Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 765–773, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

A Study on Total Rebellion Number in Graphs

V. Mohanaselvi*[1], P. Shyamala Anto Mary[2] and A. Monisha[1]

[1] *Department of Mathematics, Nehru Memorial College(Autonomous), Trichy, India*
[2] *Department of Mathematics, Jayaram college of Engineering and Technology, Thuraiyur, India*
Corresponding author: avmonishasri@gmail.com

Abstract. A set $R \subseteq V$ of a graph $G = (V,E)$ is said to be a 'rebellion set' (rb-set) of G, if $|N_R(v)| \le |N_{V/R}(v)|$, for all $v \in R$, $|R| \ge |V/R|$ and $\langle R \rangle$ has no isolated vertices. The total rebellion number $trb(G)$ is the minimum cardinality of any total rebellion set in G. A total rebellion set with cardinality $trb(G)$ is denoted by $trb(G)$-set. In this paper, we defined the total rebellion number for simple graphs. Also, we determined its tight bounds for some standard graph and characterize this parameters.

Keywords. Rebellion number and Total rebellion number

MSC. 05C38; 05C69

Received: January 22, 2017 **Accepted:** March 16, 2017

1. Introduction

A graph $G = (V,E)$ is a finite undirected connected simple graph with p vertices and q edges. All the terms defined here are used in the sense of Harary [1]. The word alliance means a bound or connection between individuals, families, states or parties. The union of individuals in an alliance is thought of to be stronger than the sole individual. Suppose nations at war, the individual nations are represented as vertices and the relations between them are represented as edges. Alliance in graphs was first introduced by Kristiansen et al. in [2]. Further it was studied by Hanyes et al. in [3]. For studying the co-alliance set the authors already introduced the rebellion and independent concept by interchanging the inequalities in the alliance set in [4] and [5]. A set $R \subseteq V$ of a graph $G = (V, E)$ is said to be a 'rebellion set' (rb-set) of G, if $|N_R(v)| \leq |N_{V/R}(v)|$, for all $v \in R$ and $|R| \geq |V/R|$. The rebellion number $rb(G)$ is the minimum cardinality of any rebellion set in G. A total dominating set D of G is a dominating set such that the induced subgraph $\langle R \rangle$ has no isolated vertices. The total domination number $r_t(G)$ of G is the minimum cardinality of a total dominating set of G [6].

In this paper, we study the total property of the rebellion set and defined the total rebellion number for simple graphs. Also, we determined its tight bounds for some standard graph and characterize this parameters.

2. The Total Rebellion Number in Graphs

Definition 2.1. A rebellion set $R \subseteq V$ of a graph $G = (V, E)$ is said to be a 'total rebellion set' (trb-set) of G, if $|N_R(v)| \leq |N_{V/R}(v)|$, for all $v \in R$, $|R| \geq |V/R|$ and $\langle R \rangle$ has no isolated vertices. The total rebellion number $trb(G)$ is the minimum cardinality of any total rebellion set in G. A total rebellion set with cardinality $trb(G)$ is denoted by $trb(G)$-set.

Example 2.2.

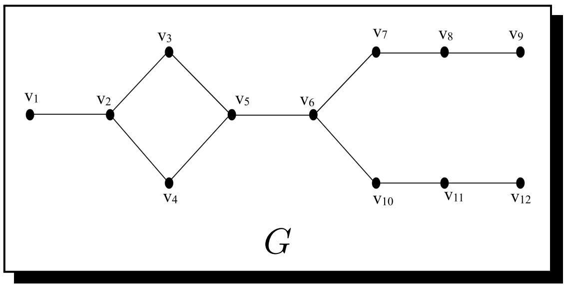

Figure 1

For the graph G in Figure 1, the set $R = \{v_2, v_3, v_6, v_7, v_{11}, v_{12}\}$ is trb-set and hence $trb(G) = 6$.

Theorem 2.3. *For the cycle C_n,*

$$trb(C_n) = \begin{cases} \lceil n/2 \rceil, & n \equiv 0,3 (\text{mod } 4) \\ \lceil n/2 \rceil + 1, & n \equiv 1,2 (\text{mod } 4), n > 5 \\ 0, & n \equiv 5. \end{cases}$$

Proof. Let G be the cycle graph C_n with atleast three vertices and R be a total rebellion set in G. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, e_3, \ldots, e_n\}$ be the vertex and edge set of G.

Case(i): $n \equiv 0,3 (\text{mod } 4)$

In this case, the set $R = \{v_{3i+2}, v_{3i+1} / i = 1,2,\ldots,\lceil \frac{n}{4} \rceil\}$ is a total rebellion set of G. Since $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) \leq |R| = 2\left\lceil \frac{n}{4} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil. \tag{2.1}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{n}{2} \rceil$ vertices.

Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n}{2} \right\rceil. \tag{2.2}$$

From (2.1) and (2.2) we have, $trb(G) = \lceil \frac{n}{2} \rceil$.

Case(ii): $n \equiv 1,2 (\text{mod } 4)$

In this case, the set $R = \{v_{3i+2}, v_{3i+1} / i = 1,2,\ldots,\lceil \frac{n}{4} \rceil\}$ is a total rebellion set of G. Since $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) \leq |R| = 2\left\lceil \frac{n}{4} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{2.3}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{n}{2} \rceil + 1$ vertices. Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{2.4}$$

From (2.3) and (2.4) we have, $trb(G) = \lceil \frac{n}{2} \rceil + 1$.

Example 2.4.

