begin{aligned}0&<\epsilon(1-k_2)\|(S-I)Az_n\|^2\leq\rho_n(1-k_2)\|(S-I)Az_n\|^2\\&\leq\|z_n-\bar{x}\|^2-\|y_n-\bar{x}\|^2+\rho_n^2\|A^*(S-I)Az_n\|^2\\&\leq\|(1-\alpha_n)x_n-x^*\|^2+\rho_n^2\|A^*(S-I)Az_n\|^2\\&\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|x_n\|^2+\rho_n^2\|A^*(S-I)Az_n\|^2\to 0\end{aligned}$$

as $n\to\infty$. This implies that

$$\|(S-I)Az_n\|\to 0 \tag{3.18}$$

as $n\to\infty$, we have

$$\|z_n-x_n\|=\|(1-\alpha_n)x_n-x_n\|\leq\alpha_n\|x_n\|\to 0$$

and

$$\|x_n-y_n\|\leq\|y_n-z_n\|+\|z_n-x_n\|\to 0$$

as $n\to\infty$. Since $\|x_n-y_n\|\to 0$ and $\|y_n-Ty_n\|\to 0$ as $n\to\infty$, we obtain

$$\|x_n-Ty_n\|\leq\|x_n-y_n\|+\|y_n-Ty_n\|\to 0$$

as $n\to\infty$. Since $\|z_n-y_n\|\to 0$ and $\|y_n-Ty_n\|\to 0$ as $n\to\infty$, we obtain

$$\|z_n-Ty_n\|\leq\|z_n-y_n\|+\|y_n-Ty_n\|\to 0.$$

Therefore, from Algorithm 3.1, it follows that

$$\begin{aligned}\|x_{n+1}-Ty_n\|^2 &= \|(1-\beta_n)(z_n-Ty_n)+\beta_n\|(1-\gamma_n)y_n+\gamma_nTy_n-Ty_n\|^2\\ &\le (1-\beta)\|z_n-Ty_n\|^2+\beta_n\|(1-\gamma_n)y_n+\gamma_nTy_n-Ty_n\|^2\\ &= (1-\beta)\|z_n-Ty_n\|^2+\beta_n\|(1-\gamma_n)(y_n-Ty_n)\|^2\\ &\le (1-\beta)\|z_n-Ty_n\|^2+\beta_n(1-\gamma_n)\|(y_n-Ty_n)\|^2\to 0\end{aligned}$$

as $n\to\infty$, which implies that

$$\|x_{n+1}-x_n\|\le \|x_{n+1}-Ty_n\|+\|x_n-Ty_n\|\to 0$$

as $n\to\infty$. Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with $z_{n_j}\rightharpoonup v\in H_1$. Thus, by $z_{n_j}\rightharpoonup v\in H_1$ and $\|y_n-z_n\|\to 0$ as $n\to\infty$, it follows that $y_{n_j}\rightharpoonup v\in H_1$. By the demiclosedness principle of $T-I$ at 0 and (3.9), we have $v\in F(T)=C$. Since A is a linear bounded operator and $z_{n_j}\rightharpoonup v\in H_1$, we have $Az_{n_j}\rightharpoonup Av\in H_2$. Hence, by (3.18), we have

$$\|SAz_{n_j}-Az_{n_j}\|\to 0$$

as $j\to\infty$. Since $S-I$ is demiclosed at 0, it follows that $Av\in F(S)=Q$ and so $v\in\Omega$.

Next, we prove that the sequence $\{x_n\}$ converges strongly to the point v. From Lemma 3.3 and Lemma 3.4, it follows that

$$\begin{aligned}\|x_{n+1}-v\|^2 &\le \|z_n-v\|^2\\ &= \|(1-\alpha_n)(x_n-v)-\alpha_nv\|^2\\ &= (1-\alpha_n)^2\|x_n-v\|^2+\alpha_n^2\|v\|^2-2\alpha_n(1-\alpha_n)\langle x_n-v,v\rangle\\ &\le (1-\alpha_n)\|x_n-v\|^2+\alpha_n(\alpha_n\|v\|^2-2(1-\alpha_n)\langle x_n-v,v\rangle). \end{aligned}\tag{3.19}$$

Since $\alpha_n\|v\|^2-2(1-\alpha_n)\langle x_n-v,v\rangle\to 0$ as $n\to\infty$. From 2.4 and (3.19), it follows that $\|x_n-v\|\to 0$, that is, $x_n\to v$ as $n\to\infty$.

