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Research Article

Some Fixed Point Theorems in C-complete Complex Valued Metric Spaces

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Abstract. In this paper, we prove common fixed point theorems for a pair of mappings satisfying rational inequality in C-complete complex valued metric spaces. The results of this paper generalize and extend the known results in C-complete complex valued metric spaces.

Keywords. Complex valued metric spaces; Common fixed points; C-complete complex valued metric spaces

MSC. 47H09; 47H10

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1. Introduction and Preliminaries

The concept of complex valued metric space was introduced by Azam *et al.* [1], proving some fixed point results for mappings satisfying a rational inequality in complex valued metric spaces. Afterwards, several papers have dealt with fixed point theory in complex valued metric spaces (see [3], [4], [6] and references therein).

Recently, Sintunavarat *et al.* [7] introduced the notion of a C-cauchy sequence in C-complete complex valued metric space and established the existence of common fixed point theorems in C-complete complex valued metric spaces. In sequel, Kumar *et al.* [5] proved common fixed point theorems for weakly compatible maps, weakly compatible along with (CLR) and E.A. Properties in C-complete complex valued metric spaces.

The aim of this paper is to establish and prove common fixed point theorems for a pair of mappings satisfying rational expressions having control functions as coefficients in C-complete complex valued metric spaces. Our results generalize and extend the results of Dubey *et al*. [2], Kumar *et al*. [5], and Sintunavarat *et al*. [7].

Consistent with Azam *et al*. [1], the following definitions and results will be needed in the sequel. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$, that is $z_1 \precsim z_2$ if one of the following holds:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C_2), (C_3) and (C_4) is satisfied and we will write $z_1 \prec z_2$ if only (C_4) is satisfied.

Remark 1.1. We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \le b \Longrightarrow az \precsim bz \ \forall \ z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$.

Definition 1.2 ([1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions;

(d1) $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i \ |y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X,d) is a complex-valued metric space.

Definition 1.4 ([1]). Let (X,d) be a complex valued metric space.

(1) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(2) A point $x \in X$ is called a limit point of A whenever, for all $0 \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A - \{x\}) \neq \phi.$$

(3) A set $A \subseteq X$ is called open set whenever each element of A is an interior point of A.

(4) A set $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

(5) A sub-basis for a Hausdorff topology τ on X is the family

$$F = \{B(x,r) : x \in X \text{ and } 0 \prec r\}.$$

Definition 1.5 ([1]). Let (X,d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and let $x \in X$.

(1) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \prec c$, then $\{x_n\}$ is said to be convergent to a point $x \in X$ or $\{x_n\}$converges to a point $x \in X$ and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) If for every Cauchy sequence in X is convergent, then (X,d) is said to be complete complex valued metric space.

Lemma 1.6 ([1]). *Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 1.7 ([1]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.*

Further, In 2013, Sintunavarat *et al.* [7] introduced the notion of a C-Cauchy sequence in C-complete complex valued metric space as follows:

Definition 1.8 ([7]). Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $k \in \mathbb{N}$ such that for all $m,n > k$, $d(x_n,x_m) \prec c$, then $\{x_n\}$ is called a C-Cauchy sequence in X.

(ii) If every C-Cauchy sequence in X is convergent, then (X,d) is said to be a C-complete complex valued metric space.

2. Main Results

Throughout this paper, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{C}_+$ denotes a set $\{c \in \mathbb{C} : 0 \precsim c\}$ and Γ denotes the class of all functions $\mu : \mathbb{C}_+ \times \mathbb{C}_+ \to [0,1)$ which satisfies the condition:

for (x_n,y_n) in $\mathbb{C}_+ \times \mathbb{C}_+$,

$$\mu(x_n,y_n) \to 1 \Longrightarrow (x_n,y_n) \to 0.$$

In 2013, Sintunavarat *et al.* [7] proved the following fixed point result:

Let S and T be self mappings of a C-complete complex valued metric space (X,d). If there exists mappings $\alpha,\beta : \mathbb{C}_+ \to [0,1)$ such that for all $x,y \in X$:

(a) $\alpha(x) + \beta(x) < 1$,

(b) the mapping $\gamma : \mathbb{C}_+ \to [0,1)$ defined by $\gamma(x) = \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ,

(c) $d(Sx,Ty) \preceq \alpha(d(x,y))d(x,y) + \beta(d(x,y))\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$.

