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Research Article

Convergence Theorem for Nonexpansive Semigroups in q-Uniformly Smooth Banach Spaces

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Abstract. In this paper, we present the iterative scheme nonexpansive semigroups in the framework of q-uniformly smooth and uniformly convex Banach spaces. Furthermore, we propose the strong convergence theorem for finding fixed points problem of nonexpansive semigroups under some appropriate conditions. Our results extend the recent ones of some authors.

Keywords. Nonexpansive semigroup; q-uniformly smooth; Banach space

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1. Introduction

Let E be a real Banach space, let C be a nonempty closed convex subset of E. A mapping T of C is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : x = Tx\}$.

Let $\{T(t) : t \ge 0\}$ be a family of mappings from a subset C of E into itself. We call it a *nonexpansive semigroup* on C if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \ge 0$;

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\|T(t)x - T(t)y\| \le \|x - y\|$ for all $x, y \in C$ and $t \ge 0$.

We denote by $F(\mathfrak{T})$ the set of all common fixed points of $\mathfrak{T}$, i.e., $F(\mathfrak{T}) = \{x \in C : T(t)x = x, 0 \le s < \infty\}$. It is known that $F(\mathfrak{T})$ is closed and convex. A mapping $f : C \to E$ is said to be *k-Lipschitzian*, if there exists a constant $k > 0$ such that

$$\|fx - fy\| \le L\|x - y\|, \quad \forall\, x, y \in C.$$

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \ge \varepsilon\right\}. \tag{1.1}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. The *modulus of smoothness* $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : \|x\| = \|y\| = 1\right\}. \tag{1.2}$$

It is also said to be *uniformly smooth* if $\lim_{n\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$, E is said to be *q-uniformly smooth* if there exists a constant $C_q > 0$ such that $\rho_E(\tau) \le C_q \tau^q$ for all $\tau > 0$ where q is a fixed real number with $1 < q < 2$.

The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\} \tag{1.3}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping* and

(i) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \ne 0$;

(ii) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$;

(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.

If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Let $V : C \to E$ be a nonlinear mapping. Then V is called

(i) *accretive* if there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Vx - Vy, j_q(x - y) \rangle \ge 0, \quad \forall\, x, y \in C;$$

(ii) *η-strongly accretive* if for some $\eta > 0$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Vx - Vy, j_q(x-y)\rangle \geq \eta \|x-y\|^q, \quad \forall\, x,y \in C.$$

In 2011, Sunthrayuth *et al.* [8] introduced a composite iterative algorithm $\{x_n\}$ in a Banach space as follows: $x_1 = x \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n, \quad \forall\, n \geq 0, \end{cases} \tag{1.4}$$

where $f: C \to C$ is a contraction mapping, $T(t_n)$ is a nonexpansive semigroup and A is a strongly positive linear bounded operator, and prove, under some conditions than $\{x_n\}$ converges strongly to a common fixed point, which solves some variational inequality in Banach spaces.

In 2013, Song and Ceng [7] introduced a new iterative algorithm for finding a common element of the set of solutions of a system of variational inequalities and the set of fixed points of a family nonexpansive mappings in a q-uniformly smooth Banach space as the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu V)T_n y_n], \\ y_n = (1-\beta_n)x_n + \beta_n k_n, \\ k_n = Q_C(z_n - \lambda A z_n), \\ z_n = Q_C(x_n - \sigma B x_n), \end{cases} \tag{1.5}$$

then, we prove a strong convergence theorem for the iterative sequence generated by (1.5) under some conditions.

For its importance of this topic, many authors tried to solve this problem by expanding and developing the iterative algorithm in various ways including generalizing their mapping for the wide range of using their algorithms. For more details please see in [1, 4–6].

In this paper, motivated and inspired by Sunthrayuth *et al.* [8] and Song and Ceng [7], we introduce the algorithm defined by:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T(t_n)x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n \mu V)T(t_n)y_n], \end{cases} \tag{1.6}$$

where $T(t_n)$ be nonexpansive semigroup. Under some appropriate different conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (1.6) converges strongly to a point x^*, where x^* is the unique solution in $F(\mathfrak{T})$.

2. Preliminaries

A mapping $T: C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty.

Lemma 2.1. *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 2.2 (Xu [10]). *Let E be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda y)\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.1}$$

for all $x, y \in \{z \in E : \|z\| \le r\}$ and $\lambda \in [0,1]$.

