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Lexicographic Preferences Induced in Asymmetric Information Environments

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Abstract. The current paper derives from a purely set-theoretical approach to the problem of strategic information transmission between an informed sender and an uninformed decision maker. The information provided by the sender is encoded in a multifunction that forces the decision maker to behave according to the preference relation induced by the encoded information. The main purpose of the paper is to study sufficient conditions for an information multifunction to alter an originally given additive preference relation of a decision maker and induce a lexicographic one. In doing so, we build several examples of sets of multifunctions inducing additive and lexicographic preferences.

1. Introduction

The strategic analysis implied by unverifiable information being transmitted between asymmetrically informed agents was first introduced by Crawford and Sobel [3] in the theoretical economic literature. In their seminal model an unilaterally informed agent, who observes privately a onedimensional signal realization regarding the state of the world, sends a message (not necessarily corresponding to the realization observed) to a receiver, who takes an action determining the wealth of them both. Optimality is given by the Bayesian Nash equilibrium of the corresponding game, with increments in the similarity of agents' preferences leading to improvements in the informativeness of messages.

Recent multidimensional extensions of the Crawford and Sobel game concentrate efforts on the design of mechanisms that allow a receiver to elicit as much information as possible from a sender. Within this setting, Chakraborty and Harbaugh [2] provide complementarity conditions on preferences that induce both parties to agree on a ranking of multiple onedimensional alternatives. A similar

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structure was developed by Battaglini [1] in an environment with multiple senders and a single multidimensional alternative.

If the state of the world is verifiable after playing a game, the reputation of an information sender would depend on its realization. Within the described framework, reputation effects would reflect the learning abilities of decision makers, who may modify their strategies based on the reputation of a given information sender. In this regard, a partial verifiability model is presented by Glazer and Rubinstein [8]. These authors design an optimal mechanism under the assumption that the decision maker is allowed to verify one of the two characteristics defining the state of the world.*

On an empirical basis, Huffman et al. [9] analyze the modification of decision makers' choices through the arrival of new information. The main conclusion derived from their paper is that decision makers are susceptible to information from interested third parties. Thus, it is possible to manipulate decision makers' choices even if information is assumed to be verifiable.

We present an integrated framework encompassing the main features of the aforementioned models, and define a theoretical structure that allows for the analysis of preference manipulation in a multiple multidimensional alternatives setting when information is *verifiable*. That is, we do not allow the information sender to lie, but only to display the information subsets he finds more convenient. As a consequence, all reputation related effects are excluded from the analysis.

Moreover, in our setting, an alternative consists of a finite dimensional tuple of characteristics. Thus, the multidimensional framework of Chakraborty and Harbaugh [2] would be equivalent to ranking a set of onedimensional objects in our theoretical setting, while the model of Battaglini [1] would correspond to preference coordination within a unique multidimensional object.

Building on the set-theoretical approach to the problem of strategic information transmission developed by the authors (see [6]), we allow for the information provided by the sender to be encoded in a multifunction and to be the only information available to the receiver/decision maker. After receiving the multifunction, the decision maker will *naturally decompose* the available information dimension by dimension using coordinate functions on the set of alternatives. Our multifunctions are mechanisms that force the decision maker to choose according to the preferences induced by their encoded information. In this way, a sender who knows the utility function as well as the set of probability distributions, or *beliefs*, of a decision maker can manipulate the choice made by

*In general, the concepts of information acquisition and manipulation are treated separately. The strategic considerations defining optimal information acquisition, i.e. signal extraction, relate to the design of incentive compatible mechanisms in procurement within onedimensional state environments, see [10]. On he other hand, the idea of manipulation is usually relegated to social choice theory and the design of voting mechanisms, where the preference profiles of voters are exogenously given and must not be learned by decision makers, see [12].

the receiver. This assumption is perfectly in line with the common knowledge of utilities and beliefs defining the set of Bayesian Nash equilibria in Crawford and Sobel [3], and the subsequent literature. The preferences induced in this way will not generally coincide with those defined in a complete information environment.

The purpose of the paper is to show how it is possible within our theoretical framework to generate lexicographic preferences starting from non-lexicographic ones. More precisely, we study sufficient conditions for an information multifunction to alter the original additive preference relation of a decision maker and induce a lexicographic one. In doing so, we build several examples of sets of multifunctions inducing additive and lexicographic preferences.

The paper proceeds as follows. Sections 2 and 3 introduce the notations and basic assumptions needed to develop the model. Sections 4 and 5 define info-multifunctions, info-maps and their corresponding induced preferences. Section 6 provides sufficient conditions on sets of info-multifunctions for the induced preferences to be lexicographic.

2. **Preliminaries and Notations**

Let X be a nonempty set. A *preference relation* on X is a binary relation $\succsim \subseteq X \times X$ satisfying reflexivity, completeness and transitivity.