Case II. Suppose that $\{\|x_{n+1}-\bar{x}\|\}$ is not monotonically decreasing. Let $\Gamma_k=\|x_n-\bar{x}\|^2$ and $\tau:\mathbb{N}\to\mathbb{N}$ be a a function defined by

$$\tau(n):=\max\{k\in\mathbb{N}:k\ge n,\ \Gamma_k\le\Gamma_{k+1}\}$$

for all $n\ge n_0$ (for some n_0 large enough). Clearly, τ is a nondecreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and

$$\Gamma_{\tau(n)+1}-\Gamma_{\tau(n)}\ge 0$$

for all $n\ge n_0$. From (3.12), it follows that

$$\|y_{\tau(n)}-Ty_{\tau(n)}\|^2\le\frac{\alpha_{\tau(n)}\mathscr{M}}{\beta_{\tau(n)}\gamma_{\tau(n)}(1-k_1-\gamma_{\tau(n)})}\to 0$$

as $n\to\infty$ and so

$$\|y_{\tau(n)}-Ty_{\tau(n)}\|\to 0$$

as $n\to\infty$.

Next, we show that $\|(S-I)Az_{\tau(n)}\|\to 0$ as $n\to\infty$,

$$\|y_{\tau(n)}-z_{\tau(n)}\|=\rho_{\tau(n)}\|A^*(S-I)Az_{\tau(n)}\|\le\rho_{\tau(n)}\|A^*\|\|(S-I)Az_{\tau(n)}\|\to 0$$

and

$$\|v_{\tau(n)} - x_{\tau(n)}\| \to 0, \quad \|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$$

as $n \to \infty$. Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{\tau(n)}\}$ of $\{v_n\}$ which converges weakly to a point $v \in H_1$. Since $\|z_{\tau(n)} - x_{\tau(n)}\| \to 0$ as $n \to \infty$ and $\|y_n - z_n\| \to 0$ as $n \to \infty$, it follows that

$$x_{\tau(n)} \rightharpoonup v \in H_1, \quad y_{\tau(n)} \rightharpoonup v \in H_1.$$

By the demiclosedness principle of $T - I$ at 0 and $\|y_{\tau(n)} - Ty_{\tau(n)}\| \to 0$ as $n \to \infty$, we have $v \in F(T) = C$.

Similarly, we can show that $v \in F(S) = Q$. Therefore, $v \in \Omega$. Note that, for all $n \geq n_0$,

$$\begin{aligned}0 &\leq \|x_{\tau(n)+1} - v\|^2 \\ &\leq \|y_{\tau(n)} - v\|^2 + \|z_{\tau(n)} - v\|^2 \\ &\leq \alpha_{\tau(n)}[-2\langle z_{\tau(n)} - v, v\rangle - \|x_{\tau(n)} - v\|^2],\end{aligned}$$

which implies that

$$\|x_{\tau(n)} - v\|^2 \leq -2\langle z_{\tau(n)} - v, v\rangle.$$

Thus we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - v\| = 0.$$

Hence we have

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for all $n \geq n_0$, we have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$) since $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Therefore, it follows that, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$$

and so $\lim_{n\to\infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to v. This completes the proof. □

Corollary 3.6. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be an adjoint operator of A. Let $T : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $T - I$ is demiclosed at 0 and $C := F(T) \neq \emptyset$. Let $S : H_2 \to H_2$ be a quasi-nonexpansive mapping such that $S - I$ is demiclosed at 0 and $Q := F(S) \neq \emptyset$. Assume that the problem* (SCFPP) *has a nonempty solution set Γ. Let $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm* 3.1 *converges strongly to an element $\bar{x}$ of Ω, where $\bar{x}$ is the minimum-norm solution of the problem* (SCFPP)*, for each $n \geq 1$, the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq 1$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n \leq 1$;

(4) $1 - k_1 - \gamma_n \geq \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Proof. The conclusion follows from Theorems 3.5. □

4. Numerical Examples

In this section, we give a numerical example to demonstrate the convergence of our algorithm. All codes were written in MATLAB 2017b and run on Dell i-5 Core laptop.