Lemma 2.3 (Kamimure and Takahashi [2]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x-y\|) \le \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in \{z \in E : \|z\| < r\}$.*

Lemma 2.4 (Sunthrayuth and Kumam [9]). *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space X. Let $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < \left(\frac{q\eta}{C_q k^q}\right)\frac{1}{q-1}$ and $\tau = \mu\left(\eta - \frac{C_q\mu^{q-1}k^q}{q}\right)$. Then for $t \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$, the mapping $S : C \to E$ define by $S := (I - t\mu V)$ is a contraction with constant $1 - t\tau$.*

Lemma 2.5 (Song and Ceng [7]). *Let C be a nonempty, closed and convex subset of a real reflexive and q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping J_q from E into E^*. Let Q_C be a sunny nonexpansive retraction from E onto C, $V : C \to E$ a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$. Suppose $f : C \to E$ is a L-Lipschitzian mapping with constant $L > 0$ and $T : C \to C$ a nonexpansive mapping such that $F(T) \ne \emptyset$. Let $0 < \mu < \left(\frac{q\eta}{C_q k^q}\right)^{\frac{1}{q-1}}$ and $0 \le \gamma L < \tau$, where $\tau = \mu\left(\eta - \frac{C_q\mu^{q-1}k^q}{q}\right)$. Then $\{x_t\}$ defined by $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t]$ converges strongly to some point $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, J_q(p - x^*)\rangle \le 0, \quad \forall\, p \in F(T).$$

Lemma 2.6 (Song and Ceng [7]). *Let C be a closed convex subset of a smooth Banach space E. Let $\tilde{C}$ be a nonempty subset of C. Let $Q : C \to \tilde{C}$ be a retraction and let J, J_q be the normalized duality mapping and generalized duality mapping on E, respectively. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive;*

(b) *$\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$, $\forall\, x, y \in E$;*

(c) *$\langle x - Qx, J(y - Qx)\rangle \le 0$, $\forall\, x \in C$, $y \in \tilde{C}$;*

(d) *$\langle x - Qx, J_q(y - Qx)\rangle \le 0$, $\forall\, x \in C$, $y \in \tilde{C}$.*

Lemma 2.7 (Mitrinovìc [3]). *Let $q > 1$. Then the following inequality holds:*

$$ab \le \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.8. (*Xu* [11]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \ge 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *satisfy the conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \le 0$;

(iii) $\gamma_n \ge 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 (Xu [10])**.** *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q, \quad \forall\, x,y \in E.$$

In particular, if E is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2, \quad \forall\, x,y \in E.$$

3. Main Results

In this section, we propose the iterative scheme for solving a fixed pint problem of nonexpansive semigroup in the framework of q-uniformly smooth and uniformly convex Banach spaces. Under the appropriate assumptions, the strong convergence theorem is proved as follows.

Theorem 3.1. *Let E be a q-uniformly smooth and uniformly convex Banach spaces which admit a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$, C be a sunny nonexpansive retract and nonempty closed and convex subsets of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\mathfrak{T} = \{T(t) : 0 \le t < \infty\}$ be a nonexpansive semigroup from C into itself such that $F(\mathfrak{T}) \ne \emptyset$. Let $f : C \to E$ be a L-Lipschitzian mapping with constant $L \ge 0$ and $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constant $k,\eta > 0$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ be the sequence in $(0,1)$ and $\{t_n\}_{n=1}^{\infty}$ be a positive real divergent sequence such that, $0 < \mu < \left(\frac{q\eta}{C_q k^q}\right)^{\frac{1}{q-1}}$ which C_q is a positive real number, $0 \le \gamma L < \tau$ where $\tau = \mu\left(\eta - \frac{C_q \mu^{q-1} k^q}{q}\right)$. Define a sequence $\{x_n\}$ by the following algorithm: $x_1 \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T(t_n)x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n \mu V)T(t_n)y_n], \end{cases} \tag{3.1}$$

which satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$,

(ii) $0 < \liminf_{n\to\infty} \delta_n < \limsup_{n\to\infty} \delta_n < 1$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(iii) $\sum_{n=1}^{\infty} \sup_{z \in x_n} \|T(t_{n+1})x - T(t_n)x\| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$, $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$.