Reflexivity: $\forall\, x \in X,\ (x,x) \in \succsim$;

Completeness: $\forall\, x,y \in X,\ (x,y) \in \succsim \vee (y,x) \in \succsim$;

Transitivity: $\forall\, x,y,z \in X,\ (x,y) \in \succsim \wedge (y,z) \in \succsim \Rightarrow\ (x,z) \in \succsim$.

We usually write $x \succsim y$ in place of $(x,y) \in \succsim$ and read: x is (weakly) preferred to y.

The strict preference and the indifference relations associated to a preference relation $\succsim$ are defined as follows:

$$x \succ y \overset{\text{def}}{\iff} x \succsim y \wedge y \not\succsim x, \qquad x \sim y \overset{\text{def}}{\iff} x \succsim y \wedge y \succsim x.$$

We read $x \succ y$ as x is *strictly preferred* to y, while $x \sim y$ is read x is *indifferent to* y.

A *utility function representing a preference relation* $\succsim$ *on* X is a function $u : X \to \mathbb{R}$ such that:

$$\forall\, x,y \in X, \quad x \succsim y \quad \iff \quad u(x) \geq u(y).$$

The symbols $\geq$ and $>$ will denote the standard partial and linear order on the reals, respectively.

Given two natural numbers i, $n \in \mathbb{N}$, $i \leq n$ will be a short for $i \in \{1,2,\ldots,n\}$.

The Cartesian product of n nonempty sets $X_1,\ldots,X_n$ will be denoted by $\prod_{i\leq n} X_i$.

Henceforth, all Cartesian products are to be considered non-trivial (that is, $n \geq 2$).

A preference relation $\succsim$ on $\prod_{i\leq n} X_i$ is called *additive* (see [13]) if it is representable by an additive utility function, that is, there exist $u : \prod_{i\leq n} X_i \to \mathbb{R}$ and $u_i : X_i \to \mathbb{R}$, where $i \leq n$, such that $\forall\ (x_1,\dots,x_n),(y_1,\dots,y_n) \in \prod_{i\leq n} X_i$,

$$u(x_1,\dots,x_n) = u_1(x_1) + \dots + u_n(x_n)$$

and

$$(x_1,\dots,x_n) \succsim (y_1,\dots,y_n) \iff u(x_1,\dots,x_n) \geq u(y_1,\dots,y_n).$$

If $u : \prod_{i\leq n} X_i \to \mathbb{R}$ is an additive utility function, then for every nonempty set Y and every function $p : Y \to \prod_{i\leq n} X_i$, we have $(u \circ p) = \sum_{i\leq n}(u_i \circ p_i)$, where p_i is the i-th coordinate[†] function of p. Clearly, $(u \circ p)$ satisfies an additive-like property. Thus, abusing notation but in order to be formally consistent, we can extend the notion of additivity to a preference relation defined on a generic nonempty set.

Definition 2.1. Let $\prod_{i\leq n} X_i$ be the Cartesian product of n nonempty sets endowed with a preference relation $\succsim$. Given a nonempty set Y and a function $f : Y \to \prod_{i\leq n} X_i$, a preference relation can be defined on Y as follows:

$$\forall\ y_1, y_2 \in Y,\ y_1 \succsim_f y_2 \overset{\text{def}}{\iff} f(y_1) \succsim f(y_2).$$

The preference relation $\succsim_f$ will be called the f-relation induced by $\succsim$.

Definition 2.2. Let $\prod_{i\leq n} X_i$ be the Cartesian product of n nonempty sets endowed with a preference relation $\succsim$. Let Y be a nonempty set and fix $f : Y \to \prod_{i\leq n} X_i$. The f-relation $\succsim_f$ will be called additive if the inducing relation $\succsim$ is additive on $\prod_{i\leq n} X_i$.

A preference relation $\succsim$ on $\prod_{i\leq n} X_i$ is called *lexicographic* (see [7]) if there exist a function $u : \prod_{i\leq n} X_i \to \mathbb{R}$ and n real-valued functions $u_1,\dots,u_n$, respectively on $X_1,\dots,X_n$, with $n \in \mathbb{N}$, such that $\forall\ (x_1,\dots,x_n),(y_1,\dots,y_n) \in \prod_{i\leq n} X_i$,

$$u(x_1,\dots,x_n) \succ u(y_1,\dots,y_n) \iff (u_1(x_1),\dots,u_n(x_n)) >_{\text{Lex}} (u_1(y_1),\dots,u_n(y_n)).$$

For the sake of completeness, recall that for every $(a_1,\dots,a_n),(b_1,\dots,b_n) \in \mathbb{R}^n$, where $n \in \mathbb{N}$:

$$\begin{aligned}&(a_1,\dots,a_n) >_{\text{Lex}} (b_1,\dots,b_n)\\ &\quad\iff a_1 > b_1 \vee [a_1 = b_1 \wedge a_2 > b_2] \vee \dots \vee\ [(\forall\ i \leq m-1, a_i = b_i) \wedge a_n > b_n].\end{aligned}$$

