Communications in Mathematics and Applications
Vol. 10, No. 2, pp. 325–336, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i2.1082

Special Issue of the Conference
The 10th Asian Conference on Fixed Point Theory and Optimization
Chiang Mai, Thailand, July 16 – 18, 2018
***Editors*: Poom Kumam, Parin Chaipunya and Nantaporn Chuensupantharat**

Research Article

Iterative Methods for Solving the Proximal Split Feasibility Problems

Manatsawin Mamat[1], Raweerote Suparatulatorn[2] and Prasit Cholamjiak[1,*]

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*
[2] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[*]**Corresponding author:** prasitch2008@yahoo.com

Abstract. In this work, we study the proximal split feasibility problem. We introduce a new algorithm with inertial technique for solving this problem in Hilbert spaces. We also prove the strong convergence theorem under some suitable conditions. Finally, we give some numerical experiments to support our results.

Keywords. Proximal split feasibility problem; Inertial; Hilbert space; Strong convergence theorem

MSC. 47H04; 47H10; 54H25

Received: August 31, 2018 **Revised:** October 2, 2018 **Accepted:** October 5, 2018

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Suppose that $f : H_1 \to \mathbb{R} \cup \{+\infty\}$, $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functions and $A : H_1 \to H_2$ is a bounded linear operator. In this research, we shall consider the following *Proximal Split Feasibility Problem* (PSFP):

Find a solution $x^* \in H_1$ such that

$$\min_{x\in H_1} f(x)+g_\lambda(Ax), \tag{1.1}$$

where $g_\lambda : H_2 \to \mathbb{R}\cup\{+\infty\}$, is the Moreau-Yosida approximate [11] defined by

$$g_\lambda(y) := \min_{u\in H_2} g(u)+\frac{1}{2\lambda}\|u-y\|^2.$$

Problem (1.1) includes many nonlinear problems in applied sciences, engineering and economics. For example, if we take $f=\delta_C$ [defined as $\delta_C(x)=0$ if $x\in C$ and $+\infty$ otherwise], the indicator function of nonempty, closed and convex subset C of H_1 and $g=\delta_Q$, the indicator function of nonempty, closed and convex subset Q of H_2, then Problem (1.1) reduces to the following *Split Feasibility Problem* (SFP):

Find

$$x\in C \text{ such that } Ax\in Q. \tag{1.2}$$

The SPFP attracts the attention of many authors due to its application in signal processing, medical image reconstruction and modeling inverse problems which arise from phase retrievals. Various algorithms have been invented to solve it (see, e.g., [1–5, 10, 13–17, 19–21] and references therein). If $A=I$, $H_1=H_2=H$ and g be differentiable, where I is an identity mapping on H, then Problem (1.1) reduces to the following minimization problem:

$$\min_{x\in H} f(x)+g(x). \tag{1.3}$$

Suppose that the problem (1.1) has at least a solution and denote by Γ the solution set of (1.1).

Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2+\|\nabla l(x)\|^2}$ with $h(x)=\frac{1}{2}\|(I-prox_{\lambda g})Ax\|^2$, $l(x)=\frac{1}{2}\|(I-prox_{\lambda f})x\|^2$, $x\in H_1$, where $prox_{\lambda g}(y)=\arg\min_{u\in H_1}\left\{g(u)+\frac{1}{2\lambda}\|u-y\|^2\right\}$. Moudafi and Thakur [8] introduced the following split proximal algorithm for solving (1.1):

Split Proximal Algorithm. Given an initial point $x_1\in H_1$. Assume that x_n has been constructed and $\theta(x_n)\neq 0$, then compute x_{n+1} via the rule

$$x_{n+1}=prox_{\lambda u_n f}(x_n-\mu_n A^*(I-prox_{\lambda g})Ax_n), \quad n\ge 1, \tag{1.4}$$

where step size $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ and $0<\rho_n<4$. If $\theta(x_n)=0$, then $x_{n+1}=x_n$ is a solution of (1.1) and the iterative process stop. Otherwise, we set $n:=n+1$ and go to (1.4). Using the split proximal algorithm (1.4), Moudafi and Thakur [8] proved a weak convergence theorem for approximating a solution of (1.1) under the condition that $\epsilon\le\rho_n\le\frac{4h(x_n)}{h(x_n)+l(x_n)}-\epsilon$ for some $\epsilon>0$.

Remark 1.1. It is observed in Shehu and Iyiola [12] that the above condition means that the convergence is ensured only in the very restrictive case when a condition imposed on the sequence $\{x_n\}$ itself is fulfilled. Thus, it is of practical computational importance to introduce a new iterative scheme in which this condition is avoided and replaced with a much simple condition on the step size and convergence result is still achieved.