Then, the sequence $\{x_n\}$ converges strongly to x^*, where x^* is the unique solution in $F(\mathfrak{T})$ of the variational inequality

$$\langle \gamma f(x^*) - \mu V x^*, J_q(z - x^*)\rangle \le 0, \quad \forall\, z \in F. \tag{3.2}$$

Proof. First, we show that $\{x_n\}$ is bounded. Let $p \in F$, we have

$$\begin{aligned}\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)T(t_n)x_n - p\|\\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|T(t_n) - p\|\\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\|\\ &\le \|x_n - p\|.\end{aligned} \tag{3.3}$$

From (3.1) and (3.3), we have

$$\begin{aligned}\|x_{n+1} - p\| &= Q_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n] - p\|\\ &\le \|[(1-\delta_n)I - \alpha_n\mu V][T(t_n)y_n - p] + \alpha_n(\gamma f(x_n) - \mu V p) + \delta_n(x_n - p)\|\\ &\le (1-\delta_n - \alpha_n\tau)\|T(t_n)y_n - p\| + \alpha_n\|\gamma f(x_n) - \mu V p\| + \delta_n\|x_n - p\|\\ &\le (1-\delta_n - \alpha_n\tau)\|y_n - p\| + \alpha_n\gamma L\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu V p\| + \delta_n\|x_n - p\|\\ &\le (1-\delta_n - \alpha_n\tau)\|y_n - p\| + \alpha_n\gamma\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu V p\| + \delta_n\|x_n - p\|\\ &= [1 - \alpha_n(k + \gamma L)]\|x_n - p\| + \alpha_n(k+\gamma L)\frac{\|\gamma f(p) - \mu V p\|}{k + \gamma L}.\end{aligned}$$

By induction, we get

$$\|x_{n+1} - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - \mu V p\|}{k+\gamma L}\right\},$$

for $n \ge 1$. Hence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{f(x_n)\}$, $\{T(t_n)x_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (3.1)

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)T(t_n)x_n,\\ y_{n+1} &= \beta_{n+1}x_{n+1} + (1-\beta_{n+1})T(t_{n+1})x_{n+1},\end{aligned} \tag{3.4}$$

we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= (1-\beta_{n+1})\|T(t_{n+1})x_{n+1} - T(t_n)x_n\| + \beta_{n+1}\|x_{n+1} - x_n\|\\ &\quad + |\beta_{n+1} - \beta_n|\|x_n - T(t_n)x_n\|.\end{aligned} \tag{3.5}$$

We consider the first term on this right side of (3.5), we get

$$\begin{aligned}\|T(t_{n+1})x_{n+1} - T(t_n)x_n\| &\le \|T(t_{n+1})x_{n+1} - T(t_{n+1})x_n\| + \|T(t_{n+1})x_n - T(t_n)x_n\|\\ &\le \|x_{n+1} - x_n\| + \|T(t_{n+1})x_n - T(t_n)x_n\|.\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le (1-\beta_{n+1})\|x_{n+1} - x_n\| + (1-\beta_{n+1})\|T(t_{n+1})x_n - T(t_n)x_n\| + \beta_{n+1}\|x_{n+1} - x_n\|\\ &\quad + |\beta_{n+1} - \beta_n|\|x_n - T(t_n)x_n\| \qquad (3.7)\\ &\le \|x_{n+1} - x_n\| + \|T(t_{n+1})x_n - T(t_n)x_n\| + |\beta_{n+1} - \beta_n|\|x_n - T(t_n)x_n\|\\ &= \|x_{n+1} - x_n\| + \sup_{z\in\{x_n\}}\|T(t_{n+1})z - T(t_n)z\| + |\beta_{n+1} - \beta_n|\|x_n - T(t_n)x_n\|. \qquad (3.8)\end{aligned}$$

Similarly, from definition of $\{x_n\}$, observing that

$$x_{n+1} = Q_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n],$$
$$x_{n+2} = Q_C[\alpha_{n+1}\gamma f(x_{n+1}) + \delta_{n+1}x_{n+1} + ((1-\delta_{n+1})I - \alpha_{n+1}\mu V)T(t_{n+1})y_{n+1}],$$