In the lexicographic case, the order of the factors $X_1,\dots,X_n$, that is the enumeration of them, is tacitly very important. In a preference sense, the existence of a lexicographic utility roughly means that X_1 dominates X_2, X_2 dominates X_3,

[†]Given $i \leq n$, the i-th coordinate function of $p : Y \to \prod_{i\leq n} X_i$ is the function $p_i : Y \to X_i$ such that $p_i(y)$ is the i-th component of the n-tuple $p(y)$.

and so on. Such an order also implies that, in a costly information gathering environment with a binding budget constraint, decision makers would start acquiring information about the dominant factors, proceeding with the remaining ones only if indifferent among the former factors.

As in the additive case, we generalize the property of being lexicographic in a way that can be applied to generic preference relations, not only to those defined on Cartesian products.

Definition 2.3. Let $\prod_{i\leq n} X_i$ be the Cartesian product of n nonempty sets endowed with a preference relation $\succsim$. Let Y be a nonempty set and fix $f : Y \to \prod_{i\leq n} X_i$. The f-relation $\succsim_f$ will be called lexicographic if the inducing relation $\succsim$ is lexicographic on $\prod_{i\leq n} X_i$.

3. **Main Assumptions and Basic Results**

Henceforth, we will let $\mathscr{G}$ denote the set of all goods, or alternatives, and fix $n \geq 2$. Moreover, for every $i \leq n$, X_i will represent the set of all possible variants for the i-th characteristic or attribute of any commodity in $\mathscr{G}$, while X will stand for the Cartesian product $\prod_{i\leq n} X_i$.

Thus, an element $x_i^G \in X_i$ specifies the i-th characteristic of a given good $G \in \mathscr{G}$, while an n-tuple $(x_1^G, \dots, x_n^G)$ lists all its characteristics.

Definition 3.1. For every $i \leq n, X_i$ will be called the i-th characteristic factor. The Cartesian product $X = \prod_{i\leq n} X_i$ will be referred to as the characteristic space.

The preference relation on X will depend on the preference relations defined on the characteristic factors according to the following assumptions, which will hold through the paper.

Assumption 1. *For every $i \leq n$, let $\succsim_i$ be a preference relation on X_i and u_i be a bounded (above and below) utility function representing $\succsim_i$.*

Let $u : X \to \mathbb{R}$ be defined by:

$$\forall\ x = (x_1, \dots, x_n) \in X, \quad u(x) = \sum_{i\leq n} u_i(x_i).$$

Since each u_i is an increasing real function, the sum function u is increasing and it induces a preference relation $\succsim_u$ on X, defined as follows:

$$\forall\ x, y \in X,\ x \succsim_u y \overset{\text{def}}{\iff} u(x) \geq u(y).$$

Assumption 2. *Endow X with the preference relation $\succsim_u$.*

Proposition 3.2. *The preference relation $\succsim_u$ is additive on X.*

Proof. By definition of u. □

Proposition 3.3. *For every $f : \mathscr{G} \to X$, the f-relation $\succsim_f$, induced by $\succsim_u$, is additive on $\mathscr{G}$ and represented by $u \circ f$.*

Proof. The additivity of $\succsim_f$ follows from Definition 2.2. Furthermore, by Assumption 1 and the definition of u, we have:

$$\forall\ G, H \in \mathscr{G} \succsim_f H \iff u(f(G)) \geq u(f(H)) \iff \sum_{i \leq n} u_i(f_i(G)) \geq \sum_{i \leq n} u_i(f_i(H)),$$

where $f_i : X \to X_i$ is the i-th coordinate function of f. □

Let $\varphi : \mathscr{G} \to X$ be defined by $\varphi(G) = (x_1^G, x_2^G, \ldots, x_n^G)$, for every $G \in \mathscr{G}$. In the case when complete information on the set of all goods is available, this correspondence associates to every good a n-tuple that characterizes the good itself.

Note that X may contain tuples of characteristics that do not necessarily describe any existing good. Therefore, φ is injective, but not necessarily bijective.

Without loss of generality, we will work under the assumption that $X = \varphi(G)$, that is:

Assumption 3. *φ is bijective.*

Clearly, by Assumption 3, every G in $\mathscr{G}$ corresponds to exactly one n-tuple of X.

Moreover, by means of the map φ, the relation $\succsim_u$ induces the preference relation $\succsim_\varphi$ on $\mathscr{G}$ (see Definition 2.1) which is additive (Proposition 3.3).