Ochs *et al.* [9] proposed the following forward-backward splitting algorithm by using the inertial technique for solving convex minimization problem for the sum of a smooth convex function g and a non-smooth convex function $f : x_0, x_1\in H_1$

$$x_{n+1}=prox_{\alpha f}(x_n-\alpha\nabla g(x_n)+\beta(x_n-x_{n-1})), \tag{1.5}$$

where $\alpha > 0$ and $\beta \in [0,1)$. Recently, Shehu and Iyiola [12] obtained weak convergence results for solving PSFP by replacing the condition imposed in with a much simpler condition on the step size. They studied convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method. They introduced the following inertial extrapolation split proximal algorithm:

Given initial points $x_0, x_1 \in H_1$. Assume that x_n has been constructed and $\theta(y_n) \neq 0$, then compute x_{n+1} via the rule

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\theta^2(y_n)}(\nabla h(y_n) + \nabla l(y_n)), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n z_n, \quad n \geq 1 \end{cases} \tag{1.6}$$

where $0 < \rho_n < 4$. If $\nabla h(y_n) = 0 = \nabla l(y_n)$ and $y_n = x_n$, then the sequence $\{x_n\}$ is a solution of PSFP and the iterative process stops. Otherwise, we set $n := n+1$ and go to (1.6).

In this research, we introduce a new algorithm for solving the proximal split feasibility problem. We prove strong convergence theorems under some suitable conditions. Some numerical experiments are shown in Section 4.

2. Preliminaries and Lemmas

In this section, we give some preliminaries which will be used in our proof. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\ x, y \in C.$$

For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \quad \forall\ y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for all $x, y \in C$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \tag{2.2}$$

for all $y \in C$. A mapping $T : H \to H$ is said to be *firmly nonexpansive* if and only if $2T - I$ is nonexpansive, or equivalently

$$\langle x - Tx, Tx - y \rangle \geq \|Tx - Ty\|^2, \quad \forall\ x, y \in H.$$

For example, projections and proximal mappings are firmly nonexpansive.

Let the proximal operator of the scaled function λf, where $\lambda > 0$, which can be expressed as $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous.

$$prox_{\lambda f}(v) = \arg\min_{x \in H} \{f(x) + (1/2\lambda)\|x - v\|_2^2\}. \tag{2.3}$$

This is also called the proximal operator of f with parameter λ. In a real Hilbert space H, we have the following equality:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{2.4}$$

and the subdifferential inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \tag{2.5}$$

for all $x, y \in H$.

Lemma 2.1 ([7]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\psi(n)\}_{n\ge n_0}$ of integers as follows:*

$$\psi(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}. \tag{2.6}$$

where $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\Gamma(n_0) \le \Gamma(n_0+1) \le \dots$ *and* $\Gamma(n) \to \infty$*;*

(ii) $\Gamma_{\psi(n)} \le \Gamma_{\psi(n)+1}$ *and* $\Gamma_n \le \Gamma_{\psi(n)+1}$, $\forall\ n \ge n_0$.

Lemma 2.2 ([6, 18]). *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1-\delta_n)a_n + b_n + c_n, \quad n \ge 1, \tag{2.7}$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n \le \delta_n M$ for some $M \ge 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \le 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In this section, we give strong convergence result using inertial extrapolation for solving (1.1), which is the main result of this paper. Now, set $\theta(x) := \sqrt{\|\nabla h(x) + \nabla l(x)\|^2}$ with $h(x) = \frac{1}{2}\|(I - prox_{\lambda g})Ax\|^2$, $l(x) = \frac{1}{2}\|(I - prox_{\lambda f})x\|^2$, $x \in H_1$ and we introduce the following inertial extrapolation split proximal algorithm:

Algorithm 3.1. *Given initial points $x_0, x_1, u \in H$. Assume that x_n has been constructed and and $\theta(y_n) \neq 0$, then compute x_{n+1} by the following manner:*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}) \\ z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\theta^2(y_n)}(\nabla h(y_n) + \nabla l(y_n)) \\ x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \quad n \ge 1 \end{cases} \tag{3.1}$$

where $0 < \alpha_n < 1$, $0 \le \beta_n < 1$ and $0 < \rho_n < 4$. If $\nabla h(y_n) = 0 = \nabla l(y_n)$ and $y_n = x_n$ then x_n is a solution of (1.1) and then iterative process stops. Otherwise, we set $n := n+1$ and go to (3.1).