we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|Q_C[\alpha_{n+1}\gamma f(x_{n+1}) + \delta_{n+1}x_{n+1} + ((1-\delta_{n+1})I - \alpha_{n+1}\mu V)T(t_{n+1})y_{n+1}] \\
&\quad - Q_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n]\| \\
&\le \|[\alpha_{n+1}\gamma f(x_{n+1}) + \delta_{n+1}x_{n+1} + ((1-\delta_{n+1})I - \alpha_{n+1}\mu V)T(t_{n+1})y_{n+1}] \\
&\quad - [\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n]\| \\
&= \|[\alpha_{n+1}\gamma f(x_{n+1}) + \delta_{n+1}x_{n+1} + ((1-\delta_{n+1})I - \alpha_{n+1}\mu V)T(t_{n+1})y_{n+1}] \\
&\quad - [\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n] + \alpha_{n+1}\gamma f(x_n) - \alpha_{n+1}\gamma f(x_n) \\
&\quad + \delta_{n+1}x_n - \delta_{n+1}x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n - ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n\| \\
&\le \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + \delta_{n+1}\|x_{n+1}-x_n\| + \|[(1-\delta_{n+1})I - \alpha_{n+1}\mu V] \\
&\quad \cdot [T(t_{n+1})y_{n+1} - T(t_n)y_n]\| + |\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\| + |\alpha_{n+1}-\alpha_n|\mu\|VT(t_n)y_n\| \\
&\quad + |\delta_{n+1}-\delta_n|\|T(t_n)y_n - x_n\| \\
&\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \delta_{n+1}\|x_{n+1}-x_n\| \\
&\quad + [(1-\delta_{n+1})I - \alpha_{n+1}\mu V]\|T(t_{n+1})y_{n+1} - T(t_n)y_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\gamma[\|f(x_n)\| + \mu\|VT(t_n)y_n\|] + |\delta_{n+1}-\delta_n|\|T(t_n)y_n - x_n\|. \qquad (3.9)
\end{aligned}$$

We note that

$$\begin{aligned}
\|T(t_{n+1})y_{n+1} - T(t_n)y_n\| &\le \|T(t_{n+1})y_{n+1} - T(t_{n+1})y_n\| + \|T(t_{n+1})y_n - T(t_n)y_n\| \\
&\le \|y_{n+1}-y_n\| + \|T(t_{n+1})y_n - T(t_n)y_n\|. \qquad (3.10)
\end{aligned}$$

Substituting (3.10) into (3.9), we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \delta_{n+1}\|x_{n+1}-x_n\| \\
&\quad + [(1-\delta_{n+1})I - \alpha_{n+1}\mu V][\|y_{n+1}-y_n\| + \|T(t_{n+1})y_n - T(t_n)y_n\|] \\
&\quad + |\alpha_{n+1}-\alpha_n|\gamma[\|f(x_n)\| + \mu\|VT(t_n)y_n\|] + |\delta_{n+1}-\delta_n|\|T(t_n)y_n - x_n\| \\
&\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \delta_{n+1}\|x_{n+1}-x_n\| \\
&\quad + [(1-\delta_{n+1})I - \alpha_{n+1}\tau][\|x_{n+1}-x_n\| + \sup_{z\in\{x_n\}}\|T(t_{n+1})z - T(t_n)z\| \\
&\quad + |\beta_{n+1}-\beta_n|\|x_n - T(t_n)x_n\| + \sup_{y\in\{y_n\}}\|T(t_{n+1})y - T(t_n)y\|] \\
&\quad + |\alpha_{n+1}-\alpha_n|\gamma[\|f(x_n)\| + \mu\|VT(t_n)y_n\|] + |\delta_{n+1}-\delta_n|\|T(t_n)y_n - x_n\| \\
&= [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\| + [(1-\delta_{n+1})I - \alpha_{n+1}\tau][\sup_{z\in\{x_n\}}\|T(t_{n+1})z - T(t_n)z\| \\
&\quad + |\beta_{n+1}-\beta_n|\|x_n - T(t_n)x_n\| + \sup_{y\in\{y_n\}}\|T(t_{n+1})y - T(t_n)y\|] \\
&\quad + |\alpha_{n+1}-\alpha_n|\gamma[\|f(x_n)\| + \mu\|VT(t_n)y_n\|] + |\delta_{n+1}-\delta_n|\|T(t_n)y_n - x_n\| \\
&\le \|x_{n+1}-x_n\| + \sup_{z\in\{x_n\}}\|T(t_{n+1})z - T(t_n)z\| + \sup_{y\in\{y_n\}}\|T(t_{n+1})y - T(t_n)y\| \\
&\quad + |\beta_{n+1}-\beta_n|\|x_n - T(t_n)x_n\| + |\alpha_{n+1}-\alpha_n|\gamma[\|f(x_n)\| + \mu\|VT(t_n)y_n\|]
\end{aligned}$$