Example 4.1. Let $H_1 = (\mathbb{R}^3, \|\cdot\|_2) = H_2$. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be two mappings defined by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}.$$

It is clear that both T and S are 0-demicontractive mappings.

The stopping criterion for our testing method is taken as

$$\|x_{n+1} - x_n\|_2 < 10^{-4},$$

where $x_1 = \begin{pmatrix} a_1 \\ b_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$.

Let us assume that $A = \begin{pmatrix} 7 & -3 & -5 \\ -8 & 4 & -8 \\ -5 & -8 & 7 \end{pmatrix}$.

Then Algorithm 3.1 becomes as follows:

Algorithm 4.2.

Initialization. Given $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{1}{80\sqrt{n+2}}$, $\gamma_n = \frac{1}{5}\left[1 + \frac{3}{100\sqrt{n+1}}\right]$.

Let $x_1 = x \in H_1$ be arbitrary.

Step 1. Set $n = 1$ and compute

$$z_n = \left(1 - \frac{1}{\sqrt{n+1}}\right)x_n, \quad y_n = z_n + \rho_n A^*(S-I)Az_n,$$

where the step size ρ_n be chosen in such a way that

$$\rho_n = \left(\epsilon, \frac{(1-k_2)\|(S-I)Az_n\|^2}{\|A^*(S-I)Az_n\|^2} - \epsilon\right), \quad SAz_n \neq Az_n, \tag{4.1}$$

for small enough $\epsilon > 0$, otherwise $\rho_n = \rho$ (ρ being any nonnegative value).

Step 2. Compute

$$x_{n+1} = \left(1 - \frac{1}{80\sqrt{n+2}}\right)z_n + \frac{1}{80\sqrt{n+2}}\left[\left(1 - \frac{1}{5}\left[1 + \frac{3}{100\sqrt{n+1}}\right]\right)y_n + \frac{1}{5}\left[1 + \frac{3}{100\sqrt{n+1}}\right]Ty_n\right].$$

If $y_n = z_n$ and $x_{n+1} = z_n$, then $z_n \in \Omega$.

Set $n \leftarrow n+1$ and go to *Step* 1.

Case I: Take $\rho = 0.01$. Then we have the numerical analysis tabulated in Table 1 and show in Figure 1.

Table 1. Example 4.1, *Case* I

ρ	Time taken	Iterations	a_n	b_n	c_n	$\|x_{n+1}-x_n\|_2$
0.01	0.174088	2	1.1656	0.2959	1.4584	4.5905
		3	0.4905	0.1261	0.6142	1.0942
		4	0.2443	0.0635	0.3061	0.3993
		5	0.1345	0.0354	0.1687	0.1781
		6	0.0793	0.0211	0.0996	0.0896
		7	0.0492	0.0132	0.0618	0.0490
		8	0.0317	0.0086	0.0399	0.0284
		9	0.0211	0.0057	0.0265	0.0173
		10	0.0144	0.0039	0.0181	0.0109
		11	0.0100	0.0028	0.0126	0.0071
		12	0.0071	0.0020	0.0090	0.0047
		13	0.0051	0.0014	0.0065	0.0032
		14	0.0037	0.0011	0.0047	0.0022
		15	0.0028	0.0008	0.0035	0.0016
		16	0.0021	0.0006	0.0026	0.0011
		17	0.0016	0.0004	0.0020	0.0008
		18	0.0012	0.0003	0.0015	0.0006
		19	0.0009	0.0003	0.0012	0.0005
		20	0.0007	0.0002	0.0009	0.0003
		21	0.0006	0.0002	0.0007	0.0003
		22	0.0004	0.0001	0.0006	0.0002
		23	0.0003	0.0001	0.0004	0.0002
		24	0.0003	0.0001	0.0003	0.0001

