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Research Article

Delay Dependent Robust Stability of A Discrete Time Recurrent Neural Network with Time Varying Delays

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Abstract. In this paper, the robust stability analysis of a problem is investigated for a class of discrete recurrent neural networks with distributed time varying delays for delay dependent case. The problem is to determine the robust stability by employing Lyapunov–Krasovskii stability theory. The class of neural network under some consideration is globally asymptotically stable if the quadratic matrix inequality involving several parameters is less than zero. Furthermore, a *Linear Matrix Inequality* (LMI) approach is provided to show the stability analysis. The numerical examples are given to show the usefulness of the proposed robust stability conditions. The numerical simulation is proved using MATLAB.

Keywords. Delay dependent; Recurrent neural network; Lyapunov–Krasovskii; Linear matrix inequality; Robust stability

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1. Introduction

Recurrent neural networks have been extensively studied in the past decades. They have been successfully applied to signal processing, pattern recognition, associative memories, combinatorial optimization, and other engineering and scientific areas [5]. In these applications, stability and convergence of neural networks are very important. Therefore, the stability analysis of recurrent neural networks has received much attention and many results on this topic have been reported in literature [1–14]. A system is said to have a delay, when the rate of variation in the system state depends on past states. Such a system is called a time-delay system. Delays appear frequently in real-world engineering systems. They are often a source of instability and poor performance, and greatly increase the difficulty of stability analysis and control design. So, many researchers in the field of control theory and engineering study the robust control of time-delay systems. The study of such systems has been very active for the last 20 years; and new developments, such as fixed model transformations based on the Newton Leibnitz formula and parameterized model transformations, are continually appearing. Although these methods are a great improvement over previous ones, they still have their limitations [7].

It should be noted, the existing stability criteria for RNNs with time delays can be classified into the delay-independent and the delay-dependent criteria. In general, when the time delay is small, the delay-dependent stability criteria are less conservative than delay-independent one [1, 5–8]. For the delay-dependent stability criteria, the maximum delay bound is a very important index for checking the criterion's conservatism. From the Lyapunov stability theory, there are two effective ways to reduce the conservatism within in stability analysis of networks and systems. One is the choice of suitable *Lyapunov-Krasovskii Functional* (LKF) and the other one is the estimation of its time derivative [9].

There are many cases in recurrent neural network have been assumed to performance in a continuous time manner. For the sake of computer based simulation, experimentation or computation, it is useful to discretize the given continuous time neural networks.

In this paper we discussed about the delay dependent robust stability of a discrete time recurrent neural network with distributed time varying delays. Some new lyapunov krosovki functions are assumed to analysis the stability of the given systems.

This paper organized as follows, in Section 2, we have discussed some suitable preliminaries for the main results, in Section 3, we have formed the discrete time neural networks with distributed time varying delays, in Section 4, we deals with stability analysis of the class of neural network. Section 5, derives some numerical results and concluding remarks.

2. Preliminaries

Before enter into the main result we need the following assumption and definition.

Assumption 1. The quantities a_i and b_{ij} can be initiated as follows:

$$A_I = \{A = \mathrm{diag}(a_i) : 0 < \underline{A} \le A \le \overline{A}\}$$

$$B_I = \{B = \mathrm{diag}(b_i) : 0 < \underline{B} \le B \le \overline{B}\}.$$

Denote $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

It is clear that A^* is a nonnegative matrix, and the interval matrix $[\overline{A}, \underline{A}]$ can be written equivalently as $[A^* - A_*, A^* + A_*]$; therefore, we have $A = A^* + \Delta A$ with $\Delta A \in [-A_*, A_*]$ [10].

Definition 1. The neural network defined by (1) with parameter ranges defined by assumption (1) is globally asymptotically robust stable if unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of the neural system (1) is globally asymptotically stable for all $A \in A_I$, $B \in B_I$.