$$
\begin{aligned}
&+|\delta_{n+1}-\delta_n|\|T(t_n)y_n-x_n\| \\
&\le \|x_{n+1}-x_n\|+[|\beta_{n+1}-\beta_n|+|\alpha_{n+1}-\alpha_n|+|\delta_{n+1}-\delta_n|]M_1 \\
&\quad+\sup_{z\in\{x_n\}}\|T(t_{n+1})z-T(t_n)z\|+\sup_{y\in\{y_n\}}\|T(t_{n+1})y-T(t_n)y\|,
\end{aligned}
$$

where $M_1=\sup_{n\ge 0}\{\|x_n-T(t_n)x_n\|,\|f(x_n)\|+\mu\|VT(t_n)y_n\|,\|T(t_n)y_n-x_n\|\}<\infty$.

By condition (i), (ii), (iv) and Lemma 2.8, we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}
$$

Next, we show that $\lim_{n\to\infty}\|T(t_n)x_n-x_n\|=0$, by the convexity of $\|\cdot\|^q$ for all $q>1$, Lemma 2.9 and (3.1), we obtain

$$
\begin{aligned}
\|y_n-p\|^q &= \|\beta_n x_n+(1-\beta_n)T(t_n)x_n-p\|^q \\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|T(t_n)x_n-p\|^q \\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|x_n-p\|^q \\
&= \|x_n-p\|^q
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
\|x_{n+1}-p\|^q &= \|Q_C[\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n\mu V)T(t_n)y_n]-p\|^q \\
&\le \|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)+\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n)\|^q \\
&\le \|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)\|^q \\
&\quad+q\langle\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n),J_q(\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p))\rangle \\
&\quad+C_q\|\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n\|^q \\
&\le \delta_n\|x_n-p\|^q+(1-\delta_n)\|T(t_n)y_n-p)\|^q \\
&\quad+q\alpha_n\|\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n\|\|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)\|^{q-1} \\
&\quad+C_q\alpha_n^q\|\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n\|^q \\
&\le \delta_n\|x_n-p\|^q+(1-\delta_n)\|y_n-p)\|^q+\alpha_n M_2 \\
&\le \|x_n-p\|^q+\alpha_n M_2,
\end{aligned} \tag{3.13}
$$

where $M_2=\sup_{n\ge 0}\{q\|\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n\|\|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)\|^{q-1}+C_q\alpha_n^{q+1}\|\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n\|^q\}<\infty$.

By (3.1), again

$$
\begin{aligned}
\|y_n-p\|^2 &= \|\beta_n x_n+(1-\beta_n)T(t_n)x_n-p\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|T(t_n)x_n-p\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2 \\
&= \|x_n-p\|^2
\end{aligned} \tag{3.14}
$$

and

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|Q_C[\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n\mu V)T(t_n)y_n]-p\|^2 \\
&\le \|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)+\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n)\|^2 \\
&\le \|\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\langle \alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n), J(\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p) \\
&+\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n))\rangle \\
\le{}& \delta_n\|x_n-p\|^2+(1-\delta_n)\|T(t_n)y_n-p\|^2-\delta_n(1-\delta_n)g(\|x_n-T(t_n)y_n\|)+\alpha_n M_3 \\
\le{}& \delta_n\|x_n-p\|^2+(1-\delta_n)\|y_n-p\|^2-\delta_n(1-\delta_n)g(\|x_n-T(t_n)y_n\|)+\alpha_n M_3 \\
\le{}& \delta_n\|x_n-p\|^2+(1-\delta_n)\|x_n-p\|^2-\delta_n(1-\delta_n)g(\|x_n-T(t_n)y_n\|)+\alpha_n M_3 \\
\le{}& \|x_n-p\|^2-\delta_n(1-\delta_n)g(\|x_n-T(t_n)y_n\|)+\alpha_n M_3,
\end{aligned}
\tag{3.15}
$$

where $M_3 = \sup_{n\ge 0}\{2\langle \gamma f(x_n)-\alpha_n\mu VT(t_n)y_n, J(\delta_n(x_n-p)+(1-\delta_n)(T(t_n)y_n-p)+\alpha_n(\gamma f(x_n)-\alpha_n\mu VT(t_n)y_n))\rangle\} < \infty$.

