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Research Article

Some Fixed Point of Hardy-Rogers Contraction in Generalized Complex Valued Metric Spaces

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Abstract. In this work, we defined the generalized complex valued metric space for some partial order relation and give some example. Then we study and established a fixed point theorem for general Hardy-Rogers contraction. The results extend and improve some results of Elkouch and Marhrani [5].

Keywords. General Kannan condition; Hardy-Rogers contraction; Class of generalized complex valued metric space

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1. Introduction

The axiomatic development of a metric space was essentially carried out by Frence mathematician Freehen in the year 1960. Recently, Banach [2], introduce the Banach fixed point theorem in a complex valued metric space, has been generalized in many space. In 2011 Azam [1], introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed point of a pair of mappings satisfying a contractive condition.

Let us recall that a mapping T on a metric space (X,d) is a Kannan contraction [8] if there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le \alpha d(x,Tx) + \alpha d(y,Ty) \tag{1.1}$$

for all $x,y \in X$.

In 1968, Kannan [8] proved the existance results for a mapping defined by (1.1), this the following result.

Theorem 1.1 ([8]). *Let (X,d) be a complete metric space, $\lambda \in [0,1)$ and T is a self-mapping on X satisfying* (1.1)*. Then T has a unique fixed point.*

Recently, Jleli and Samet [7] introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on.

In 2017, Elkouch and Marhrani [5], extend the Theorem 1.1 to generalized metric space and they proved existence results for the Kannan contraction defined by (1.1), and they introduced the Chatterjea contraction with $\lambda = \frac{1}{3}$ in (X,D) such that

$$D(fx,fy) \le \lambda(D(y,fx) + D(x,fy))$$

for all $x,y \in X$. Then they proved that a mapping f has a fixed point in X. The final of their work they introduce the Hardy-Rogers contraction.

Definition 1.2. Let (X,D) be a generalized metric space. A self-mapping f on X is called a Hardy-Rogers contraction if there exist nonnegative real numbers λ_i for $i = 1,2,3,4,5$ such that $\lambda = \sum_{i=1}^{5} \lambda_i \in [0,1)$ and satisfying

$$D(fx,fy) \le \lambda_1 D(x,y) + \lambda_2 D(x,fx) + \lambda_3 D(y,fy) + \lambda_4 D(y,fx) + \lambda_5 D(x,fy), \tag{1.2}$$

for all $x,y \in X$. Then they proved the uniqueness of the fixed point of f in X.

In 2018, Saipara, Kumam and Cho [10], prove some random fixed point theorems for Hardy-Rogers self-random operators in separable Banach spaces and, as some applications, we show the existence of a solution for random nonlinear integral equations in Banach spaces. In this year, Khammahawong and Kumam [9], establish a new best proximity point theorem for Roger-Hardy type generalized F-contraction mappings and nonexpansive mappings in complete metric spaces.

Motivate by Elkouch and Marhrani [5], and Khammahawong and Kumam [9], we defined and introduce the generalized complex valued metric space and consider the general Hardy-Rogeds contraction for some partial order relation, if there exist nonnegative real numbers λ_i for $i = 1,2,3,4,5$ such that $\lambda = \sum_{i=1}^{5} \lambda_i \in [0,1)$ and satisfying

$$D(fx,fy) \precsim \lambda_1 D(x,y) + \lambda_2 D(x,fx) + \lambda_3 D(y,fy) + \lambda_4 D(y,fx) + \lambda_5 D(x,fy), \tag{1.3}$$

for all $x,y \in X$. Then we claim that a mapping f is satisfying (1.3) has a fixed point on X.

2. Preliminaries

In this section, we give some definitions and lemmas for this work. Let $\mathbf{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order relation $\precsim$ on $\mathbf{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2) \text{ and } Im(z_1) \le Im(z_2).$$

Thus $z_1 \preceq z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precneqq z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.1. We can easily to check the following:

(i) If $a, b \in \mathbf{R}$, $0 \le a \le b$ and $z_1 \preceq z_2$ then $az_1 \preceq bz_2$, for all $z_1, z_2 \in \mathbf{C}$.