Theorem 3.2. *Assume that $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,4)$, $\{\beta_n\} \subset [0,\beta]$, where $\beta \in [0.1)$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;* (C2) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$*;* (C3) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) strongly converges to $x^* = P_\Gamma u$ which is a solution in Γ.

Proof. Let $x^* = P_\Gamma u \in \Gamma$. Observe that $\nabla h(x) = A^*(I - prox_{\lambda g})Ax$ and $\nabla l(x) = (I - prox_{\lambda f})x$. Since $prox_{\lambda g}$ is firmly nonexpansive, the mapping $(I - prox_{\lambda g})$ is also firmly nonexpansive. It follows that

$$\langle \nabla h(y_n), y_n - x^* \rangle = \langle (I - prox_{\lambda g})Ay_n, Ay_n - Ax^* \rangle \geq \|(I - prox_{\lambda g})Ay_n\|^2 = 2h(y_n).$$

From (3.1), we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 &= \left\| y_n - x^* - \rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} (\nabla h(y_n) + \nabla l(y_n)) \right\|^2 \\
&= \|y_n - x^*\|^2 + \rho_n^2 \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \frac{\|\nabla h(y_n) + \nabla l(y_n)\|^2}{\theta^2(y_n)} \\
&\quad - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \langle \nabla h(y_n) + \nabla l(y_n), y_n - x^* \rangle \\
&= \|y_n - x^*\|^2 + \rho_n^2 \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \\
&\quad - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \langle A^*(I - prox_{\lambda g})A(y_n) + (I - prox_{\lambda f})y_n, y_n - x^* \rangle \\
&\leq \|y_n - x^*\|^2 - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} [2h(y_n) + 2l(y_n)] + \rho_n^2 \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \\
&= \|y_n - x^*\|^2 - 4\rho_n \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} + \rho_n^2 \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \\
&= \|y_n - x^*\|^2 - \rho_n(4 - \rho_n) \left(\frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \right) \\
&\leq \|y_n - x^*\|^2. \qquad (3.2)
\end{aligned}$$

So, we obtain $\|z_n - x^*\| \leq \|y_n - x^*\|$ for all $n \geq 0$. Moreover, we have

$$\|y_n - x^*\| = \|x_n - x^* + \beta_n(x_n - x_{n-1})\| \leq \|x_n - x^*\| + \beta_n \|x_n - x_{n-1}\|. \qquad (3.3)$$

It follows that, by (3.3)

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(\alpha_n u + (1 - \alpha_n)z_n) - x^*\| \\
&\leq \alpha_n \|u - x^*\| + (1 - \alpha_n)\|y_n - x^*\| \\
&\leq \alpha_n \|u - x^*\| + (1 - \alpha_n)[\|x_n - x^*\| + \beta_n \|x_n - x_{n-1}\|] \\
&= \alpha_n \|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| + (1 - \alpha_n)\beta_n \|x_n - x_{n-1}\| \\
&= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \left[\|u - x^*\| + (1 - \alpha_n)\frac{\beta_n}{\alpha_n} \|x_n - x_{n-1}\| \right]. \qquad (3.4)
\end{aligned}$$

By Lemma 2.2, we conclude that $\{x_n\}$ is bounded. By (2.4), we have

$$\begin{aligned}
\langle x_n - x^*, x_n - x_{n-1} \rangle &= \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_n - x^* - x_n + x_{n-1}\|^2 \\
&= \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_{n-1} - x^*\|^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^* + \beta_n(x_n - x_{n-1})\|^2 \\
&= \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x_{n-1}\|^2 + 2\beta_n \langle x_n - x^*, x_n - x_{n-1} \rangle \\
&= \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x_{n-1}\|^2 + 2\beta_n \left(\frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_{n-1} - x^*\|^2 \right) \\
&= \|x_n - x^*\|^2 + \beta_n^2 \|x_n - x_{n-1}\|^2 + \beta_n \|x_n - x^*\|^2 + \beta_n \|x_n - x_{n-1}\|^2 - \beta_n \|x_{n-1} - x^*\|^2
\end{aligned}$$

$$\leq \|x_n - x^*\|^2 + \beta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) + 2\beta_n\|x_n - x_{n-1}\|^2. \tag{3.5}$$

Now, using (2.5) and (3.1), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*)\|^2 \\
&\leq (1-\alpha_n)^2\|z_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|y_n - x^*\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{(h(y_n) + l(y_n)^2)}{\theta^2(y_n)} + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n)\beta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) \\
&\quad + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{(h(y_n) + l(y_n)^2)}{\theta^2(y_n)} \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.6}$$