Then, we get

$$
\begin{aligned}
\delta_n(1-\delta_n)g(\|x_n-T(t_n)y_n\|) &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_3 \\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\alpha_n M_3.
\end{aligned}
$$

By (3.11), condition (i) and (ii), we have

$$
\lim_{n\to\infty} g(\|x_n-T(t_n)y_n\|)=0.
$$

From property of g that

$$
\lim_{n\to\infty}\|x_n-T(t_n)y_n\|=0. \tag{3.16}
$$

By (3.1), we have

$$
\begin{aligned}
\|x_{n+1}-T(t_n)y_n\| &= \|Q_C[\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n\mu V)T(t_n)y_n]-T(t_n)y_n\| \\
&\le \|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n\mu V)T(t_n)y_n-T(t_n)y_n\| \\
&= \|\alpha_n\gamma f(x_n)+\delta_n x_n-\delta_n T(t_n)y_n-\alpha_n\mu VT(t_n)y_n\| \\
&= \|\alpha_n(\gamma f(x_n)-\mu VT(t_n)y_n)+\delta_n(x_n-T(t_n)y_n)\| \\
&\le \alpha_n\|\gamma f(x_n)-\mu VT(t_n)y_n\|+\delta_n\|x_n-T(t_n)y_n\|.
\end{aligned}
\tag{3.17}
$$

By condition (i) and (3.16), we have

$$
\lim_{n\to\infty}\|x_{n+1}-T(t_n)y_n\|=0. \tag{3.18}
$$

Observe that

$$
\begin{aligned}
\|x_n-T(t_n)x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)y_n\|+\|T(t_n)y_n-T(t_n)x_n\| \\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)y_n\|+\|y_n-x_n\| \\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)y_n\|+\|\beta_n x_n+(1-\beta_n)T(t_n)x_n-x_n\| \\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)y_n\|+\beta_n\|x_n-x_n\| \\
&\quad +(1-\beta_n)\|T(t_n)x_n-x_n\|.
\end{aligned}
\tag{3.19}
$$

It follow that

$$
\beta_n\|x_n-T(t_n)x_n\| \le \|x_n-x_{n+1}\|+\|x_{n+1}-T(t_n)y_n\|.
$$

By condition (iv), (3.11)and (3.18), we have

$$
\lim_{n\to\infty}\|T(t_n)x_n-x_n\|=0. \tag{3.20}
$$

Next, we show that $z \in F(\mathfrak{T})$. We can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$. Let $t_{n_i} \geq 0$ such that $t_{n_i} \to 0$ and $\frac{\|T(t_{n_i})x_{n_i} - x_{x_i}\|}{t_{n_i}} \to 0$, $i \to \infty$. Fix $s > 0$, we have

$$\begin{aligned}
\|x_{n_i} - T(s)z\| &\leq \sum_{j=0}^{[s/t_{n_j}]-1} \|T((j+1)t_{n_i})x_{n_i} - T(jt_{n_i})x_{n_i}\| + \|T([s/t_{n_i}]t_i)x_{n_i} - T([s/t_{n_i}]t_{n_i})z\| \\
&\quad + \|T([s/t_{n_i}]t_{n_i})z - T(s)z\| \\
&\leq [s/t_{n_i}]\|T(t_{n_i})x_{n_i} - x_{n_i}\| + \|x_{n_i} - z\| + \|T(s - [s/t_{n_i}]t_{n_i})z - z\| \\
&\leq s\frac{T(t_{n_i})x_{n_i} - x_{n_i}\|}{t_{n_i}} + \|x_{n_i} - z\| + \max\{\|T(t)z - z\| : 0 \leq t \leq t_{n_i}\}.
\end{aligned}$$

For all $i \in \mathbb{N}$, we have

$$\limsup_{i\to\infty} \|x_{n_i} - T(s)z\| \leq \limsup_{i\to\infty} \|x_{n_i} - z\|.$$

Since E is a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies $T(s)z = z$.