(ii) $0 \preceq z_1 \precneqq z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \preceq z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam *et al.* [1] defined the complex valued metric space in the following way:

Definition 2.2 ([1])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbf{C}$ satisfies the following conditions:

(C1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

In 2017, Elkouch and Marhrani [5] defined a new class of metric space, let X be a nonempty set, and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}.$$

Definition 2.3 ([7])**.** A mapping D is called a generalized metric if it satisfies the following conditions:

(1) For every $(x, y) \in X \times X$, we have

$$D(x, y) = 0 \Leftrightarrow x = y.$$

(2) For every $(x, y) \in X \times X$, we have

$$D(x, y) = D(y, x).$$

(3) There exists a real constant $C > 0$ such that for all $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, we have

$$D(x, y) \le C \limsup_{n\to\infty} D(x_n, y).$$

The pair (X, D) is called a generalized metric space.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbf{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n, x)| = 0 \right\}.$$

Definition 2.4. Let X be a nonempty set, a mapping $D : X \times X \to \mathbf{C}$ is called a generalized complex value metric if it satisfies the following conditions:

(1) for every $x, y \in X$, we have
$$0 \preceq D(x,y).$$

(2) for every $x, y \in X$, we have
$$D(x,y) = 0 \Rightarrow x = y.$$

(3) for all $x, y \in X$, we have
$$D(x,y) = D(y,x).$$

(4) there exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D,X,x)$, we have
$$D(x,y) \preceq r \limsup_{n\to\infty} |D(x_n,y)|.$$

Then, a pair (X,D) is called a generalized complex valued metric space.

Definition 2.5. Let (X,D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D,X,x)$. We denote by $\lim_{n\to\infty} x_n = x$.

Example 2.6. Let $X = [0,1]$ and let $D : X \times X \to \mathbf{C}$ be the mapping define by for any $x, y \in X$
$$\begin{cases} D(x,y) = (x+y)i; \ x \neq 0 \text{ and } y \neq 0 \\ D(x,0) = D(0,x) = \frac{x}{2} i. \end{cases}$$

Proof. Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$.
If $D(x,y) = (x+y)i = 0 + (x+y)i \preceq 0 + 0i = 0$.
If $D(x,0) = \frac{x}{2}i = 0 + \frac{x}{2}i \preceq 0 + 0i = 0$.
Hence $D(x,y) \preceq 0$.
If $D(x,y) = 0$, then $\frac{x}{2}i = 0$ and $y = 0$. Hence, $x = 0 = y$.
If $x \neq 0$ and $y \neq 0, D(x,y) = (x+y)i = (y+x)i = D(y,x)$ and $D(x,0) = D(0,x)$.
Let $\{\frac{(n-1)x}{n}\} \subseteq X$, we see that $\limsup_{n\to\infty} |D(x_n,x)| = 0$ and put $r = i$, then, we have
$$D(0,y) = \frac{y}{2} i \quad \text{and} \quad \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \left(\sqrt{\left(\frac{(n-1)x}{n} + y \right)^2} \right) = x + y.$$
Hence, $D(0,y) = \frac{y}{2} i \preceq (x+y)i$, and we see that
$$D(x,y) = (x+y)i \quad \text{and} \quad \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \left(\sqrt{\left(\frac{(n-1)x}{n} + y \right)^2} \right) = x + y.$$
Hence, $D(x,y) = (x+y)i \preceq (x+y)i = r \limsup_{n\to\infty} |D(x_n,y)|$. □

Definition 2.7. Let (X,D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to Cauchy sequence in X, if
$$\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0.$$

Definition 2.8. Let (X,D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X,D) is called a complete complex valued metric space.

Lemma 2.9. *Let λ is a real number such that $0 \le \lambda < 1$, and let $\{b_n\}$ be a sequence of positives reals numbers such that $\lim_{n\to\infty} b_n = 0$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying*

$$a_{n+1} \le a_n + b_n, \quad \text{for all } n \in \mathbf{N}$$

we have $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 ([3])**.** *Let $\{a_n\}$ be a sequence of nonnegative, and let $\{\lambda_n\}$ be a real sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} \lambda_n = \infty.$$

If, for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$a_{n+1} \le (1-\lambda_n)a_n + \varepsilon\lambda_n, \quad \text{for all } \ n \ge n_0$$

then $0 \le \limsup_{n\to\infty} a_n \le \varepsilon$.