Lemma 1 ([10]). *$S(x) : R^n \to R^n$ is a homeomorphism $S(x)$ satisfies the following conditions:*

(1) *$S(x)$ is infective, that is, $H(x) \neq H(y)$ for all $x \neq y$,*

(2) *$S(x)$ is proper, that is, $\|S(x)\| \to \infty$ as $\|x\| \to \infty$.*

3. Mathematical Formulation

Consider the discrete-time neural networks with distributed time varying delays

$$y(k+1) = Ay(k) + Bf(y(k - d(k))) + J, \tag{1}$$
$$y(k) = \varphi(k), \quad k = -d_M, -d_M + 1, \dots, 0,$$

where $y(k) = [y_1(k), \dots, y_n(k)]^T \in \mathbb{R}^n$ denotes the state neuron,

$$g(y(k)) = [g_1(y_1(k - d(k)), \dots, g_n(y_n(k - d(k))]^T \in \mathbb{R}^n,$$

denotes delayed activation function, $\varphi(k)$ is the given initial condition sequence of $y(k)$, $J = [J_1, \dots, J_n] \in \mathbb{R}^n$ denotes the external input vector, $A = \mathrm{diag}\{a_1, \dots, a_n\} \in \mathbb{R}^{n\times n}$ with $|a_i| < 1$ describes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs, $B = B_{ij} \in \mathbb{R}^{n\times n}$ is the time delay connection weight matrix. The delay $d(k)$ represents the time varying delay satisfying $d_m \le d(k) \le d_M$, where d_m and d_M are prescribed positive integers representing the lower and upper bounds of the time varying delay, respectively.

The following assumptions are very important, which are useful to our main results.

Assumption 2. For any $x, y \in \mathbb{R}, x \neq y$, each activation function $f_i(\cdot)$ described in (1) satisfy

$$k_i^- \le \frac{f_i(x) - f_i(y)}{x - y} \le k_i^+, \quad i = 1, 2, \dots, n \tag{2}$$

where k_i^- and k_i^+ are known constant values. The activation function condition mentioned in Assumption 1 is widely used in many literatures [1–14].

In stability analysis of the neural networks (1), the equilibrium point $y^* \in \mathbb{R}^n$ whose uniqueness under Assumption 1 can be assured by using Brouwer's fixed point theorem. This states that the equilibrium point shifted to the origin by using $x_i(k) = y_i(k) - y^*$ and $f_i(x_i(k)) = g_i(y_i(k) + y_i^* - g_i(y_i^*)$ with$f_i(0) = 0$. Using this condition the equation (1) becomes

$$x(k+1) = Ax(k) + Bf(x(k - d(k))),$$

$$x(k) = \varnothing(k), \quad k = -d_M, -d_M + 1, \dots, 0, \tag{3}$$

where $x(k) = [x_1(k), \dots, x_n(k)]^T \in \mathbb{R}^n$ denotes the state neuron,

$$f(x(k)) = [f_1(x_1(k - d(k)), \dots, f_n(x_n(k - d(k))]^T \in \mathbb{R}^n,$$

denotes delayed activation function, $\varnothing(k)$ is the given initial condition sequence of $x(k)$. From (2), we have

$$k_i^- \le \frac{f_i(x)}{x} \le k_i^+, \quad i = 1, 2, \dots, n.$$

4. Main Results

Before proofing the theorem we have to mention the following assumptions:

$$\varnothing_1(k) = \begin{cases} x(k - d(k)) - x(k - d_M) - \sum_{i=k-d_M}^{k-d(k)-1} \eta(i) = 0, & \text{when } d(k) \neq d_M, \\ x(k - d(k)) - x(k - d_M) = 0, & \text{when } d(k) = d_M. \end{cases}$$

Robust Stability Analysis

Theorem 1. *Let us assume that the Assumption* 2 *to be true, then the neural network model* (1) *is globally asymptotically robust stable, if*

$$\gamma = 2A_m - \vartheta_M(\|\widehat{B}\|_1 + \|\widehat{B}\|_\infty) > 0,$$

where $A_m = \min(\underline{c}_m)$, $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $\widehat{B} = (b_{ij}^*)_{n \times n}, b_{ij}^* = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$ *and* $\vartheta_M = \max(\vartheta_i)$.

