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On Order of Banach Valued Analytic Functions

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Abstract. We consider Banach valued analytic functions and show that the order of a Banach valued analytic function is the same as that of its derivative.

1. Introduction

If f is defined in a domain $\Omega \subset \mathbb{C}$ taking values in a Banach space X, then it is analytic if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in \Omega$.

The mapping

$$u : z \to u(z) = \log^+ \|f(z)\|$$

is upper semi-continuous if for all $z_0 \in \Omega$

$$\lim_{z \to z_0} \sup u(z) \leq u(z_0).$$

Let Ω be a domain of $\mathbb{C}$. A function u from Ω to $\mathbb{R} \cup \{\infty\}$ is said to be subharmonic on Ω if it is upper semi-continuous and satisfies the mean inequality

$$u(z_0) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) d\theta$$

whenever the closed disc $\overline{B}(z_0, r)$ is contained in Ω.

It is harmonic if both u and $-u$ are subharmonic.

We now state the following results [2, p. 38] for subharmonic functions.

Theorem A. *Suppose f is analytic from a domain Ω to a Banach space X. Then the functions $\|f\|$ and $\log \|f\|$ are subharmonic in Ω.*

Following theorem shows that subharmonic functions satisfy a maximum principle.

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Theorem B. *If u is subharmonic on a domain Ω and for some $z_0 \in \Omega$ holds $u(z_0) \geq u(z)$ for all $z \in \Omega$, then u is constant in Ω.*

In the present note we first prove maximum principle for Banach valued analytic functions. Then we study the order for Banach valued entire functions in front of the usual operations of sum and product and establish its invariance under derivation. Throughout this note we shall assume f, f_1, f_2, etc., are non-constant Banach valued analytic functions from a domain $\Omega \subset \mathbb{C}$ to a Banach space X, unless otherwise stated.

2. Maximum Principle for Banach Valued Analytic Functions

Theorem 1. *Suppose f is analytic from a domain Ω bounded by a closed curve C to a Banach space X. If $\sup\limits_{z\in\Omega\cup C} \|f(z)\| = M$, then either $\|f(z)\| \equiv M$ or $\|f(z)\| < M$ in Ω.*

Proof. Since f is analytic, so by Theorem A, $\|f\|$ is subhormonic in Ω and $\|f\|$ being continuous in $\Omega \cup C$, its assumes its maximum M in $\Omega \cup C$. Now if f be non-constant then by Theorem B for any $z_0 \in \Omega$ $\exists$ $z_1 \in \Omega$ such that

$$\|f(z_0)\| < \|f(z_1)\|.$$

This implies that $\|f(z)\| < M$ in Ω, unless $f(z)$ is constant. Otherwise $\|f(z)\| \equiv M$.

This completes the proof. $\square$

For an analytic Banach valued function f one can define analogously

$$M_\infty(r, f) = \sup_{|z|\leq r} \|f(z)\|.$$

Then clearly $M_\infty(r, f)$ is an increasing function of r.

We now introduce the following definition.

Definition 1. Let f be an entire Banach valued function.Then the order of f is denoted by $\rho_\infty(f)$ and is defined by

$$\begin{aligned}\rho_\infty(f) &= \limsup_{r\to\infty} \frac{\log\log M_\infty(r, f)}{\log r} \\ &= \inf\{\mu > 0 : \log M_\infty(r, f) < r^\mu, \text{ for all } r > r_0(\mu) > 0\}.\end{aligned}$$

3. Sum and Product Theorems

Theorem 2. *Let f_1 and f_2 be two entire Banach valued functions having respective orders $\rho_\infty(f_1)$ and $\rho_\infty(f_2)$. Then*

$$\rho_\infty(f_1 \pm f_2) \leq \max\{\rho_\infty(f_1), \rho_\infty(f_2)\}.$$

Proof. We may suppose that $\rho_\infty(f_1)$ and $\rho_\infty(f_2)$ both are finite. Because if one of $\rho_\infty(f_1), \rho_\infty(f_2)$ or both are infinite, the inequality is evident. Let $\rho_\infty(f_1) \le \rho_\infty(f_2)$. Then for arbitrary $\epsilon > 0$ and for all large r we have

$$M_\infty(r, f_1) < e^{r^{\rho_\infty(f_1)+\epsilon}} \quad \text{and} \quad M_\infty(r, f_2) < e^{r^{\rho_\infty(f_2)+\epsilon}}.$$

So for all large r,

$$\begin{aligned} M_\infty(r, f_1 \pm f_2) &= \sup_{|z|\le r} \|f_1(z) \pm f_2(z)\| \\ &\le e^{r^{\rho_\infty(f_1)+\epsilon}} + e^{r^{\rho_\infty(f_2)+\epsilon}} \\ &\le e^{r^{\rho_\infty(f_2)+2\epsilon}}. \end{aligned}$$

Therefore, $\rho_\infty(f_1 \pm f_2) \le \rho_\infty(f_2) + 2\epsilon$.