3. Fixed Point for General Hardy-Rogers Contraction

In this section we prove some propositions for use in the main theorem and prove some fixed point theorem in generalized complex valued metric space.

Definition 3.1. Let (X,D) be a generalized complex valued metric space. A self-mapping f on X is called a general Hardy-Rogers contraction if there exists nonnegative real constants λ_i for $i = 1,2,3,4,5$ such that $\lambda = \sum_{i=1}^{i=5} \lambda_i \in [0,1)$ and

$$D(fx,fy) \preceq \lambda_1 D(x,y) + \lambda_2 D(x,fx) + \lambda_3 D(y,fy) + \lambda_4 D(y,fx) + \lambda_5 D(x,fy) \tag{3.1}$$

for all $x, y \in X$.

Proposition 3.2. *Let (X,D) be a generalized complex valued metric space, and let $f : X \to X$ be a Hardy-Rogers contraction. Then any fixed point $\omega \in X$ of f satisfies*

$$|D(\omega,\omega)| < \infty \quad \Rightarrow \quad D(\omega,\omega) = 0.$$

Proof. Let $\omega \in X$ be a fixed point f such that $|D(\omega,\omega)| < \infty$ and $f\omega = \omega$. To show that $D(\omega,\omega) = 0$.

$$\begin{aligned} D(\omega,\omega) &= D(f\omega,f\omega) \\ &\preceq \lambda_1 D(f\omega,f\omega) + \lambda_2 D(\omega,f\omega) + \lambda_3 D(\omega,f\omega) + \lambda_4 D(\omega,f\omega) + \lambda_5 D(\omega,f\omega) \\ &= \lambda_1 D(\omega,\omega) + \lambda_2 D(\omega,\omega) + \lambda_3 D(\omega,\omega) + \lambda_4 D(\omega,\omega) + \lambda_5 D(\omega,\omega). \end{aligned}$$

By Remark 2.1(ii), we have

$$\begin{aligned} |D(\omega,\omega)| &\le \lambda_1 |D(\omega,\omega)| + \lambda_2 |D(\omega,\omega)| + \lambda_3 |D(\omega,\omega)| + \lambda_4 |D(\omega,\omega)| + \lambda_5 |D(\omega,\omega)| \\ &= (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)|D(\omega,\omega)| \\ &= \lambda |D(\omega,\omega)|. \end{aligned}$$

Since $\lambda = \sum_{i=1}^{i=5} \lambda_i \in [0,1)$ it follows that $|D(\omega,\omega)| = 0$. Hence $D(\omega,\omega) = 0$. □

Theorem 3.3. *Let (X,D) be a complete generalized complex valued metric space, and let f be a self-mapping on X satisfying* (3.1)*. let $|r|\lambda_3 + \lambda_5 < 1$, and there exists element $x_0 \in X$ such that*

$\delta(D,f,x_0)<\infty$. Then the sequence $\{f^n x_0\}$ converges to some $\omega\in X$ and ω is a fixed point of f. Moreover, If ω' is a fixed point of f in X such that $|D(\omega',\omega')|<\infty$ and $|D(\omega,\omega)|<\infty$ then $\omega=\omega'$.

Proof. Let $n\in\mathbf{N}$, for all $i,j\in\mathbf{N}$, we have

$$D(f^{n+i}x_0,f^{n+j}x_0)=D(f(f^{n+i-1}x_0),f(f^{n+j-1}x_0)).$$

By (3.1), we have

$$\begin{aligned}D(f^{n+i}x_0,f^{n+j}x_0)\preceq{}&\lambda_1 D(f^{n+i-1}x_0,f^{n+j-1}x_0)+\lambda_2 D(f^{n+i-1}x_0,f^{n+i}x_0)\\&+\lambda_4 D(f^{n+j-1}x_0,f^{n+i}x_0)\\&+\lambda_5 D(f^{n+i-1}x_0,f^{n+j}x_0).\end{aligned}$$