Proof. We first prove the existence and uniqueness of the equilibrium point. To this end, define the following mapping associated with (1):

$$S(x) = Ax + Bf(x) + J. \tag{4}$$

Let x^* be an equilibrium point of (1). Therefore, it follows from assumption that, for the system defined by (1), there exists a unique equilibrium point for every input vector J, if $S(x)$ is a homeomorphism of R^n. We will now prove that $S(x)$ is a homeomorphism of R^n. To this end, we choose two vectors $x, y \in \mathbb{R}^n$ such that $x \neq y$. For $S(x)$ defined by (7), we can write

$$S(x) - S(y) = A(x - y) + B(f(x) - f(y)).$$

If we multiply both sides of equation (above) by $2(x - y)^T$, then, we get

$$\begin{aligned} 2(x-y)^T(S(x) - S(y)) &= 2(x-y)^T A(x-y) + 2(x-y)^T B(f(x) - f(y)) \\ &= \sum_{i=1}^n 2c_i(x_i - y_i) + \sum_{i=1}^n \sum_{j=1}^n 2b_{ij}(x_i - x_i)(f_j(x_j) - f_j(y_j)) \\ &\le 2A_m\|x-y\|_2^2 + 2\vartheta_M(\|A^*\|_2 + \|A_*\|_2)\|x-y\|_2^2 \\ &\quad + \vartheta_M(\|\widehat{B}\|_1 + \|\widehat{B}\|_\infty)\|x-y\|_2^2. \end{aligned} \tag{5}$$

For any $x - y \neq 0$,

$$2A_m\|x-y\|_2^2 + 2\vartheta_M(\|A^*\|_2 + \|A_*\|_2)\|x-y\|_2^2 + \vartheta_M(\|\widehat{B}\|_1 + \|\widehat{B}\|_\infty)\|x-y\|_2^2 < 0.$$

Using the assumption, $(x-y)^T(S(x)-S(y))<0$, for which it can concluded that $S(x)\neq S(y)$ for all $x\neq y$.

Put $y=0$ in equation (4), we have

$$2(x)^T(S(x)-S(0))\leq(2A_m+2\vartheta_M(\|A^*\|_2+\|A_*\|_2)+\vartheta_M(\|\widehat{B}\|_1+\|\widehat{B}\|_\infty))\|x\|_2^2. \tag{6}$$

From this, we have

$$\|x\|_\infty\|S(x)-S(0)\|_1\geq\alpha\|x\|_2^2,$$

where

$$\alpha=2A_m+2\vartheta_M(\|A^*\|_2+\|A_*\|_2)+\vartheta_M(\|\widehat{B}\|_1+\|\widehat{B}\|_\infty).$$

Since $\|S(0)\|_1$ is finite, we conclude that $\|S(x)\|\to\infty$ as $\|(x)\|\to\infty$. Hence by the given statement, the system (1) has a unique equilibrium point.

Theorem 2. *Given positive integers $d_m>0$ and $d_M>0$. Then discrete time delay system in* (1) *is asymptotically stable for any time delay $d(k)$ satisfying $d_m\leq d(k)\leq d_M$, if there exist a matrix $P=P^T>0$, $Q=Q^T>0$, $R=R^T>0$ and any appropriately dimensioned matrices Y and T, such that the following LMI holds:*

$$\varphi=\begin{bmatrix}\varphi_{11} & \varphi_{12} & dA^TR\\ * & \varphi_{22} & dB^TR\\ * & * & -dR\end{bmatrix}<0, \tag{7}$$

$$\psi=\begin{bmatrix}X_{11} & X_{12} & Y\\ * & X_{22} & T\\ * & * & Z\end{bmatrix}\geq 0, \tag{8}$$

where

$$\varphi_{11}=A^TPA-A+Y+Y^T+Q+d_MX_{11},$$

$$\varphi_{12}=PB+PB^T+T^T-Y+d_MX_{12},$$

$$\varphi_{22}=-T-T^T-(d_M-d_m)Q+d_MX_{22}.$$

Proof. Consider the following hypothesis for any appropriately dimensioned matrices Y, T and for any semi positive definite matrix $X=\begin{bmatrix}X_{11} & X_{12}\\ * & X_{22}\end{bmatrix}\geq 0$, the following condition holds:

$$d_M\delta^T(k)X\delta(k)-\sum_{i=k-d(k)}^{k-1}\delta^T(k)X\delta(k)\geq 0, \tag{9}$$

where $\delta^T(k)=[x^T(k)\quad x^T(k-d(k))]$.