Then S and T have a unique common fixed point. Next, we prove our main results.

Theorem 2.1. *Let S and T be self mappings of a C-complete complex valued metric space (X,d). If there exists mappings $\alpha,\beta,\gamma : \mathbb{C}_+ \times \mathbb{C}_+ \to [0,1)$ such that for all x,y in X:*

(i) $\alpha(x,y) + \beta(x,y) + \gamma(x,y) < 1,$ (2.1)

(ii) *the mapping $\mu : \mathbb{C}_+ \times \mathbb{C}_+ \to [0,1)$ defined by $\mu(x,y) := \frac{\alpha(x,y)}{1-\beta(x,y)}$ belongs to Γ,* (2.2)

(iii) $d(Sx,Ty) \preceq \alpha(x,y)d(x,y) + \beta(x,y)\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)} + \gamma(x,y)\frac{d(y,Sx)[1+d(x,Ty)]}{1+d(x,y)}$. (2.3)

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1}, \quad \text{for all } n \geq 0. \tag{2.4}$$

For $n \geq 0$, we get

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\preceq \alpha(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1}) \\ &\quad + \beta(x_{2n},x_{2n+1})\frac{d(x_{2n+1},Tx_{2n+1})[1+d(x_{2n},Sx_{2n})]}{1+d(x_{2n},x_{2n+1})} \\ &\quad + \gamma(x_{2n},x_{2n+1})\frac{d(x_{2n+1},Sx_{2n})[1+d(x_{2n},Tx_{2n+1})]}{1+d(x_{2n},x_{2n+1})} \\ &= \alpha(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1}) \\ &\quad + \beta(x_{2n},x_{2n+1})\frac{d(x_{2n+1},x_{2n+2})[1+d(x_{2n},x_{2n+1})]}{1+d(x_{2n},x_{2n+1})} \\ &\quad + \gamma(x_{2n},x_{2n+1})\frac{d(x_{2n+1},x_{2n+1})[1+d(x_{2n},x_{2n+2})]}{1+d(x_{2n},x_{2n+1})} \\ &\preceq \alpha(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1}) + \beta(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2}), \end{aligned}$$

which implies that

$$d(x_{2n+1},x_{2n+2}) \preceq \mu(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1}) \tag{2.5}$$

where $\mu(x,y) = \frac{\alpha(x,y)}{1-\beta(x,y)}$.

Similarly, for $n \geq 0$, we get

$$d(x_{2n+2},x_{2n+3}) \preceq \mu(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+2}). \tag{2.6}$$

From (2.5) and (2.6), we get

$$d(x_n,x_{n+1}) \preceq \mu(x_{n-1},x_n)d(x_{n-1},x_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we get

$$|d(x_n,x_{n+1})| \leq \mu(x_{n-1},x_n)|d(x_{n-1},x_n)| \leq |d(x_{n-1},x_n)|, \tag{2.7}$$

for all $n \in \mathbb{N}$.

This implies that the sequence $\{|d(x_{n-1},x_n)|\}$, $n\in\mathbb{N}$ is monotone non-increasing and bounded below, therefore,

$$|d(x_{n-1},x_n)|\to r \text{ for some } r\geq 0.$$

Next, we claim that $r=0$. Assume to the contrary that $r>0$. Proceeding limit as $n\to\infty$, we have from (2.7), $\mu(x_{n-1},x_n)\to 1$. Since $\mu\in\Gamma$, we get $(x_{n-1},x_n)\to 0$, that is

$$|d(x_{n-1},x_n)|\to 0, \text{ which is a contradiction.}$$

Therefore, we have $r=0$, that is

$$|d(x_{n-1},x_n)|\to 0. \tag{2.8}$$

Next, we show that $\{x_n\}$ is a C-Cauchy sequence. According to (2.8), it is sufficient to prove that the subsequence $\{x_{2n}\}$ is a C-Cauchy sequence. Let, if possible, $\{x_{2n}\}$ is not a C-Cauchy sequence. So there is $c\in\mathbb{C}$ with $0\prec c$, for which, for all $k\in\mathbb{N}$, there exists $m(k)>n(k)\geq k$, such that