Assumption 4. *Endow $\mathscr{G}$ with the φ-induced additive preference relation $\succsim_\varphi$.*

Assumption 3 and Assumption 4 yield the following proposition.

Proposition 3.4. *The map φ is an order isomorphism of $(\mathscr{G}, \succsim_\varphi)$ into $(X \succsim_u)$.*

We also assume the decision maker to be endowed with a subjective probability (density) function over each characteristic factor X_i. Abusing notation, each X_i can be considered a random variable.

Assumption 5. *For every $i \leq n$, $\mu_i : X_i \to [0, 1]$ is a non-atomic probability density function if X_i is absolutely continuous, and a non-degenerate probability function if X_i discrete.*

Clearly, we do not consider atomic probability density functions or degenerate probability functions, since they do not necessarily induce risk on the choices made by the decision marker.

The functions $\mu_1, \ldots, \mu_n$ must be interpreted as the subjective "beliefs" of the decision maker. For $i \leq n$, $\mu_i(Y_i)$ is the subjective probability that a randomly observed good from $\mathscr{G}$ displays an element $x_i \in Y_i \subseteq X_i$ as its i-th characteristic. [‡]

[‡]Note that the functions $\mu_1, \ldots, \mu_n$ can be assumed either independent or correlated, without this fact affecting our results.

Finally, following the standard economic theory of choice under uncertainty (see [11]), we assume that every decision maker assigns to each unknown i-th characteristic $x_i \in X_i$ the i-th certainty equivalent value induced by her subjective probability (density) function μ_i.

Definition 3.5. Let $i \leq n$. The certainty equivalent of μ_i and u_i, denoted by c_i, is a characteristic in X_i that the decision maker is indifferent to accept in place of the expected one to be obtained through (μ_i, u_i).

In other words, for every $i \leq n, c_i$ is an element of X_i whose utility $u_i(c_i)$ equals the expected value of u_i. Hence, $c_i \in u_i^{-1}(\int_{X_i} u_i(x_i)\mu_i(x_i)dx_i)$, if X_i is absolutely continuous, and $c_i \in u_i^{-1}(\sum_{x_i \in X_i} u_i(x_i)\mu_i(x_i))$, if X_i is discrete.

The existence of the i-th certainty equivalent characteristic defined by the decision maker in X_i is trivially equivalent to $u_i^{-1}(\int_{X_i} u_i(x_i)\mu_i(x_i)dx_i)$, or $u_i^{-1}(\sum_{x_i \in X_i} u_i(x_i)\mu_i(x_i))$, being a nonempty set. It is not difficult to provide examples of pairs (μ_i, u_i) on the set X_i such that c_i does not exist. In these cases, the decision maker can fix an element of X_i whose utility provides the subjectively closest approximation to the expected value $\big($that is, $\int_{X_i} u_i(x_i)\mu_i(x_i)dx_i$, or $\sum_{x_i \in X_i} u_i(x_i)\mu_i(x_i)\big)$; see [4] and [5]. Clearly, any approximation process generates a bias on the choice of the decision maker. However, as it will become evident below, our results remain unaffected by this fact. Hence, without loss of generality, we will work under the following assumption.

Assumption 6. *For every $i \leq n$, c_i exists.*

The use of certainty equivalent values implies that if the known characteristic delivers a higher (lower) utility than the corresponding subjective certainty equivalent value, the decision maker prefers the good defined by the former (latter) one.

4. **Info-multifunctions and Info-maps**

Consider a **multifunction** $T : \mathscr{G} \rightrightarrows \{1, 2, \ldots, n\}$, that is, a map that associates to each good G a (possibly empty) finite set of indices. Denote by $\mathrm{Dom}(T)$ the domain of T, that is, the set $\{G \in \mathscr{G} : T(G) \neq \emptyset\}$. In particular, $\mathrm{Dom}(T) = \mathscr{G}$ means that T takes only nonempty values.

We interpret each image $T(G)$ as the set of indices corresponding to the known characteristics of the good G. Following this interpretation, a multifunction T becomes a mechanism describing which information and from which good is made available to the decision maker by the information sender.

Assigning a multifunction $T : \mathscr{G} \rightrightarrows \{1, 2, \ldots, n\}$ implies assuming that the sender releases an information set of the form:

$$I_T = \{x_i^G : G \in \mathscr{G} \wedge i \in T(G)\}$$

Assigning a multifunction T, however, is weaker than assigning an information set, since the multifunction does not specify the value of each of the known characteristics. As a consequence, the information set associated with T does not need to be unique: the number of information sets that can be associated with T will depend on the cardinalities of the X_i's.

Definition 4.1. An information multifunction, or info-multifunction, is a *multi-function from the set* $\mathscr{G}$ *into* $\{1, 2, \ldots, n\}$. We will denote by $\mathscr{M}(\mathscr{G}, n)$ the set of all info-multifunctions.