We next consider two cases. Set $\Gamma_n = \|x_n - x^*\|^2$.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\{\Gamma_n\}$ is convergent. From (C1) and (C2), we can find a constant σ such that $(1-\alpha_n)\rho_n(4-\rho_n) \geq \sigma > 0$ for all $n \in \mathbb{N}$, it follows that, by (3.6)

$$\begin{aligned}
\Gamma_{n+1} &\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 \\
&\quad - (1-\alpha_n)\rho_n(4-\rho_n)\frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle,
\end{aligned} \tag{3.7}$$

which gives

$$\begin{aligned}
\sigma\frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} &\leq (1-\alpha_n)\Gamma_n + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) - \Gamma_{n+1} \\
&\quad + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (\Gamma_n - \Gamma_{n+1}) + (1-\alpha_n)\beta_n(\Gamma_n - \Gamma_{n-1}) + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.8}$$

We see that (C3) implies $\beta_n\|x_n - x_{n-1}\|^2 \to 0$ since $\{\alpha_n\}$ is bounded. Since $\{\Gamma_n\}$ converges and $\alpha_n \to 0$,

$$\frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \to 0. \tag{3.9}$$

Consequently, we have

$$\lim_{n\to\infty}(h(y_n) + l(y_n)) = 0 \Leftrightarrow \lim_{n\to\infty} h(y_n) = 0 \text{ and } \lim_{n\to\infty} l(y_n) = 0,$$

since $\theta^2(x_n) = \|\nabla h(x_n) + \nabla l(x_n)\|^2$ is bounded. This follows from the fact that ∇h is Lipschitz continuous with constant $\|A\|^2$, ∇l is nonexpansive and $\{y_n\}$ is bounded. Indeed

$$\|\nabla h(y_n)\| = \|\nabla h(y_n) - \nabla h(x^*)\| \leq \|A\|^2\|y_n - x^*\| \text{ and } \|\nabla l(y_n)\| = \|\nabla l(y_n) - \nabla l(x^*)\| \leq \|y_n - x^*\|.$$

Now, let z be a weak cluster point of $\{x_n\}$. So there exists a subsequence $\{x_{n_j}\}$ which weakly converges to z. Since $x_{n_j} \rightharpoonup z$, $j \to \infty$ and $\lim_{n\to\infty}\|y_n - x_n\| = 0$, we have $y_{n_j} \rightharpoonup z$, $j \to \infty$. The lower-semicontinuity of h implies that

$$0 \leq h(z) \leq \liminf_{j\to\infty} h(y_{n_j}) = \lim_{n\to\infty} h(y_n) = 0.$$

This shows that $h(z) = \frac{1}{2}\|(I - prox_{\lambda g})Az\| = 0$. Thus Az is a minimizer of g.

Also, the lower-semicontinuity of l implies that

$$0 \le l(z) \le \liminf_{j\to\infty} l(y_{n_j}) = \lim_{n\to\infty} l(y_n) = 0.$$

Hence $l(z) = \frac{1}{2}\|(I - prox_{\mu_n \lambda f})z\| = 0$ and $z \in \Gamma$. From (2.6) it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle &= \limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \\ &= \lim_{j\to\infty}\langle u - x^*, x_{n_j} - x^*\rangle \\ &= \langle u - x^*, z - x^*\rangle \\ &\le 0. \end{aligned} \tag{3.10}$$

From (3.7), it follows that

$$\Gamma_{n+1} \le (1-\alpha_n)\Gamma_n + 2(1-\alpha_n)\beta_n\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle. \tag{3.11}$$

Applying Lemma 2.2(ii) and using (3.10) and (3.11) and the conditions (C1) and (C3), we conclude that $\Gamma_n = \|x_n - x^*\|^2 \to 0$ and thus $x_n \to x^*$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\psi : \mathbb{N} \to \mathbb{N}$ as in (2.6). Then, by Lemma 2.2 we have $\Gamma_{\psi(n)} \le \Gamma_{\psi(n)+1}$. From (3.9), it follows that

$$\begin{aligned}\Gamma_{\psi(n)+1} \le\ &(1-\alpha_{\psi(n)})\Gamma_{\psi(n)} + (1-\alpha_{\psi(n)})\beta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) + 2(1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ &- \sigma\frac{(h(y_{\psi(n)}) + l(y_{\psi(n)}))^2}{\theta^2(y_{\psi(n)})} + 2\alpha_{\psi(n)}\langle u - x^*, x_{\psi(n)+1} - x^*\rangle,\end{aligned}$$