$$d\left(x_{2n(k)},x_{2m(k)}\right)\succcurlyeq c. \tag{2.9}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k)>n(k)\geq k$ satisfying (2.9). Then, we have

$$d\left(x_{2n(k)},x_{2m(k)}\right)\succcurlyeq c \tag{2.10}$$

and

$$d\left(x_{2n(k)},x_{2m(k)-2}\right)\prec c. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned} c &\preccurlyeq d\left(x_{2n(k)},x_{2m(k)}\right)\\ &\preccurlyeq d\left(x_{2n(k)},x_{2m(k)-2}\right)+d\left(x_{2m(k)-2},x_{2m(k)-1}\right)+d\left(x_{2m(k)-1},x_{2m(k)}\right)\\ &\prec c+d\left(x_{2m(k)-2},x_{2m(k)-1}\right)+d\left(x_{2m(k)-1},x_{2m(k)}\right).\end{aligned}$$

This implies that

$$|c|\leq\left|d\left(x_{2n(k)},x_{2m(k)}\right)\right|\leq|c|+\left|d\left(x_{2m(k)-2},x_{2m(k)-1}\right)\right|+\left|d\left(x_{2m(k)-1},x_{2m(k)}\right)\right|.$$

Letting $k\to\infty$, we get

$$\left|d(x_{2n(k)},x_{2m(k)})\right|\to|c|. \tag{2.12}$$

Further, we have

$$\begin{aligned} d\left(x_{2n(k)},x_{2m(k)}\right) &\preccurlyeq d\left(x_{2n(k)},x_{2m(k)+1}\right)+d(x_{2m(k)+1},x_{2m(k)})\\ &\preccurlyeq d\left(x_{2n(k)},x_{2m(k)}\right)+d\left(x_{2m(k)},x_{2m(k)+1}\right)+d\left(x_{2m(k)+1},x_{2m(k)}\right),\end{aligned}$$

implies that

$$\left|d(x_{2n(k)},x_{2m(k)})\right|\leq\left|d(x_{2n(k)},x_{2m(k)})\right|+\left|d(x_{2m(k)},x_{2m(k)+1})\right|+\left|d(x_{2m(k)+1},x_{2m(k)})\right|.$$

Letting $k\to\infty$ and using (2.8) and (2.12), we get

$$\left|d(x_{2n(k)},x_{2m(k)+1})\right|\to|c|. \tag{2.13}$$

Now

$$d(x_{2n(k)},x_{2m(k)+1})\preccurlyeq d(x_{2n(k)},x_{2n(k)+1})+d(x_{2n(k)+1},x_{2m(k)+2})+d(x_{2m(k)+2},x_{2m(k)+1})$$

$$
\begin{aligned}
&= d(x_{2n(k)}, x_{2n(k)+1}) + d(Sx_{2n(k)}, Tx_{2m(k)+1}) + d(x_{2m(k)+2}, x_{2m(k)+1}) \\
&\precsim d\left(x_{2n(k)}, x_{2n(k)+1}\right) + \alpha\left(x_{2n(k)}, x_{2m(k)+1}\right) d\left(x_{2n(k)}, x_{2m(k)+1}\right) \\
&\quad + \beta\left(x_{2n(k)}, x_{2m(k)+1}\right) \frac{d\left(x_{2m(k)+1}, Tx_{2m(k)+1}\right)\left[1 + d\left(x_{2n(k)}, Sx_{2n(k)}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)} \\
&\quad + \gamma\left(x_{2n(k)}, x_{2m(k)+1}\right) \frac{d\left(x_{2m(k)+1}, Sx_{2n(k)}\right)\left[1 + d\left(x_{2n(k)}, Tx_{2m(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)} \\
&\quad + d\left(x_{2m(k)+2}, x_{2m(k)+1}\right) \\
&= d\left(x_{2n(k)}, x_{2n(k)+1}\right) + \alpha\left(x_{2n(k)}, x_{2m(k)+1}\right) d\left(x_{2n(k)}, x_{2m(k)+1}\right) \\
&\quad + \beta\left(x_{2n(k)}, x_{2m(k)+1}\right) \frac{d\left(x_{2m(k)+1}, x_{2m(k)+2}\right)\left[1 + d\left(x_{2n(k)}, x_{2n(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)} \\
&\quad + \gamma\left(x_{2n(k)}, x_{2m(k)+1}\right) \frac{d\left(x_{2m(k)+1}, x_{2n(k)+1}\right)\left[1 + d\left(x_{2n(k)}, x_{2m(k)+2}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)} \\
&\quad + \ d\left(x_{2m(k)+2}, x_{2m(k)+1}\right)
\end{aligned}
$$