Next, we show that $\limsup_{i\to\infty}\langle \gamma f(x^*) - \mu V x^*, J_q(x_n - x^*)\rangle \leq 0$. Since E is a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$ and $x_{n_i} \rightharpoonup z$, we obtain that

$$\begin{aligned}
\limsup_{i\to\infty}\langle \gamma f(x^*) - \mu V x^*, J_q(x_n - x^*)\rangle &= \lim_{i\to\infty}\langle \gamma f(x^*) - \mu V x^*, J_q(x_{n_i} - x^*)\rangle \\
&= \langle \gamma f(x^*) - \mu V x^*, J_q(z - x^*)\rangle \leq 0. \qquad (3.21)
\end{aligned}$$

Finally, we show that $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

$$\begin{aligned}
\|y_n - x^*\|^q &\leq \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|T(t_n)x_n - x^*\|^q \\
&\leq \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|x_n - x^*\|^q \\
&\leq \|x_n - x^*\|^q, \qquad (3.22)
\end{aligned}$$

and let $w_n = \alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n\mu V)T(t_n)y_n$. From Lemma 2.4, Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \langle Q_C w_n - w_n, J_q(x_{n+1} - x^*)\rangle + \langle w_n - x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq \langle w_n - x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq \langle [(1-\delta_n)I - \alpha_n\mu V][T(t_n)y_n - x^*], J_q(x_{n+1} - x^*)\rangle \\
&\quad + \alpha_n\langle \gamma f(x_n) - \mu V x^*, J_q(x_{n+1} - x^*)\rangle + \delta_n\langle x_n - x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq (1-\delta_n - \alpha_n\tau)\|T(t_n)y_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \delta_n\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} \\
&\quad + \alpha_n\langle \gamma f(x_n) - \gamma f(x^*), J_q(x_{n+1} - x^*)\rangle + \alpha_n\langle \gamma f(x_n) - \mu V x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq (1-\delta_n - \alpha_n\tau)\|y_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \delta_n\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} \\
&\quad + \alpha_n\gamma L\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \alpha_n\langle \gamma f(x_n) - \mu V x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\tau + \alpha_n\gamma L)\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \alpha_n\langle \gamma f(x_n) - \mu V x^*, J_q(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\tau + \alpha_n\gamma L)\left[\frac{1}{q}\|x_n - x^*\|^q + \frac{q-1}{q}\|x_{n+1} - x^*\|^q\right]
\end{aligned}$$

$$+\alpha_n\langle \gamma f(x_n)-\mu V x^*, J_q(x_{n+1}-x^*)\rangle, \tag{3.23}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le \frac{1-\alpha_n(\tau+\gamma L)}{1+(q-1)\alpha_n(k-\gamma L)}\|x_n-x^*\|^q \\ &\quad+\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma f(x_n)-\mu V x^*, J_q(x_{n+1}-x^*)\rangle \\ &\le [1-\alpha_n(\tau+\gamma L)]\|x_n-x^*\|^q \\ &\quad+\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma f(x_n)-\mu V x^*, J_q(x_{n+1}-x^*)\rangle. \end{aligned} \tag{3.24}$$

From (i), (3.21) and applying Lemma 2.8 to (3.24), we have $\|x_n-x^*\|\to 0$ as $n\to\infty$. Then, the sequence $\{x_n\}$ converges strongly to $x^*\in F$. This completes the proof. □

Corollary 3.2. *Let E be a q-uniformly smooth and uniformly convex Banach spaces which admit a weakly sequentially continuous generalized duality mapping $J_q: E\to E^*$, C be a sunny nonexpansive retract and nonempty closed and convex subsets of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\mathfrak{T}=\{T(t): 0\le t<\infty\}$ be a nonexpansive semigroup from C into itself such that $F(\mathfrak{T})\neq\emptyset$. Let $\{\beta_n\}_{n=1}^\infty$, $\{\delta_n\}_{n=1}^\infty$ be the sequence in $(0,1)$ and $\{t_n\}_{n=1}^\infty$ be a positive real divergent sequence. Define a sequence $\{x_n\}$ by the following algorithm: $x_1\in C$ and*

$$\begin{cases} y_n=\beta_n x_n+(1-\beta_n)T(t_n)x_n, \\ x_{n+1}=Q_C[\delta_n x_n+(1-\delta_n)T(t_n)y_n], \end{cases} \tag{3.25}$$

which satisfy the conditions (ii), (iii) *and* (iv). *Then, the sequence $\{x_n\}$ converges strongly to x^*, where x^* is the unique solution in $F(\mathfrak{T})$ of the variational inequality*

$$\langle \gamma f(x^*)-\mu V x^*, J_q(z-x^*)\rangle \le 0, \quad \forall\, z\in F. \tag{3.26}$$