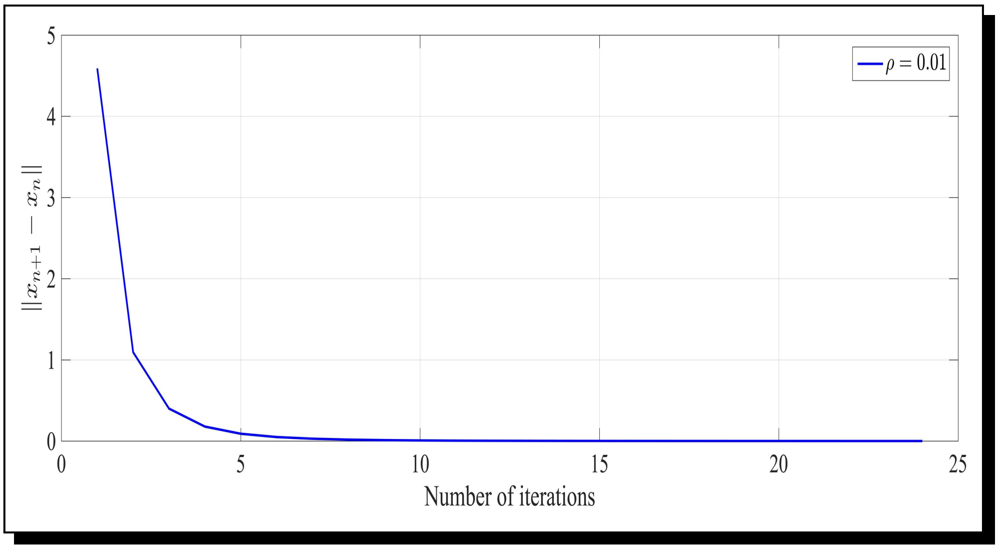

Figure 1. Example 4.1, *Case* I

Case-II: Take $\rho = 0.001$. Then we have the numerical analysis tabulated in Table 2 and show in Figure 2.

Table 2. Example 4.1, *Case* II

ρ	Time taken	Iterations	a_n	b_n	c_n	$\|x_{n+1} - x_n\|_2$
0.001	0.022494	2	1.1695	0.2943	1.4624	4.5853
		3	0.4935	0.1249	0.6173	1.0953
		4	0.2464	0.0627	0.3083	0.4005
		5	0.1360	0.0348	0.1703	0.1789
		6	0.0804	0.0206	0.1007	0.0902
		7	0.0500	0.0129	0.0626	0.0494
		8	0.0323	0.0083	0.0404	0.0287
		9	0.0215	0.0056	0.0269	0.0175
		10	0.0147	0.0038	0.0184	0.0111
		11	0.0102	0.0027	0.0128	0.0072
		12	0.0073	0.0019	0.0091	0.0048
		13	0.0053	0.0014	0.0066	0.0033
		14	0.0038	0.0010	0.0048	0.0023
		15	0.0029	0.0008	0.0036	0.0016
		16	0.0021	0.0006	0.0027	0.0012
		17	0.0016	0.0004	0.0020	0.0008
		18	0.0012	0.0003	0.0016	0.0006
		19	0.0010	0.0003	0.0012	0.0005
		20	0.0007	0.0002	0.0009	0.0003
		21	0.0006	0.0002	0.0007	0.0003
		22	0.0005	0.0001	0.0006	0.0002
		23	0.0004	0.0001	0.0005	0.0002
		24	0.0003	0.0001	0.0004	0.0001