implies that

$$
\begin{aligned}
\left|d(x_{2n(k)}, x_{2m(k)+1})\right| &\le \left|d(x_{2n(k)}, x_{2n(k)+1})\right| + \alpha\left(x_{2n(k)}, x_{2m(k)+1}\right)\left|d\left(x_{2n(k)}, x_{2m(k)+1}\right)\right| \\
&\quad + \beta\left(x_{2n(k)}, x_{2m(k)+1}\right)\left|\frac{d\left(x_{2m(k)+1}, x_{2m(k)+2}\right)\left[1 + d\left(x_{2n(k)}, x_{2n(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| \\
&\quad + \gamma\left(x_{2n(k)}, x_{2m(k)+1}\right)\left|\frac{d\left(x_{2m(k)+1},\ x_{2n(k)+1}\right)\left[1 + d\left(x_{2n(k)}, x_{2m(k)+2}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| \\
&\quad + \left|d\left(x_{2m(k)+2}, x_{2m(k)+1}\right)\right| \\
&\le \left|d(x_{2n(k)}, x_{2n(k)+1})\right| + \alpha\left(x_{2n(k)}, x_{2m(k)+1}\right)\left|d\left(x_{2n(k)}, x_{2m(k)+1}\right)\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2m(k)+2}\right)\left[1 + d\left(x_{2n(k)}, x_{2n(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2n(k)+1}\right)\left[1 + d\left(x_{2n(k)}, x_{2m(k)+2}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| + \left|d\left(x_{2m(k)+2}, x_{2m(k)+1}\right)\right| \\
&\le \left|d(x_{2n(k)}, x_{2n(k)+1})\right| + \frac{\alpha(x_{2n(k)}, x_{2m(k)+1})}{1 - \beta(x_{2n(k)}, x_{2m(k)+1})}\left|d(x_{2n(k)}, x_{2m(k)+1})\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2m(k)+2}\right)\left[1 + d\left(x_{2n(k)}, x_{2n(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2n(k)+1}\right)\left[1 + d\left(x_{2n(k)}, x_{2m(k)+2}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| + \left|d\left(x_{2m(k)+2}, x_{2m(k)+1}\right)\right| \\
&\le \left|d(x_{2n(k)}, x_{2n(k)+1})\right| + \mu(x_{2n(k)}, x_{2m(k)+1})\left|d(x_{2n(k)}, x_{2m(k)+1})\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2m(k)+2}\right)\left[1 + d\left(x_{2n(k)}, x_{2n(k)+1}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| \\
&\quad + \left|\frac{d\left(x_{2m(k)+1}, x_{2n(k)+1}\right)\left[1 + d\left(x_{2n(k)}, x_{2m(k)+2}\right)\right]}{1 + d\left(x_{2n(k)}, x_{2m(k)+1}\right)}\right| + \left|d\left(x_{2m(k)+2}, x_{2m(k)+1}\right)\right|.
\end{aligned}
$$

Letting limit as $k \to \infty$, we get

$$|c| \leq \lim_{k\to\infty} \mu(x_{2n(k)}, x_{2m(k)+1})|c| \leq |c|,$$

which implies that $\lim_{k\to\infty} \mu(x_{2n(k)}, x_{2m(k)+1}) = 1$.