which gives

$$\begin{aligned}\sigma\frac{(h(y_{\psi(n)}) + l(y_{\psi(n)}))^2}{\theta^2(y_{\psi(n)})} &\le (1-\alpha_{\psi(n)})\beta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) + 2(1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ &\quad + 2\alpha_{\psi(n)}\langle u - x^*, x_{\psi(n)+1} - x^*\rangle \\ &\le (1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|\left(\sqrt{\Gamma_{\psi(n)}} + \sqrt{\Gamma_{\psi(n)-1}}\right) \\ &\quad + 2(1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|^2 \\ &\quad + 2\alpha_{\psi(n)}\langle u - x^*, x_{\psi(n)+1} - x^*\rangle \to 0.\end{aligned}$$

This shows that $h(y_{\psi(n)}) \to 0$ and $l(y_{\psi(n)}) \to 0$ as $n \to \infty$. Moreover, we have

$$\begin{aligned}\|y_{\psi(n)} - x_{\psi(n)}\| &= \|x_{\psi(n)} - \beta_{\psi(n)}(x_{\psi(n)} - x_{\psi(n)-1}) - x_{\psi(n)}\| \\ &= \beta_{\psi(n)}\|x_{\psi(n)} - x_{\psi(n)-1}\|. \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.12}$$

From (3.1), we have

$$\begin{aligned}\|z_{\psi(n)} - x_{\psi(n)}\| &\le \|y_{\psi(n)} - x_{\psi(n)}\| + \rho_{\psi(n)}\frac{h(y_{\psi(n)}) + l(y_{\psi(n)})}{\theta^2(y_{\psi(n)})}\|\nabla h(y_{\psi(n)}) + \nabla l(y_{\psi(n)})\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}\|x_{\psi(n)+1} - x_{\psi(n)}\| &\le \alpha_{\psi(n)}\|u - x_{\psi(n)}\| + (1-\alpha_{\psi(n)})\|z_{\psi(n)} - x_{\psi(n)}\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

From (3.7), we have

$$\Gamma_{\psi(n)+1} \le (1-\alpha_{\psi(n)})\Gamma_{\psi(n)} + (1-\alpha_{\psi(n)})\beta_{\psi(n)}(\|x_{\psi(n)}-x^*\|^2 - \|x_{\psi(n)-1}-x^*\|^2) \\ + 2(1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|^2 + 2\alpha_{\psi(n)}\langle u-x^*, x_{\psi(n)+1}-x^*\rangle,$$

which implies

$$\alpha_{\psi(n)}\Gamma_{\psi(n)} \le (1-\alpha_{\psi(n)})\beta_{\psi(n)}(\|x_{\psi(n)}-x^*\|^2 - \|x_{\psi(n)-1}-x^*\|^2) \\ + 2(1-\alpha_{\psi(n)})\beta_{\psi(n)}\|x_{\psi(n)}-x_{\psi(n)-1}\|^2 + 2\alpha_{\psi(n)}\langle u-x^*, x_{\psi(n)+1}-x^*\rangle.$$

Hence

$$\Gamma_{\psi(n)} \le \frac{(1-\alpha_{\psi(n)})\beta_{\psi(n)}}{\alpha_{\psi(n)}}(\|x_{\psi(n)}-x^*\|^2 - \|x_{\psi(n)-1}-x^*\|^2) \\ + \frac{2(1-\alpha_{\psi(n)})\beta_{\psi(n)}}{\alpha_{\psi(n)}}\|x_{\psi(n)}-x_{\psi(n)-1}\|^2 + 2\langle u-x^*, x_{\psi(n)+1}-x^*\rangle. \tag{3.14}$$

Now repeating the argument of the proof in Case 1, we obtain

$$\limsup_{n\to\infty}\langle u-x^*, x_{\psi(n)}-x^*\rangle \le 0.$$

Since $\|x_{\psi(n)+1}-x_{\psi(n)}\| \to 0$, we obtain $\limsup_{n\to\infty}\langle u-x^*, x_{\psi(n)+1}-x^*\rangle \le 0$. From (3.14) it follows that

$$\limsup_{n\to\infty}\Gamma_{\psi(n)} \le 0.$$

This implies that

$$\lim_{n\to\infty}\Gamma_{\psi(n)} = \lim_{n\to\infty}\|x_{\psi(n)}-x^*\|^2 = 0.$$