Denote by $T_\emptyset$ the *empty valued info-multifunction* in $\mathscr{M}(\mathscr{G}, n)$ defined by:

$$\forall\ G \in \mathscr{G}, \quad T_\emptyset(G) = \emptyset.$$

Denote by T^* the global info-multifunction in $M(\mathscr{G}, n)$ defined by:

$$\forall\ G \in \mathscr{G}, \quad T^*(G) = \{1, 2, \ldots, n\}.$$

Clearly, $\mathrm{Dom}(T^*) = \mathscr{G}$. However, requiring $\mathrm{Dom}(T) = \mathscr{G}$ for $T \in M(\mathscr{G}, n)$ does not necessarily imply that $T = T^*$. Examples of multifunctions $T \neq T^*$ such that $\mathrm{Dom}(T) = \mathscr{G}$ can be easily given: consider, for instance, $T \in \mathscr{M}(\mathscr{G}, n)$ defined by $T(G) = \{1\}$, whenever $G \in \mathscr{G}$.

The value of each of the known characteristics remains specified by means of the info-map determined by the given info-multifunction, and defined as follows.

Definition 4.2. Let $T \in \mathscr{M}(\mathscr{G}, n)$. For every $i \leq n$, the i-th info-function determined by T is the function $\psi_i^T : \mathscr{G} \to X_i$ defined by

$$\psi_i^T(G) = \begin{cases} x_i^G & \text{if } \ i \in T(G), \\ c_i & \text{otherwise}, \end{cases}$$

with c_i being the i-th certainty equivalent of μ_i and u_i (see Definition 3.5 and Assumption 6). The product function $\prod_{i \leq n} \psi_i^T : \mathscr{G} \to X$ defined by

$$\left(\prod_{i \leq n} \psi_i^T \right)(G) = (\psi_1^T(G), \ldots, \psi_n^T(G)),$$

where $G \in \mathscr{G}$, and denoted by ψ^T is the *info-map determined by* T.

Every info-map ψ^T, determined by $T \in \mathscr{M}(\mathscr{G}, n)$, allows to describe each good as an n-tuple where all unknown characteristics are substituted by their corresponding certainty equivalent values. Clearly, info-maps are not necessarily bijective.

The info-map ψ^{T^*}, determined by the global info-multifunction T^*, equals the identification map φ; while the info-map $\psi^{T_\emptyset}$, determined by the empty valued info-multifunction $T_\emptyset$, is the constant function defined by $\psi^{T_\emptyset}(G) = (c_1, \dots, c_n)$, whenever $G \in \mathscr{G}$.

5. Preferences Induced by Info-maps

Given $T \in \mathscr{M}(\mathscr{G}, n)$, the decision maker is endowed with an incomplete information set, that may force her to change her original preference relation, $\succsim_\varphi$. Indeed, in place of $\succsim_\varphi$, the decision maker will base her choice on the ψ^T-relation induced by $\succsim_u$; see Definition 2.1.

More precisely:

$$\forall\ G, H \in \mathscr{G},\ \ G \succsim_{\psi^T} H \quad \overset{\text{def}}{\iff} \quad \psi^T(G) \succsim_u \psi^T(H).$$

Proposition 3.3 and Definition 4.2 immediately yield the following.

Proposition 5.1. *For every $T \in \mathscr{M}(\mathscr{G}, n)$, the preference relation $\succsim_{\psi^T}$ is additive on $\mathscr{G}$ and represented by $u \circ \psi^T$.*

Remark 5.2. It deserves to explicitly underline the additive character of the ψ^T-relation $\succsim_{\psi^T}$. For every $G, H \in \mathscr{G}$,

$$G \succsim_{\psi^T} H \iff u(\psi^T(G)) \geq u(\psi^T(H)) \iff \sum_{i \leq n} u_i(\psi_i^T(G)) \geq \sum_{i \leq n} u_i(\psi_i^T(H)).$$

Furthermore, the indifference relation $\sim_{\psi^T}$ associated to $\succsim_{\psi^T}$ can be described as follows.

For every $G, H \in \mathscr{G}$,

$$G \sim_{\psi^T} H \iff G, H \in \mathscr{G} \setminus \mathrm{Dom}(T) \vee \Big(G, H \in \mathrm{Dom}(T) \wedge \sum_{i \in T(G)} u_i(x_i^G) = \sum_{i \in T(H)} u_i(x_i^H)\Big).$$

The preference relation $\succsim_{\psi^T}$, induced by a $T \in \mathscr{M}(\mathscr{G}, n)$, is in general different from the preference relation $\succsim_\varphi$ induced by $\succsim_u$ on $\mathscr{G}$ under perfect information (see Section 3). Therefore, different preference relations can be induced depending on the information set presented to the decision maker. This implies that knowing the original preference relation of a decision maker, $\succsim_\varphi$, allows for displaying information sets in such a way so as to manipulate the final choice of the decision maker.