Since $\mu \in \Gamma$, we get $\left(x_{2n(k)}, x_{2m(k)}\right) \to 0$, that is $\left|d(x_{2n(k)}, x_{2m(k)+1})\right| \to 0$, which contradicts $0 \prec c$. Therefore, we can conclude that $\{x_{2n}\}$ is C-Cauchy sequence and hence $\{x_n\}$ is a C-Cauchy sequence in X and X is complete, so there exists a point z in X such that $x_n \to z$ as $n \to \infty$.

Next, we prove that $Sz = z$. If $Sz \neq z$ then $d(Sz, z) > 0$.

Now,

$$\begin{aligned}
d(z,Sz) &\precsim d(z,x_{2n+2}) + d(x_{2n+2},Sz) \\
&= d(z,x_{2n+2}) + d(Tx_{2n+1},Sz) \\
&= d(z,x_{2n+2}) + d(Sz,Tx_{2n+1}) \\
&\precsim d(x_{2n+2},z) + \alpha(z,x_{2n+1})d(z,x_{2n+1}) + \beta(z,x_{2n+1})\frac{d(x_{2n+1},Tx_{2n+1})[1+d(z,Sz)]}{1+d(z,x_{2n+1})} \\
&\quad + \gamma(z,x_{2n+1})\frac{d(x_{2n+1},\ Sz)[1+d(z,\ Tx_{2n+1})]}{1+d(z,x_{2n+1})} \\
&= d(x_{2n+2},z) + \alpha(z,x_{2n+1})d(z,x_{2n+1}) \\
&\quad + \beta(z,x_{2n+1})\frac{d(x_{2n+1},x_{2n+2})[1+d(z,Sz)]}{1+d(z,x_{2n+1})} + \gamma(z,x_{2n+1})\frac{d(x_{2n+1},\ Sz)[1+d(z,x_{2n+2})]}{1+d(z,x_{2n+1})}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$d(z,Sz) \precsim d(z,z) + \alpha(z,z)d(z,z) + \beta(z,z)\frac{d(z,z)[1+d(z,Sz)]}{1+d(z,z)} + \gamma(z,z)\frac{d(z,Sz)[1+d(z,z)]}{1+d(z,z)}$$

that is $|d(z,Sz)| \leq \gamma(z,z)|d(z,Sz)|$, which is a contradiction.

Thus, we get $Sz = z$. Similarly, we get $Tz = z$. Therefore $z = Sz = Tz$, that is, z is a common fixed point of S and T.

Finally, we show that z is the unique common fixed point of S and T. Assume that there exists another point ω such that $\omega = S\omega = T\omega$. From (2.3), we have

$$\begin{aligned}
d(z,\omega) &= d(Sz,T\omega) \\
&\precsim \alpha(z,\omega)d(z,\omega) + \beta(z,\omega)\frac{d(\omega,T\omega)[1+d(z,Sz)]}{1+d(z,\omega)} + \gamma(z,\omega)\frac{d(\omega,Sz)[1+d(z,T\omega)]}{1+d(z,\omega)} \\
&= \alpha(z,\omega)d(z,\omega) + \gamma(z,\omega)\frac{d(\omega,Sz)[1+d(z,T\omega)]}{1+d(z,\omega)} \\
&\precsim \left[\alpha(z,\omega) + \gamma(z,\omega)\right]d(z,\omega),
\end{aligned}$$

that is

$$|d(z,\omega)| \leq [\alpha(z,\omega) + \gamma(z,\omega)]|d(z,\omega)|,$$

which implies that $\alpha(z,\omega) + \gamma(z,\omega) \geq 1$, which is contradiction and hence $z = \omega$. Therefore, z is a unique common fixed point of S and T. □

Corollary 2.2. *Let S and T be self mappings of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d(Sx,Ty) \precsim ad(x,y) + b\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)} + c\frac{d(y,Sx)[1+d(x,Ty)]}{1+d(x,y)} \tag{2.14}$$

for all x,y in X, where a,b,c are non-negative reals with $a+b+c<1$. Then S and T have a unique common fixed point.