Proof. Take $\alpha_n=0$, then (3.1) is reduced to (3.25). □

Corollary 3.3. *Let E be a q-uniformly smooth and uniformly convex Banach spaces which admit a weakly sequentially continuous generalized duality mapping $J_q: E\to E^*$, C be a sunny nonexpansive retract and nonempty closed and convex subsets of E. Let Q_C be a sunny nonexpansive retraction from E onto C, T be a nonexpansive mapping from C into itself such that $F(T)\neq\emptyset$. Let $f: C\to E$ be a L-Lipschitzian mapping with constant $L\ge 0$ and $V: C\to E$ be a k-Lipschitzian and η-strongly accretive operator with constant $k,\eta>0$. Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\delta_n\}_{n=1}^\infty$ be the sequence in $(0,1)$ and $\{t_n\}_{n=1}^\infty$ be a positive real divergent sequence such that, $0<\mu<(\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ which C_q is a positive real number, $0\le\gamma L<\tau$ where $\tau=\mu(\eta-\frac{C_q\mu^{q-1}k^q}{q})$. Define a sequence $\{x_n\}$ by the following algorithm: $x_1\in C$ and*

$$\begin{cases} y_n=\beta_n x_n+(1-\beta_n)Tx_n, \\ x_{n+1}=Q_C[\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n\mu V)Ty_n], \end{cases} \tag{3.27}$$

which satisfy the conditions (i), (ii) *and* (iv). *Then, the sequence $\{x_n\}$ converges strongly to x^*, where x^* is the unique solution in $F(T)$ of the variational inequality*

$$\langle \gamma f(x^*)-\mu V x^*, J_q(z-x^*)\rangle \le 0, \quad \forall\, z\in F. \tag{3.28}$$

4. Conclusion

According to this research, we presented the generalized iterative method which extended from the recently proposed researches in the literatures. Under the necessary and simple conditions, the strong convergence theorem is proved to solve fixed point problem for nonexpansive semigroup in the framework of q-uniformly smooth and uniformly convex Banach spaces. Moreover, since many applications in sciences can be formulated in form of fixed point problems, therefore our method can be also applied to solve various problems in many disciplines.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Jaiboon and P. Kumam, Strong convergence theorems for solving equilibrium problems and fixed point problems of xi-strict pseudo-contraction mappings by two hybrid projection methods, *J. Comput. Appl. Math.* **234** (2010), 722 – 732, DOI: 10.1016/j.cam.2010.01.012.

[2] S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.* **13** (2002), 938 – 945, DOI: 10.1137/S105262340139611X.

[3] D. S. Mitrinoviċ, *Analytic Inequalities*, Springer, New York (1970), DOI: 10.1007/978-3-642-99970-3.

[4] S. Plubtieng and P. Kumam, Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings, *J. Comput. Appl. Math.* **224** (2009), 614 – 621, DOI: 10.1016/j.cam.2008.05.045.

[5] S. Saewan and P. Kumam, A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems, *Abstract and Applied Analysis* **2010** (2010), Article ID 123027, 31 pages, DOI: 10.1155/2010/123027.

[6] S. Saewan and P. Kumam, Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-phi-asymptotically nonexpansive mappings, *Abstract and Applied Analysis* **2010** (2010), Article ID 357120, 22 pages, DOI: 10.1155/2010/357120.

[7] Y. Song and L. Ceng, A general iteration scheme for variational inequality problem and common fixed point problems of nonexpansive mappings in q-uniformly smooth Banach spaces, *J. Glob. Optim.* **57** (2013), 1327 – 1348, DOI: 10.1007/s10898-012-9990-4.

[8] P. Sunthrayuth, K. Wattanawitoon and P. Kumam, Convergence theorems of a general composite iterative method for nonexpansive semigroups in Banach spaces, *ISRN Math. Anal.* **2011** (2011), Article ID 576135, 24 pages, DOI: 10.5402/2011/576135.

[9] P. Sunthrayuth and P. Kumam, Iterative methods for variational inequality problems and fixed point problems of a countable family of strict pseudo-contractions in a q-uniformly smooth Banach space, *Fixed Point Theory and Appl.* **2012** (2012), 65 pages, DOI: 10.1186/1687-1812-2012-65.

[10] H. K. Xu, Inequalities in Banach space with applications, *Nonlinear Anal.* **16** (1991), 1127 – 1138, DOI: 10.1016/0362-546X(91)90200-K.

[11] H. K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.* **66** (2002), 240 – 256, DOI: 10.1112/S0024610702003332.