Consider the Lyapunov function for equation (1),

$$V(k)=\sum_{i=1}^{3}V_i(k), \tag{10}$$

where

$$V_1(k) = x(k)^T P x(k),$$

$$V_2(k) = \sum_{i=k-d(k)}^{k-1} X^T(i) Q x(i),$$

$$V_3(k) = \sum_{j=-d_M}^{-d_m-1} \sum_{i=k+j}^{k-1} \eta(i)^T R \eta\ (i),$$

where $\eta(k) = x(k+1) - x(k)$, the matrices $P > 0$, $Q > 0$ and $Z > 0$.

Now take the derivative of equation (6).

Define $\Delta V(K) = V(k+1) - V(k)$,

$$\begin{aligned}
\Delta V_1(k) &= V_1(K+1) - V_1(k) \\
&= x^T(k)\left[A^T P A - P\right] x(k) + 2A^T x^T(k) P B f(x(k-d(k))) \\
&\quad + B^T f(x(k-d(k)))^T P B f(x(k-d(k))), \\
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&\le \sum_{i=k+1-d^1}^{k-d} x^T(i) Q x(i) + x^T(k) Q x(k) - x^T(k-d(k)) Q x(k-d(k)), \\
\Delta V_3(k) &= V_3(k+1) - V_3(k) \\
&= (d_M - d_m)\eta^T(k) R \eta(k) - \sum_{i=k-d_M}^{k-d_m-1} \eta^T(i) R \eta(i) \\
&= x^T(k)[(d_M - d_m)(A-I) R (A-I)] x(k) \\
&\quad + 2x^T(k)[(d_M - d_m)(A-I) R B] f(x(k-d(k)) \\
&\quad + f(x(k-d(k))^T \left[(d_M - d_m) B^T R B\right] f(x(k-d(k)) - \sum_{i=k-d_M}^{k-d(k)-1} \eta^T(i) R \eta(i).
\end{aligned}$$

Now, adding $\Delta V_1(k)$, $\Delta V_2(k)$, $\Delta V_3(k)$, we have

$$\begin{aligned}
\Delta V(k) &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) \\
&= x^T(k)\left[A^T P A - P\right] x(k) + 2A^T x^T(k) P B f(x(k-d(k))) \\
&\quad + B^T f(x(k-d(k)))^T P B f(x(k-d(k))) \\
&\quad + \sum_{i=k+1-d^1}^{k-d} x^T(i) Q x(i) + x^T(k) Q x(k) - x^T(k-d(k)) Q x(k-d(k)) \\
&\quad + x^T(k)[(d_M - d_m)(A-I) R (A-I)] x(k) \\
&\quad + 2x^T(k)[(d_M - d_m)(A-I) R B] f(x(k-d(k)) \\
&\quad + f(x(k-d(k))^T \left[(d_M - d_m) B^T R B\right] f(x(k-d(k)) \\
&\quad - \sum_{i=k-d_M}^{k-d(k)-1} \eta^T(i) R \eta(i) + 2\left[x^T(k) Y + f(x(k-d(k))^T(k)\right] \Psi_1(k)
\end{aligned}$$

$$+d_M\delta^T(k)X\delta(k)-\sum_{i=k-d(k)}^{k-1}\delta^T(i)X\delta(i)$$
$$=\delta^T(k)E\delta(k)-\sum_{i=k-d(k)}^{k-1}\delta^T(i)\psi\delta(i),$$

where

$$\delta(k)=[x^T(k)\quad f(x(k-d(k))^T\quad \delta^T(i)]^T,$$
$$E=\begin{bmatrix}\varphi_{11}+d_MA^TRA & \varphi_{12}+d_MA^TRB\\ * & \varphi_{22}+d_MB^TRB\end{bmatrix}.$$

If $E<0$ and $\psi\geq 0$. Then, we have

$$\Delta V(K)<0$$

for any $\delta(k)\neq 0$. Applying the Shur Complement, we have $E<0$. So equation (1) is asymptotically stable if the *Linear Matrix Inequalities* (LMI) given in equation (7) and (8) are true. This completes the proof. □

From Theorem 1 and Theorem 2, we conclude that the given neural system (1) is globally asymptotically robust stable.