More precisely, an information sender who knows u_i, for $i \leq n$ (equivalently, $\succsim_i$ for $i \leq n$), as well as μ_i, $i \leq n$, can manipulate the choice made by the receiver. As already mentioned (see Section 1), this assumption is in line with the common knowledge of utilities and beliefs defining the set of Bayesian Nash equilibria in Crawford and Sobel [3], and the subsequent literature.

Through the next section we provide the conditions for an information multifunction to modify the original preference relation of a decision maker obliging her to choose lexicographically.

6. **Inducing Lexicographic Preferences**

We start by considering the case where the only information displayed is the i-th characteristic of a certain good G, that is, where the info-multifunction $T \in \mathscr{M}(\mathscr{G}, n)$ defined by $T(G) = \{i\}$ and $T(H) = \emptyset$, for every $H \in \mathscr{G} \setminus \{G\}$, is assigned.

Suppose that $\psi^T(G) = x_i^G$; hence x_i^G is the only characteristic known to the decision maker. Then, for every $H \neq G$, $G \succsim_{\psi^T} H$ if and only $x_i^G \succsim_i c_i$. This means that G is lexicographically preferred to every other good if its i-th characteristic is preferred to the i-th certainty equivalent, otherwise any other good will be lexicographically strictly preferred to G.

Thus, providing the decision maker with x_i^G for a unique given good G suffices to manipulate her preferences and induce lexicographic choices. Note that this fact is not necessarily true if $T(G)$ consists of at least two elements (that is, $T(G) = \{i, j\}$ where $i, j \leq n$). We shall therefore study sufficient conditions for a generic into-multifunction to induce lexicographic choices.

In the case when $\prod_{i \leq n} X_i$ is the product of n finite sets, appropriate but strong independence conditions must hold for the existence of lexicographic utilities to imply that of additive utilities (see Section 4.3 in [7]). The converse is neither generally true, nor has been adequately studied. We shall investigate some natural conditions under which additive utilities happen to be lexicographic, and vice versa.

6.1. *The Set* $\mathscr{M}(\text{Lex})$

Let $\mathscr{M}(\text{Lex})$ be the set of all info-multifunctions $T \in \mathscr{M}(\mathscr{G}, n) \setminus \{T_\emptyset\}$ satisfying the following properties:

(L.1) $\forall\ G \in \text{Dom}(T)$ and $\forall\ i \in T(G)$, $x_i^G \succsim_i c_i$;

(L.2) $\exists\ i^* \in \text{Range}(T)$ such that $\forall\ G, H \in \text{Dom}(T)$,

(L.2.a) $x_{i^*}^G \not\sim_{i^*} x_{i^*}^H$;

(L.2.b) $\forall\ i \in \text{Range}(T) \setminus \{i^*\}$, $x_i^G \sim_i x_i^H$.

Range(T) denotes the range of the info-multifunction T, that is, the union set $\bigcup_{G \in \mathscr{G}} T(G)$.

Theorem 6.1. *For every $T \in \mathscr{M}(\text{Lex})$, the preference relation induced by T, $\succsim_{\psi^T}$, is additive and lexicographic on $\mathscr{G}$. In particular, $\succsim_{\psi^T}$ is a strict lexicographic preference on* $\text{Dom}(T)$.

Proof. Fix $T \in \mathscr{M}(\text{Lex})$. By Proposition 5.1, $\succsim_{\psi^T}$ is additive on $\mathscr{G}$. By Remark 5.2, $G \sim_{\psi^T} H$, whenever $G, H \in \mathscr{G} \setminus \text{Dom}(T)$. By (L.1), $G \succsim_{\psi^T} H$, whenever $G \in \text{Dom}(T)$ and $H \in \mathscr{G} \setminus \text{Dom}(T)$. By (L.2) and Remark 5.2, $\exists\ i^* \in \text{Range}(T)$ such that $\forall\ G, H \in \text{Dom}(T)$,

$$\begin{aligned} &u(\psi^T(G)) > u(\psi^T(H)) \\ &\quad \Longleftrightarrow u_{i^*}(\psi^T_{i^*}(G)) >_{i^*} u_{i^*}(\psi^T_{i^*}(H)) \\ &\quad \Longleftrightarrow (u_1(\psi^T_1(G)), \dots, u_n(\psi^T_n(G))) >_{Lex} (u_1(\psi^T_1(H)), \dots, u_n(\psi^T_n(H))). \end{aligned}$$

Hence, $\forall\ G, H \in \mathscr{G}$,

$$\begin{aligned} u(\psi^T(G)) > u(\psi^T(H)) \iff &(u_1(\psi^T_1(G)), \dots, u_n(\psi^T_n(G))) \\ &\geq_{\text{Lex}} (u_1(\psi^T_1(G)), \dots, u_n(\psi^T_n(G))). \end{aligned}$$ □