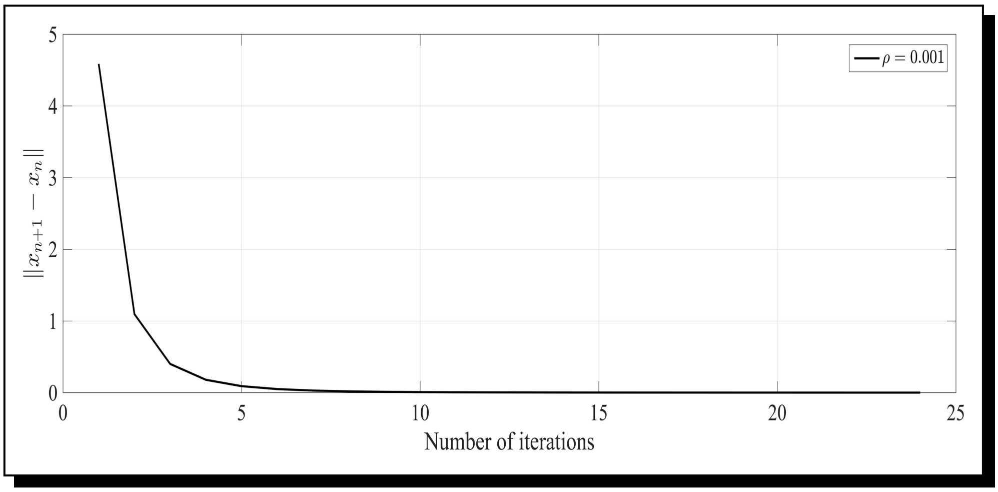

Figure 2. Example 4.1, *Case* II

Case III: Take $\rho = 0.0001$. Then we have the numerical analysis tabulated in Table 3 and show in Figure 3.

Table 3. Example 4.1, *Case* III

ρ	Time taken	Iterations	a_n	b_n	c_n	$\|x_{n+1}-x_n\|_2$
0.0001	0.015636	2	1.1699	0.2942	1.4628	4.5847
		3	0.4938	0.1248	0.6176	1.0955
		4	0.2466	0.0626	0.3085	0.4006
		5	0.1362	0.0347	0.1704	0.1790
		6	0.0805	0.0206	0.1008	0.0903
		7	0.0500	0.0128	0.0626	0.0494
		8	0.0323	0.0083	0.0405	0.0287
		9	0.0215	0.0056	0.0270	0.0175
		10	0.0147	0.0038	0.0184	0.0111
		11	0.0103	0.0027	0.0129	0.0072
		12	0.0073	0.0019	0.0091	0.0048
		13	0.0053	0.0014	0.0066	0.0033
		14	0.0039	0.0010	0.0048	0.0023
		15	0.0029	0.0008	0.0036	0.0016
		16	0.0021	0.0006	0.0027	0.0012
		17	0.0016	0.0004	0.0020	0.0008
		18	0.0012	0.0003	0.0016	0.0006
		19	0.0010	0.0003	0.0012	0.0005
		20	0.0007	0.0002	0.0009	0.0003
		21	0.0006	0.0002	0.0007	0.0003
		22	0.0005	0.0001	0.0006	0.0002
		23	0.0004	0.0001	0.0005	0.0002
		24	0.0003	0.0001	0.0004	0.0001

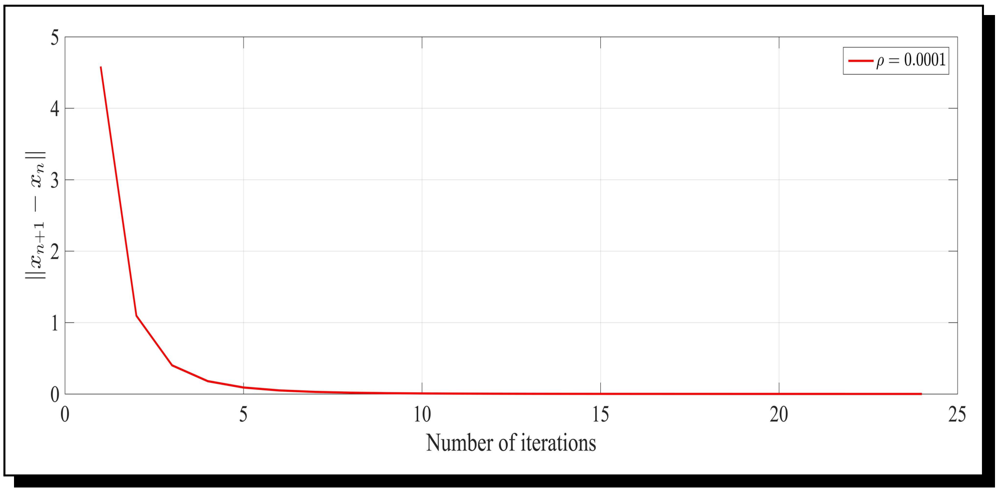

Figure 3. Example 4.1, Case III

Remark 4.3. We see that the smaller the choice of $\lambda > 0$ chosen, the less the number of iterations required.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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