By Remark 2.1(ii), we have

$$\begin{aligned}|D(f^{n+i}x_0,f^{n+j}x_0)|\le{}&\lambda_1|D(f^{n+i-1}x_0,f^{n+j-1}x_0)|+\lambda_2|D(f^{n+i-1}x_0,f^{n+i}x_0)|\\&+\lambda_3|D(f^{n+j-1}x_0,f^{n+j}x_0)|+\lambda_4|D(f^{n+j-1}x_0,f^{n+i}x_0)|\\&+\lambda_5|D(f^{n+i-1}x_0,f^{n+j}x_0)|\\\le{}&\lambda_1\delta(D,f,f^{n-1}x_0)+\lambda_2\delta(D,f,f^{n-1}x_0)+\lambda_3\delta(D,f,f^{n-1}x_0)\\&+\lambda_4\delta(D,f,f^{n-1}x_0)+\lambda_5\delta(D,f,f^{n-1}x_0)\\={}&(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)\delta(D,f,f^{n-1}x_0)\\={}&\lambda\delta(D,f,f^{n-1}x_0).\end{aligned}$$

We have

$$|D(f^{n+i}x_0,f^{n+j}x_0)|\le\lambda\delta(D,f,f^{n-1}x_0). \tag{3.2}$$

By (3.2), we see that $\lambda\delta(D,f,f^{n-1}x_0)$ is upper bound of the set $\{|D(f^{n+i}x_0,f^{n+j}x_0)|:i,j\in\mathbf{N}\}$ since $\delta(D,f,f^n x_0)$ is least upper bound of $\{|D(f^{n+i}x_0,f^{n+j}x_0)|\}$, it follows that

$$\delta(D,f,f^n x_0)\le\lambda\delta(D,f,f^{n-1}x_0).$$

Similarly, we induction to

$$\begin{aligned}\delta(D,f,f^n x_0)&\le\lambda\delta(D,f,f^{n-1}x_0)\\&\le\lambda^2\delta(D,f,f^{n-2}x_0)\\&\ \vdots\\&\le\lambda^n\delta(D,f,x_0).\end{aligned}$$

By

$$|D(f^n x_0,f^{m+n}x_0)|=|D(f(f^{n-1}x_0),f^{m+1}(f^{n-1}x_0))|,$$

we have

$$|D(f^n x_0,f^{m+n}x_0)|\le\lambda\delta(D,f,f^{n-1}x_0)\le\lambda^{n-1}\delta(D,f,x_0)$$

for all integer m. Since $\delta(D,f,x_0)<\infty$ and $\lambda\in[0,1)$, we have

$$\lim_{n\to\infty}\lambda^{n-1}\delta(D,f,x_0)=0$$

thus

$$\lim_{n\to\infty}|D(f^n x_0,f^{m+n}x_0)|=0. \tag{3.3}$$

Then, we have $\{f^n x_0\}$ is a cauchy sequence. By X be a complete, thus there exists $\omega \in X$ such that

$$\lim_{n\to\infty} |D(f^n x_0,\omega)| = 0.$$

By Definition 2.4(4), we have

$$D(f\omega,\omega) \preceq r \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)|. \tag{3.4}$$

By Remark 2.1(ii), we have

$$|D(f\omega,\omega)| \le |r| \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)|. \tag{3.5}$$

By (3.1), we have

$$\begin{aligned} D(f^{n+1}, f\omega) \preceq{}& \lambda_1 D(f^n x_0,\omega) + \lambda_2 D(f^n x_0, f^{n+1}x_0) + \lambda_3 D(\omega, f\omega) \\ &+ \lambda_4 D(\omega, f^{n+1}x_0) + \lambda_5 D(f^n x_0, f\omega). \end{aligned} \tag{3.6}$$

By Remark 2.1(ii), we have

$$\begin{aligned} |D(f^{n+1}, f\omega)| \le{}& \lambda_1 |D(f^n x_0,\omega)| + \lambda_2 |D(f^n x_0, f^{n+1}x_0)| + \lambda_3 |D(\omega, f\omega)| \\ &+ \lambda_4 |D(\omega, f^{n+1}x_0)| + \lambda_5 |D(f^n x_0, f\omega)|. \end{aligned} \tag{3.7}$$

Let

$$\begin{cases} a_n = |D(f^n x_0, f\omega)| \\ b_n = \lambda_1 |D(f^n x_0,\omega)| + \lambda_2 |D(f^n x_0, f^{n+1}x_0)| + \lambda_4 |D(\omega, f^{n+1}x_0)| \\ K = \lambda_3 |D(\omega, f\omega)|. \end{cases}$$

By (3.7), we have

$$a_{n+1} \le \lambda_5 a_n + b_n + K.$$

Consider

$$\lim_{n\to\infty} |D(f^n x_0,\omega)| = 0,$$
$$\lim_{n\to\infty} |D(f^n x_0, f^{n+1}x_0)| = 0,$$
$$\lim_{n\to\infty} |D(\omega, f^{n+1}x_0)| = 0.$$

It follows that $\lim_{n\to\infty} = b_n = 0$.