Proof. By putting $\alpha(x,y)=a$, $\beta(x,y)=b$, $\gamma(x,y)=c$ in Theorem 2.1, we get the required result. □

Corollary 2.3. *Let T be self map of a C-complete complex valued metric space (X,d). If there exists mappings $\alpha,\beta,\gamma:\mathbb{C}_+\times\mathbb{C}_+\to[0,1)$ satisfying (2.1), (2.2) and the following:*

$$d(Tx,Ty) \precsim \alpha(x,y)d(x,y) + \beta(x,y)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \gamma(x,y)\frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)} \tag{2.15}$$

for all x,y in X. Then T has a unique fixed point in X.

Proof. By putting $S=T$ in Theorem 2.1, we get the required result. □

Corollary 2.4. *Let T be self mapping of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d(Tx,Ty) \precsim ad(x,y) + b\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + c\frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)} \tag{2.16}$$

for all x,y in X, where a,b,c are non-negative reals with $a+b+c<1$. Then T has a unique fixed point in X.

Proof. By putting $\alpha(x,y)=a$, $\beta(x,y)=b$, $\gamma(x,y)=c$ in Corollary 2.3, we get the required result. □

Theorem 2.5. *Let T be self map of a C-complete complex valued metric space (X,d). If there exists mappings $\alpha,\beta,\gamma:\mathbb{C}_+\times\mathbb{C}_+\to[0,1)$ satisfying (2.1), (2.2) and the following:*

$$\begin{aligned} d\left(T^n x, T^n y\right) \precsim\ & \alpha(x,y)d(x,y) + \beta(x,y)\frac{d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)} \\ & + \gamma(x,y)\frac{d(y,T^n x)[1+d(x,T^n y)]}{1+d(x,y)} \end{aligned} \tag{2.17}$$

for all x,y in X some $n\in\mathbb{N}$. Then T has a unique fixed point in X.

Proof. From Corollary 2.3, T^n has a fixed point z. But T^n has a fixed point Tz, since $T^n(Tz)=T(T^n z)=Tz$. Therefore $Tz=z$ by the uniqueness of a fixed point T^n. Therefore, z is also a fixed point of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is also unique. □

Corollary 2.6. *Let T be self mapping of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d\left(T^n x, T^n y\right) \precsim a d(x,y) + b\frac{d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)} + c\frac{d(y,T^n x)[1+d(x,T^n y)]}{1+d(x,y)} \tag{2.18}$$

where a,b,c are non-negative reals with $a+b+c<1$. Then T has a unique fixed point in X.

Proof. By putting $\alpha(x,y)=a$, $\beta(x,y)=b$, $\gamma(x,y)=c$ in Theorem 2.5, we get the required result. □

Theorem 2.7. *Let S and T be self mappings of a C-complete complex valued metric space (X,d). If there exists mapping $\alpha,\beta:\mathbb{C}_+\to[0,1)$ such that for all x,y in X:*

(i) $\alpha(x)+\beta(x)<1,$ (2.19)

(ii) *the mapping $\mu:\mathbb{C}_+\to[0,1)$ defined by $\mu(x)=\frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ,* (2.20)

(iii) $d(Sx,Ty)\precsim\alpha(d(x,y))d(x,y)+\beta(d(x,y))\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)}.$ (2.21)

Then S and T have a unique common fixed point in X.

Proof. Define $\alpha,\beta,\gamma:\mathbb{C}_+\times\mathbb{C}_+\to[0,1)$ by

$$\alpha(x,y)=\alpha(d(x,y)),\quad \beta(x,y)=\beta(d(x,y)),\quad \gamma(x,y)=0 \text{ for all } x,y \text{ in } X.$$

Now using Theorem 2.1, we get the required result. □

Corollary 2.8. *Let S and T be self mappings of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d(Sx,Ty)\precsim a d(x,y)+b\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)}, \tag{2.22}$$

for all x,y in X, where a,b are non-negative reals with $a+b<1$. Then S and T have a unique common fixed point.