Remark 6.2. A set of info-multifunctions larger than $\mathscr{M}(\text{Lex})$ and whose elements still induce lexicographic preference relations can be obtained by replacing (L.2) with the following weaker property:

(wL.2) $\exists\ i^* \in \text{Range}(T)$ such that:

(wL.2.a) $x^G_{i^*} \not\sim_{i^*} x^H_{i^*}$ for some $G, H \in \text{Dom}(T)$.
(wL.2.b) $\forall\ G, H \in \text{Dom}(T)$, $\forall\ i \in \text{Range}(T) \setminus \{i^*\}$, $x^G_i \sim_i x^H_i$.

The corresponding of Theorem 6.1 would still hold true, but the induced preference relations would not necessarily be strict on $\text{Dom}(T)$. See also the discussion at the beginning of the current section.

Remark 6.3. Both conditions (L.2.a) and (L.2.b) are essential in order to induce a lexicographic preference relation on $\mathscr{G}$. In fact, without condition (L.2.a) the decision maker could turn out to be indifferent among all goods. On the other hand, condition (L.2.b) avoids situations where $\widehat{i} = \min\{i \leq n : x^H_i \not\sim_i x^G_i\} \leq i^*$, $x^H_{\widehat{i}} \succ_{\widehat{i}} x^G_{\widehat{i}}$ and $\sum_{i \leq n} u(x^G_i) > \sum_{i \leq n} u(x^H_i)$ implying $H \succ_{\text{Lex}} G$ but $G \succ_{\varphi^T} H$.

We present now two particularly interesting types of subsets of $\mathscr{M}(\text{Lex})$.

6.1.1. The subsets $\mathscr{M}(\Lambda)$, $\Lambda \subseteq \{1, 2, \dots, n\}$.

Let $\Lambda \subseteq \{1, 2, \dots, n\}$ and $\mathscr{M}(\Lambda)$ be the set of all info-multifunctions $T \in \mathscr{M}(\mathscr{G}, n) \setminus \{T_\emptyset\}$ satisfying the following properties:

(Λ.0) $\forall\ G \in \text{Dom}(T)$, $T(G) = \Lambda$;
(Λ.1) $\forall\ G \in \text{Dom}(T)$ and $\forall\ i \in \Lambda$, $x^G_i \succsim_i c_i$;
(Λ.2) $\forall\ G, H \in \text{Dom}(T)$,

(Λ.2.a) $x^G_{\max\Lambda} \not\sim_{\max\Lambda} x^H_{\max\Lambda}$;
(Λ.2.b) $\forall\ i \in \Lambda \setminus \{\max\Lambda\}$, $x^G_i \sim_i x^H_i$.

It is easy to check that, given any $\Lambda \subseteq \{1,2,\ldots,n\}$, $\mathscr{M}(\Lambda)$ is a subfamily of $\mathscr{M}(\text{Lex})$. In fact, properties (Λ.0) and (Λ.1) imply (L.1), while property (Λ.2) implies (L.2).

Corollary 6.4. *Let $\Lambda \subseteq \{1,2,\ldots,n\}$. For every $T \in \mathscr{M}(\Lambda)$, the preference relation induced by T, $\succsim_{\psi^T}$, is additive and lexicographic on $\mathscr{G}$. In particular, $\succsim_{\psi^T}$ is a strict lexicographic preference on* Dom(T).

Proof. For every $\Lambda \subseteq \{1,2,\ldots,n\}$, $\mathscr{M}(\Lambda) \subseteq \mathscr{M}(\text{Lex})$. Apply Theorem 6.1. □

6.1.2. The subsets $\mathscr{M}(h)$, $h \le n$.

Fix $h \le n$ and let $\mathscr{M}(h)$ be the set of all info-multifunctions $T \in \mathscr{M}(\mathscr{G},n) \setminus \{T_\emptyset\}$ satisfying the following properties:

(h.0) $\forall\ G \in \text{Dom}(T)$, $T(G) = \{1,2,\ldots,h\}$;
(h.1) $\forall\ G \in \text{Dom}(T)$ and $\forall\ i \le h$, $x^G_i \succsim_i c_i$;
(h.2) $\forall\ G,H \in \text{Dom}(T)$,
(h.2.a) $x^G_h \not\sim_h x^H_h$;
(h.2.b) $\forall\ i \le h-1$, $x^G_i \sim_i x^H_i$.

It is easy to check that, given any $h \le n$, $\mathscr{M}(h) = \mathscr{M}(\Lambda)$, provided that $\Lambda = \{1,2,\ldots,h\}$. In particular, $\mathscr{M}(h)$ is a subfamily of $\mathscr{M}(\text{Lex})$, whenever $h \le n$.