Since $\lim_{n\to\infty} = b_n = 0$ for all $\varepsilon > 0$. Let $\varepsilon > \frac{K}{1-\lambda_5} > 0$ there exists N_ε such that

$$b_n \le \varepsilon(1-\lambda_5) - K, \quad \text{for all } n \ge N_\varepsilon.$$

Then, we have

$$a_{n+1} \le \lambda_5 a_n + b_n + K \le \lambda_5 a_n + \varepsilon(1-\lambda_5).$$

By Lemma 2.10, we have

$$0 \le \limsup_{n\to\infty} a_n \le \varepsilon, \quad \text{for all } \varepsilon > \frac{K}{1-\lambda_5}$$

then

$$0 \le \limsup_{n\to\infty} a_n \le \frac{K}{1-\lambda_5}.$$

By Definition 2.4(4), we have

$$D(f\omega,\omega) \preceq r \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)| \preceq r \limsup_{n\to\infty} a_n \preceq r \frac{K}{1-\lambda_5}$$

and by $K = \lambda_3|D(f\omega,\omega)|$, we have

$$D(f\omega,\omega) \precsim r\frac{\lambda_3|D(f\omega,\omega)|}{1-\lambda_5}.$$

By Remark 2.1(ii), we have

$$|D(f\omega,\omega)| \le |r|\frac{\lambda_3|D(f\omega,\omega)|}{1-\lambda_5}.$$

Since $|r|\lambda_3+\lambda_5<1$, we have $\frac{|r|\lambda_3}{1-\lambda_5}<1$ then $|D(f\omega,\omega)|=0$ thus $D(f\omega,\omega)=0$ it follows that $f\omega=\omega$.

If ω' is any fixed point of f such that $|D(\omega,\omega)|<\infty$ and $|D(\omega',\omega')|<\infty$ then by (3.1), we have

$$\begin{aligned} D(\omega,\omega') &= D(f\omega,f\omega') \\ &\precsim \lambda_1 D(\omega,\omega')+\lambda_2 D(\omega,f\omega)+\lambda_3 D(\omega',f\omega')+\lambda_4 D(\omega',f\omega)+\lambda_5 D(\omega,f\omega') \\ &\precsim \lambda_1 D(\omega,\omega')+\lambda_2 D(\omega,\omega)+\lambda_3 D(\omega',\omega')+\lambda_4 D(\omega',\omega)+\lambda_5 D(\omega,\omega'). \end{aligned}$$

By Remark 2.1(ii), we have

$$\begin{aligned} |D(\omega,\omega')| &\le \lambda_1|D(\omega,\omega')|+\lambda_2|D(\omega,\omega)|+\lambda_3|D(\omega',\omega')|+\lambda_4|D(\omega',\omega)|+\lambda_5|D(\omega,\omega')| \\ &= (\lambda_1+\lambda_4+\lambda_5)|D(\omega,\omega')|+\lambda_2|D(\omega,\omega)|+\lambda_3|D(\omega',\omega')| \\ &= (\lambda_1+\lambda_4+\lambda_5)|D(\omega,\omega')|+\lambda_2(0)+\lambda_3(0) \\ &= (\lambda_1+\lambda_4+\lambda_5)|D(\omega,\omega')| \end{aligned}$$

then, we have

$$|D(\omega,\omega')| \le (\lambda_1+\lambda_4+\lambda_5)|D(\omega,\omega')|.$$

Since $\sum_{i=1}^{i=5}\lambda_i \in [0,1)$, we have $|D(\omega,\omega')|=0$ thus $D(\omega,\omega')=0$. Hence $\omega=\omega'$. □

4. Conclusion

In this work, we defined the generalized complex valued metric space. We study the uniqueness of fixed point of a mapping which satisfying the Hardy-Rogers contraction under some control negative real constants.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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