Hence $x_{\psi(n)} \to x^*$ as $n\to\infty$. On the other hand, we see that

$$\|x_{\psi(n)+1}-x^*\| \le \|x_{\psi(n)+1}-x_{\psi(n)}\| + \|x_{\psi(n)}-x^*\| \to 0$$

as $n\to\infty$. By Lemma 2.1, we have $\Gamma_n \le \Gamma_{\psi(n)+1}$ and thus

$$\Gamma_n = \|x_n-x^*\|^2 \le \|x_{\psi(n)+1}-z\|^2 \to 0.$$

So we conclude that $x_n \to x^*$ as $n\to\infty$. This completes the proof. □

4. Examples and Numerical Results

In this section, we give numerical experiments to support our main result this paper.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^5$. Let $f(x) := \|x\|_2$ and $g(x) := -\sum_{i=1}^{5}\log x_i$, $x = (x_1, x_2, \dots, x_5) \in \mathbb{R}^5$.

Let $A = \begin{pmatrix} 5 & 7 & 10 & 5 & 8 \\ 3 & 10 & 7 & 2 & 4 \\ 6 & 7 & 8 & 9 & 11 \\ 13 & 7 & 5 & 9 & 11 \\ 11 & 13 & 15 & 3 & 7 \end{pmatrix}$.

We aim to find $x^* \in \arg\min f$ such that $Ax^* \in \arg\min g$.

Choose $\alpha_n = \frac{1}{\sqrt{(200n)+1}}$, $\rho_n = 3.95$ and $\beta_n = \min\left\{0.5, \frac{1}{n^{1.5}\|x_n-x_{n-1}\|}\right\}$ if $x_n \ne x_{n-1}$ and $\beta_n = 0.5$ if $x_n = x_{n-1}$. The stopping criteria is defined by $E_n = \|x_n - x_{n-1}\|_2 < 10^{-5}$.

Then, we obtain the following results.

Table 1. Numerical results of Example 4.1

	Number of iterations		CPU time	
	$\beta_n = 0$	$\beta_n \neq 0$	$\beta_n = 0$	$\beta_n \neq 0$
Case 1 $x_0 = (1,3,5,7,9)$, $x_1 = (11,5,3,13,7)$, $u = (3,5,11,7,11)$	2434	2259	0.1977	0.1802
Case 2 $x_0 = (11,15,5,7,19)$, $x_1 = (11,15,5,13,3)$, $u = (13,25,11,6,11)$	2271	1994	0.1801	0.1629
Case 3 $x_0 = (25,13,27,3,23)$, $x_1 = (13,25,7,15,5)$, $u = (9,11,15,8,17)$	2262	1589	0.1714	0.1301
Case 4 $x_0 = (13,13,11,7,15)$, $x_1 = (25,5,33,31,5)$, $u = (11,13,9,8,21)$	2234	1640	0.1723	0.1339

The convergence behavior of error E_n for each cases are shown in Figures 1-4, respectively.