Remark 1. If the matrices Y, T and X in the equation (4) are set to zero and $R=\epsilon I$ (ϵ is sufficiently small positive scalar), then Theorem 1 is identical to the well known delay independent stability criterion which are stated in some literature [1–10].

Numerical Examples

Example 1. Assume that the network parameter of neural system (1) is given as follows:

$$\underline{A}=\begin{bmatrix}0&0\\-8&0\end{bmatrix},\quad \overline{A}=\begin{bmatrix}6&8\\0&6\end{bmatrix},$$
$$\vartheta_1=\vartheta_2=\vartheta_M=1,\ \underline{c}_1=\overline{c}_2=c_m,$$
$$\underline{B}=\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix},\quad \overline{B}=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

The matrices A^*, A_*, $\widehat{B}$ are obtained as follows:

$$A^*=\begin{bmatrix}3&4\\-4&3\end{bmatrix},\quad A_*=\begin{bmatrix}3&4\\4&3\end{bmatrix},\quad \widehat{B}=\begin{bmatrix}1&1\\1&1\end{bmatrix},$$

where $\|A^*\|_2=5$, $\|A_*\|_2=7$, $\|\widehat{B}\|_1=\|\widehat{B}\|_\infty=2$. Substituting the values in to equation (6), this leads to $c_m=14$. This shows that the system given in equation (1) is globally robust stability [10].

Example 2. Assume that the network parameter of neural system (1) is given as follows:

$$\underline{A}=\begin{bmatrix}0&0\\-6&0\end{bmatrix},\quad \overline{A}=\begin{bmatrix}4&6\\0&4\end{bmatrix},$$
$$\vartheta_1=\vartheta_2=\vartheta_M=1,\ \underline{c}_1=\overline{c}_2=c_m,$$

$$\underline{B} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrices A^*, A_*, $\widehat{B}$ are obtained as follows:

$$A^* = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}, \quad A_* = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

where $\|A^*\|_2 = 5$, $\|A_*\|_2 = 5$, $\|\widehat{B}\|_1 = \|\widehat{B}\|_\infty = 2$. Substituting the values in to equation (6), this leads to $c_m = 12$. This shows that the system given in equation (1) is globally robust stability.