Proof. By putting $\alpha(x)=a$, $\beta(x)=b$ in Theorem 2.7, we get the required result. □

Corollary 2.9. *Let T be a self map of a C-complete complex valued metric space (X,d). If there exists mappings $\alpha,\beta:\mathbb{C}_+\to[0,1)$ satisfying* (2.19)*,* (2.20) *and the following:*

$$d(Tx,Ty)\precsim\alpha(d(x,y))d(x,y)+\beta(d(x,y))\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \tag{2.23}$$

for all x,y in X. Then T has a unique fixed point in X.

Proof. By putting $S=T$ in Theorem 2.7, we get the required result. □

Corollary 2.10. *Let T be self mapping of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d(Tx,Ty)\precsim a d(x,y)+b\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \tag{2.24}$$

for all x,y in X, where a,b are non-negative reals with $a+b<1$. Then T has a unique fixed point in X.

Proof. By putting $\alpha(x)=a$, $\beta(x)=b$ in Corollary 2.9, we get the required result. □

Theorem 2.11. *Let T be self map of a C-complete complex valued metric space (X,d). If there exists mapping $\alpha,\beta:\mathbb{C}_+\to[0,1)$ satisfying (2.19),(2.20) and the following:*

$$d\left(T^n x,T^n y\right)\precsim \alpha(d(x,y))d(x,y)+\beta(d(x,y))\frac{d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)}, \tag{2.25}$$

for all x,y in X and some $n\in\mathbb{N}$. Then T has a unique fixed point in X.

Proof. From Corollary 2.9, T^n has a fixed point Z. Since $T^n(Tz)=T(T^n z)=Tz$, we get Tz is a fixed point of T^n. Therefore, $Tz=Z$ by the uniqueness of a fixed point T^n. Therefore, z is also a fixed point of T. Since the fixed point of T is also a fixed point of T^n, we get that fixed point of T is also unique. □

Corollary 2.12. *Let T be self mapping of a C-complete complex valued metric space (X,d) satisfying the following:*

$$d\left(T^n x,T^n y\right)\precsim ad(x,y)+b\frac{d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)}, \tag{2.26}$$

for all x,y in X and some $n\in\mathbb{N}$, where a,b are non-negative reals with $a+b<1$. Then T has a unique fixed point in X.

Proof. By putting $\alpha(x)=a$, $\beta(x)=b$ in Theorem 2.11, we get the required result. □

3. Conclusion

The aim of this paper is to investigate common fixed point theorems for a pair of mappings satisfying rational inequality in the framework of C-complex valued metric spaces. The future scope of our results, to obtain the existence and uniqueness of a common solution of the system of Urysohn integral equations. The integral equation plays very significant and important role in mathematical analysis and has various applications in real world problems.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.* **32**(3) (2011), 243 – 253.

[2] A.K. Dubey, M. Tripathi and R.P. Dubey, Various fixed point theorems in complex valued b-metric spaces, *International Journal of Engineering Mathematics* **2016** (2016), Article ID 7072606, 7 pages, DOI: 10.1155/2016/7072606.

[3] F. Rouzkard and M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Computer and Mathematics with Applications* **64** (2012), 1866 – 1874, DOI: 10.1016/j.camwa.2012.02.063.

[4] K. Sitthikul and S. Saejung, Some fixed point theorems in complex valued metric spaces, *Fixed Point Theory and Applications* **2012** (2012), 189, DOI: 10.1186/1687-1812-2012-189.

[5] M. Kumar, P. Kumar, S. Kumar and S. Araci, Weakly compatible maps in complex valued metric spaces and an application to solve Urysohn integral equation, *Filomat* **30**(10) (2016), 2695 – 2709, DOI: 10.2298/FIL1610695K.

[6] W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *Journal of Inequalities and Applications* **2012**(1) (2012), 84, DOI: 10.1186/1029-242X-2012-84.

[7] W. Sintunavart, Y.J. Cho and P. Kumam, Urysohn integral equation approach by common fixed points in complex valued metric spaces, *Advances in Difference Equations* **2013** (2013), 49, DOI: 10.1186/1687-1847-2013-49.