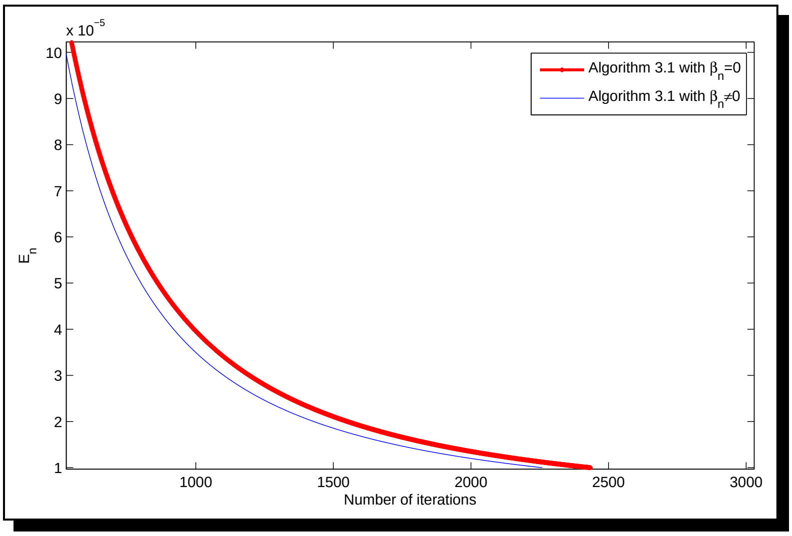

Figure 1. Error plotting E_n for Case 1 in Example 4.1

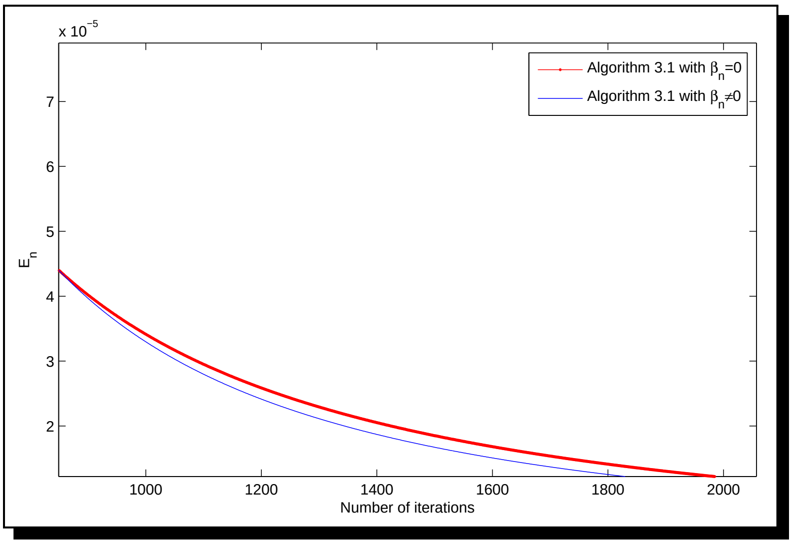

Figure 2. Error plotting E_n for Case 2 in Example 4.1

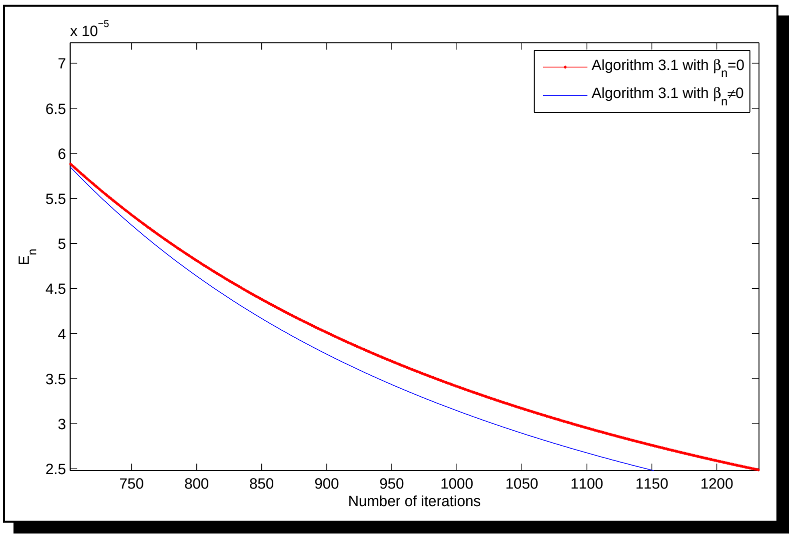

Figure 3. Error plotting E_n for Case 3 in Example 4.1

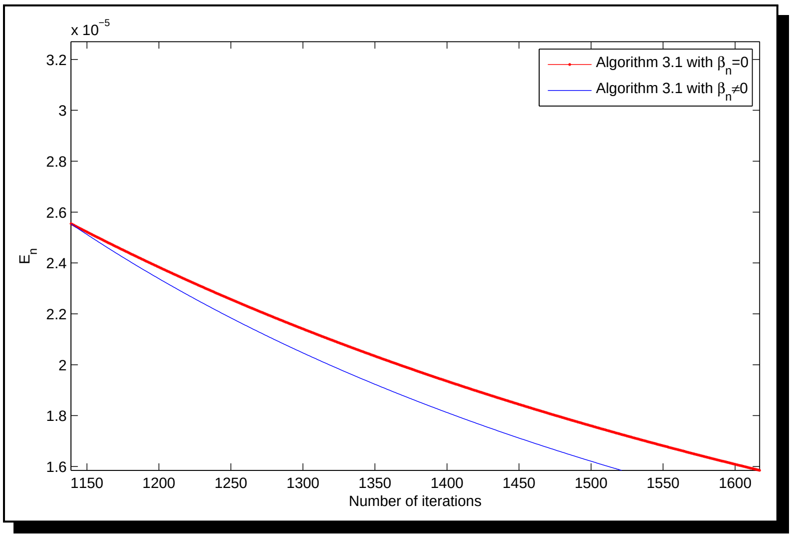

Figure 4. Error plotting E_n for Case 4 in Example 4.1

Acknowledgment

The authors thank the referees for valuable suggestions and University of Phayao.
P. Cholamjiak was supported by The Thailand Research Fund and University of Phayao under granted RSA6180084.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] J. Y. Bello Cruz and Y. Shehu, An iterative method for split inclusion problems without prior knowledge of operator norms, *Journal of Fixed Point Theory and Applications* **19**(3) (2017), 2017 – 2036, DOI: 10.1007/s11784-016-0387-8.