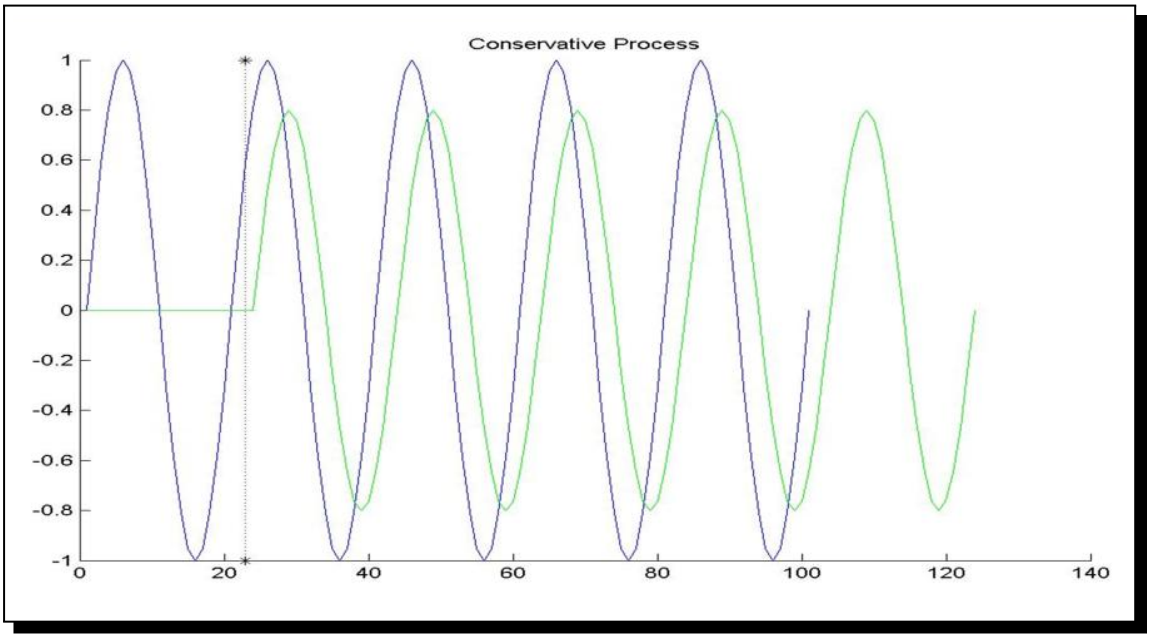

Figure 1. Conservative process of the system

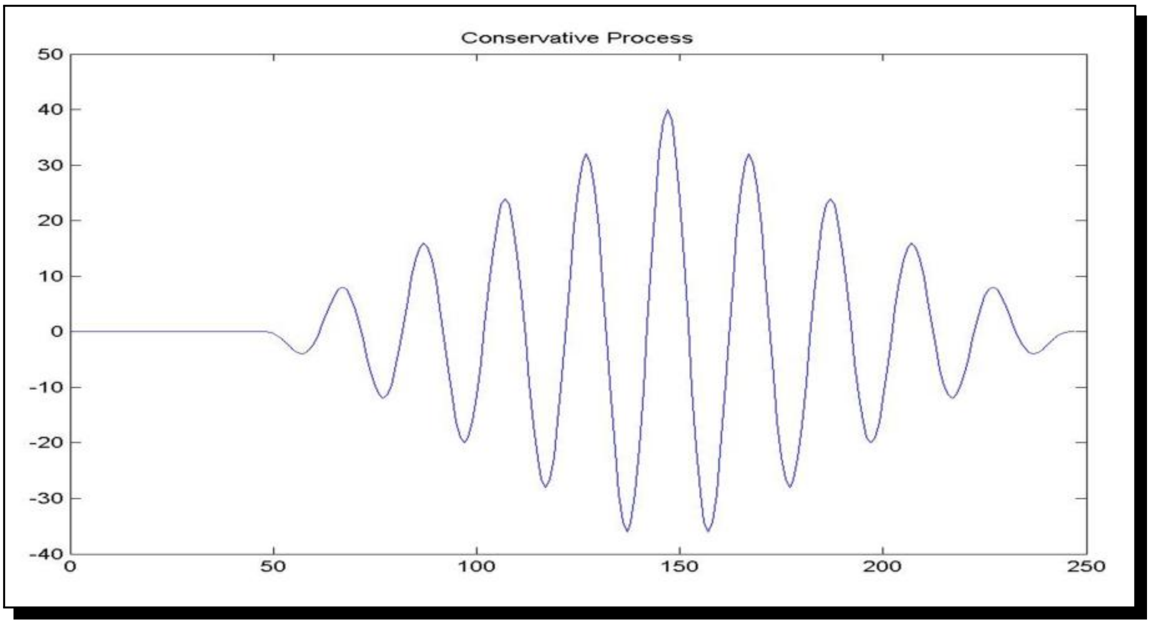

Figure 2. Stability analysis of the system

Figure 1 Represents the conservative process of system(1). After applying theorem (2) conditions, the system representation given in Figure 2. The conservative process of Figure 2 shows the stability analysis of system (1).

5. Conclusion

In this paper, we have studied the global robust stability problem for a class of discrete time recurrent neural networks with time delays. By using Laypunov krosvskii and some well known inequalities, we have established a stability analysis. And also we have proved the existence and uniqueness and global asymptotically robust stable for discrete case. The obtained condition can be easily verified in terms of neural parameters only. The numerical simulation is provided through MATLAB.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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