Since $\epsilon > 0$ was arbitrary, we have

$$\rho_\infty(f_1 \pm f_2) \le \rho_\infty(f_2) = \max\{\rho_\infty(f_1), \rho_\infty(f_2)\}.$$ □

Theorem 3. *Let f_1 be a Banach valued entire function and f_2 be an entire function having respective orders $\rho_\infty(f_1)$ and $\rho(f_2)$. Then*

$$\rho_\infty(f_1 f_2) \le \max\{\rho_\infty(f_1), \rho(f_2)\},$$

where $\rho(f_2) = \inf\{\mu > 0 : \log M(r, f) < r^\mu$, *for all* $r > r_0(\mu) > 0\}$ *and* $M(r, f) = \sup_{|z|\le r} |f(z)|$.

Proof. Suppose $\rho_\infty(f_1) \le \rho(f_2)$. Now for arbitrary $\epsilon > 0$ and for all large r,

$$\log M_\infty(r, f_1) < r^{\rho_\infty(f_1)+\epsilon} \quad \text{and} \quad \log M(r, f_2) < r^{\rho(f_2)+\epsilon}.$$

So for all large r,

$$\begin{aligned} M_\infty(r, f_1 f_2) &= \sup_{|z|\le r} \|f_1(z).f_2(z)\| \\ &= \sup\{||f_1(z)|| \cdot |f_2(z)|\} \\ &\le e^{r^{\rho_\infty(f_1)+\epsilon}} \cdot e^{r^{\rho(f_2)+\epsilon}} \\ &\le e^{r^{\rho(f_2)+2\epsilon}}. \end{aligned}$$

Therefore, $\rho_\infty(f_1.f_2) \le \rho(f_2) + 2\epsilon$.

Since $\epsilon > 0$ was arbitrary, we have

$$\rho_\infty(f_1 \cdot f_2) \le \rho(f_2) = \max\{\rho_\infty(f_1), \rho(f_2)\}.$$ □

4. Order of the Derivative

Theorem 4. *If f is a Banach valued entire function then*

$$\rho_\infty(f') = \rho_\infty(f).$$

Proof. If $f(z)$ be a Banach valued polynomial of degree m, then for $\epsilon > 0$ and for $r > r_0(\epsilon)$, $M_\infty(r,f) \leq \|a_m\| r^m(1+\epsilon)$, a_m is the coefficient of the highest degree term. So in this case

$$\rho_\infty(f) = 0 = \rho_\infty(f').$$

Next we suppose that f be a Banach valued entire function other than a polynomial. As in classical case we can easily prove that its maximum modulus increases faster than the maximum modulus of any polynomials. Let z be such that $|z| = r(< R)$ and Γ be the circle $|w - z| = R - r$. Then we have by Cauchy's integral formula [1, p. 104]

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^2}dw \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+(R-r)e^{i\theta})}{\{(R-r)e^{i\theta}\}^2} i(R-r)e^{i\theta}d\theta \end{aligned}$$

So,

$$\begin{aligned} \|f'(z)\| &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\|f(z+(R-r)e^{i\theta})\|}{R-r}d\theta \\ &\leq \frac{M_\infty(R,f)}{R-r}. \end{aligned}$$

Therefore

$$M_\infty(r,f') \leq \frac{M_\infty(R,f)}{R-r}. \tag{1}$$

We may write $f(z) = \int_0^z f'(t)dt + f(0)$, where the line of integration is the segment from $z=0$ to $z = re^{i\theta}$, $r > 0$. Let $z_1 = re^{i\theta_1}$ be such that $\|f(z_1)\| = M_\infty(r,f)$. Then

$$\begin{aligned} M_\infty(r,f) &= \|f(z_1)\| \\ &= \left\| \int_0^{z_1} f'(t)dt + f(0) \right\| \\ &\leq rM_\infty(r,f') + \|f(0)\| \\ &\leq 2rM_\infty(r,f'). \end{aligned} \tag{2}$$

Combining (1) and (2) by putting $R = 2r$, we get

$$\begin{aligned} M_\infty(r,f') &\leq \frac{M_\infty(2r,f)}{r} \\ &\leq 4M_\infty(2r,f'). \end{aligned}$$

So for all large r,

$$\begin{aligned} \log\log M_\infty(r,f') + O(1) &\leq \log\log M_\infty(2r,f) \\ &\leq \log\log M_\infty(2r,f') + O(1) \end{aligned}$$

i.e.,

$$\rho_\infty(f') \le \rho_\infty(f) \le \rho_\infty(f').$$

This proves the theorem. □

References

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