Corollary 6.5. *Let $h \le n$. For every $T \in \mathscr{M}(h)$, the preference relation induced by T, $\succsim_{\psi^T}$, is additive and lexicographic on $\mathscr{G}$. In particular, $\succsim_{\psi^T}$ is a strict lexicographic preference on* Dom(T).

Proof. Fix $h \le n$. Then, $\mathscr{M}(h) = \mathscr{M}(\Lambda)$, where $\Lambda = \{1,2,\ldots,h\}$. Apply Corollary 6.4. □

6.2. *The Set* $\mathscr{M}(\sharp\Lambda)$

Another approach to the problem of inducing preference relations that are both additive and lexicographic is the following.

Fix $\Lambda \subseteq \{1,2,\ldots,n\}$. Let $\mathscr{M}(\sharp\Lambda)$ be the set of all $T \in \mathscr{M}(\mathscr{G},n) \setminus \{T_\emptyset\}$ satisfying the following properties:

(♯Λ.1) $\forall\ G \in \text{Dom}(T)$, $\exists\ h \in \Lambda$ such that $T(G) = \{1,\ldots,h\}$ and $\forall\ i \in T(G)$, $x^G_i \succ_i c_i$;
(♯Λ.2) $\forall\ G,H \in \text{Dom}(T)$,
(♯Λ.2.a) $T(G) = T(H) = \{1,\ldots,h\}$ implies that $\forall\ i \le h-1$, $x^G_i \sim_i x^H_i$ and $x^G_h \not\sim_i x^H_h$
(♯Λ.2.b) $T(G) = \{1,\ldots,h\}$ and $T(H) = \{1,\ldots,k\}$, with $h > k$, imply that $\forall\ i \le k-1$, $x^G_i \sim_i x^H_i$ and $x^G_k \succsim_k x^H_k$.

Remark 6.6. It is easy to check that $\mathscr{M}(\sharp\Lambda) = \mathscr{M}(h)$ if and only if $\Lambda = \{h\}$. That is, $\mathscr{M}(\sharp\Lambda) \neq \bigcup_{h\in\Lambda} \mathscr{M}(h)$ unless Λ has cardinality one.

Theorem 6.7. *Let $\Lambda \subseteq \{1,2,\ldots,n\}$. For every $T \in \mathscr{M}(\sharp\Lambda)$, the preference relation induced by T, $\succsim_{\psi^T}$, is additive and lexicographic on $\mathscr{G}$. In particular, $\succsim_{\psi^T}$ is a strict lexicographic preference on* Dom(T).

Proof. Fix $T \in \mathscr{M}(\sharp\Lambda)$. By Proposition 5.1, $\subseteq_{\psi^T}$ is additive on $\mathscr{G}$.

By Remark 5.2, $G \sim_{\psi^T} H$, whenever $G,H \in \mathscr{G} \setminus \mathrm{Dom}(T)$. By ($\sharp\Lambda$.1) and Remark 5.2, $G \succ_{\psi^T} H$, whenever $G \in \mathrm{Dom}(T)$ and $H \in \mathscr{G} \setminus \mathrm{Dom}(T)$. By ($\sharp\Lambda$.2), if $T(G) = T(H) = \{1,\ldots,h\}$, then

$$u(\psi^T(G)) > u(\psi^T(H)) \iff u_h(\psi_h^T(G)) > u_h(\psi_h^T(H)).$$

while, if $T(G) = \{1,\ldots,h\}$ and $T(H) = \{1,\ldots,k\}$, with $h > k$, then

$$u(\psi^T(G)) > u(\psi^T(H)) \iff u_k(\psi_k^T(G)) \geq u_k(\psi_k^T(H)).$$

Hence, $\forall\, G,H \in \mathscr{G}$,

$$u(\psi^T(G)) > u(\psi^T(H)) \iff (u_1(\psi_1^T(G)),\ldots,u_n(\psi_n^T(G))) \geq_{\mathrm{Lex}} (u_1(\psi_1^T(H)),\ldots,u_n(\psi_n^T(H))). \qquad \square$$

By Remark 6.6, Corollary 6.5 is also a consequence of Theorem 6.7.

Corollary 6.8 (Corollary 6.5). *Let $\Lambda = \{h\}$, for some $h \leq n$. For every $T \in \mathscr{M}(\sharp\Lambda)$, the preference relation induced by T, $\succsim_{\psi^T}$, is additive and lexicographic on $\mathscr{G}$. In particular, $\succsim_{\psi^T}$ is a strict lexicographic preference on* Dom(T).

Remark 6.9. For the sake of completeness, note that our set of assumptions (refer to Section 3) does not suffice to define info-multifunctions inducing continuously representable preference relations. The issue of continuity for utility functions representing induced preference relations is studied in [6], to which the interested reader is referred.

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