[2] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Probl.* **18**(2) (2002), 441 – 453, DOI: 10.1088/0266-5611/18/2/310.

[3] P. Cholamjiak and S. Suantai, A new CQ algorithm for solving split feasibility problems in Hilbert spaces, *Bulletin of the Malaysian Mathematical Sciences Society* (2018), 1 – 18, DOI: 10.1007/s40840-018-614-0.

[4] J. Deepho and P. Kumam, A viscosity approximation method for the split feasibility problems, in: *Transactions on Engineering Technologies*, G. C. Yang, S. I. Ao, X. Huang and O. Castillo (eds.), Springer, Dordrecht, (2014), pp. 69 – 77, DOI: 10.1007/978-94-017-9588-3_6.

[5] G. Lopez, V. Martin-Marquez, F. Wang and H. K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Probl.* **28** (2012), 085004, DOI: 10.1088/0266-5611/28/8/085004.

[6] P. E. Maingé, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **325** (2007), 469 – 479, DOI: 10.1016/j.jmaa.2005.12.066.

[7] P. E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.* **16** (2008), 899 – 912, DOI: 10.1007/s11228-008-0102-z.

[8] A. Moudafi and B. S. Thakur, Solving proximal split feasibility problems without prior knowledge of operator norms, *Optim. Lett.* **8**(7) (2014), 2099 – 2110, DOI: 10.1007/s11590-013-0708-4.

[9] P. Ochs, T. Brox and T. Pock, iPiasco: inertial proximal algorithm for strongly convex optimization, *J. Math. Imaging Vision* **53** (2015), 171 – 181, DOI: 10.1007/s10851-015-0565-0.

[10] B. Qu and N. Xiu, A note on the CQ algorithm for the split feasibility problem, *Inverse Probl.* **21**(5) (2005), 1655 – 1665, DOI: 10.1088/0266-5611/21/5/009.

[11] R. T. Rockafellar and R. Wets, *Variational Analysis*, Springer, Berlin (1988), DOI: 10.1007/978-3-642-02431-3.

[12] Y. Shehu and O. S. Iyiola, Convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method, *Journal of Fixed Point Theory and Applications* **19**(4) (2017), 2483 – 2510, DOI: 10.1007/s11784-017-0435-z.

[13] K. Sitthithakerngkiet, J. Deepho and P. Kumam, Modified hybrid steepest method for the split feasibility problem in image recovery of inverse problems, *Numerical Functional Analysis and Optimization* **38**(4) (2017), 507 – 522, DOI: 10.1080/01630563.2017.1287084.

[14] S. Suantai, N. Pholasa and P. Cholamjiak, The modified inertial relaxed CQ algorithm for solving the split feasibility problems, *J. Indust. Manag. Optim.* **14** (2018), 1595 – 1615, DOI: 10.3934/jimo.2018023.

[15] S. Suthep, Y. Shehu and P. Cholamjiak, Nonlinear iterative methods for solving the split common null point problem in Banach spaces, *Optimization Methods and Software* **34** (2018), 1 – 22, DOI: 10.1080/10556788.2018.1472257.

[16] U. Witthayarat, Y. J. Cho and P. Cholamjiak, On solving proximal split feasibility problems and applications, *Ann. Funct. Anal.* **9** (2018), 111 – 122, DOI: 10.1007/s11590-013-0708-4.

[17] H.-K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split, *Inverse Problems* **22**(6), (2006), 2021 – 2034, DOI: 10.1088/0266-5611/22/6/007.

[18] H.-K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.* **66** (2002), 240 – 256, DOI: 10.1112/S0024610702003332.

[19] Q. Yang, The relaxed CQ algorithm solving the split feasibility problem, *Inverse Probl.* **20**(4) (2004), 1261 – 1266, DOI: 10.1088/0266-5611/20/4/014.

[20] Q. Yang and J. Zhao, Generalized KM theorems and their applications, *Inverse Probl.* **22**(3) (2006), 833 – 844, DOI: 10.1088/0266-5611/22/3/006.

[21] Y. Yao, W. Jigang and Y.-C. Liou, Regularized methods for the split feasibility problem, *Abstr. Appl. Anal.* Article ID 140679, 13 (2012), DOI: 10.1155/2012/140679.