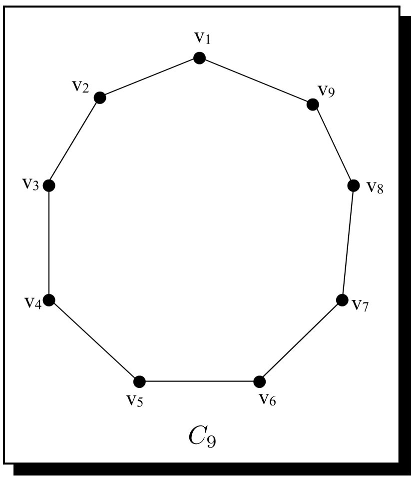

Figure 2

For the graph C_9 in Figure 2, the set $R = \{v_1, v_2, v_4, v_5, v_7, v_8\}$ is a trb-set and hence $trb(G) = 6$.

Theorem 2.5. *For the path P_n, $n \geq 3$,*

$$trb(P_n) = \begin{cases} \lceil n/2 \rceil, & n \equiv 0,3 (\text{mod } 4) \\ \lceil n/2 \rceil + 1, & n \equiv 1,2 (\text{mod } 4) \end{cases}$$

Proof. Let G be the path graph P_n with atleast three vertices and R be a total rebellion set in G. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, e_3, \ldots, e_{n-1}\}$ be the vertex and edge set of G.

Case (i): $n \equiv 0,3 (\text{mod } 4)$
In this case, the set $R = \{v_{3i+2}, v_{3i+1} / i = 1, 2, \ldots, \lceil \frac{n}{4} \rceil\}$ is a total rebellion set of G.
Since $|R| \geq |V/R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) \leq |R| = 2\left\lceil \frac{n}{4} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil. \tag{2.5}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{n}{2} \rceil$ vertices. Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n}{2} \right\rceil. \tag{2.6}$$

From (2.5) and (2.6) we have, $trb(G) = \lceil \frac{n}{2} \rceil$.

Case (ii): $n \equiv 1,2 (\text{mod } 4)$
In this case, the set $R = \{v_{3i+2}, v_{3i+1} / i = 1, 2, \ldots, \lceil \frac{n}{4} \rceil\}$ is a total rebellion set of G. Since $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) \leq |R| = 2\left\lceil \frac{n}{4} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{2.7}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives such that $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{n}{2} \rceil + 1$ vertices.
Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{2.8}$$

From (2.7) and (2.8) we have, $trb(G) = \lceil \frac{n}{2} \rceil + 1$.

Example 2.6.

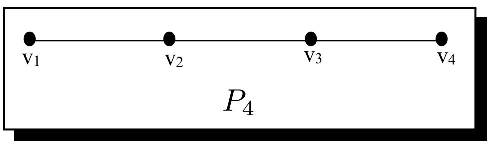

Figure 3

For the graph P_4 in Figure 3, the set $R = \{v_2, v_3\}$ is a trb-set and hence $trb(G) = 2$.

Theorem 2.7. *For the complete K_n, $trb(G) = \lceil \frac{n}{2} \rceil$, $n \geq 3$.*

Proof. Let G be the complete graph K_n with atleast three vertices and R be a total rebellion set in G. Let $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(G) = \{e_1, e_2, e_3, \dots, e_{\frac{n-1}{2}}\}$ be the vertex and edge set of G. The set $R = \{v_i/i = 1, 2, \dots, \lceil \frac{n}{2} \rceil\}$ is a total rebellion set of G, Since $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) \leq |R| = \left\lceil \frac{n}{2} \right\rceil. \tag{2.9}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives such that $|R| \geq |V/R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{n}{2} \rceil$ vertices.
Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n}{2} \right\rceil. \tag{2.10}$$

From (2.9) and (2.10), we have $trb(G) = \lceil \frac{n}{2} \rceil$. □

Example 2.8.

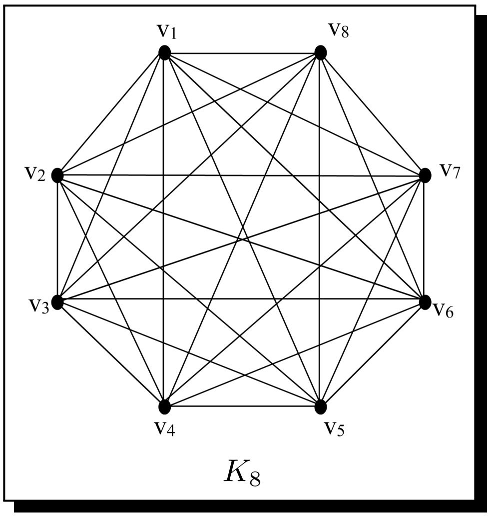

Figure 4

For the graph K_8 in Figure 4, the set $R = \{v_1, v_2, v_4, v_5\}$ is a trb-set and hence $trb(G) = 4$.

Theorem 2.9. *For the complete bipartite graph $K_{m,n}$*

(i) $trb(K_{m,n}) = \lceil \frac{n+m}{2} \rceil$, *when $m, n \geq 2$ and even*

(ii) $trb(K_{m,n}) = 0$, *when $m = n = odd$.*

Proof. Let G be the complete bipartite graph $K_{m,n}$ with atleast four vertices and R be a total rebellion set in G. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{m+n}\}$ and $E(G) = \{e_1, e_2, e_3, \dots, e_{mn}\}$ be the vertex and edge set of G.

Case (i): m and n are even

In this case, the set $R = \{u_1\} \cup \{v_i / i = 1, 2, \ldots, \frac{n}{2}\}$ is a total rebellion set of G. Since $|R| \geq |V/R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) = |R| \leq \left\lceil \frac{n}{2} + 1 \right\rceil = \left\lceil \frac{n+2}{2} \right\rceil \leq \left\lceil \frac{n+m}{2} \right\rceil \tag{2.11}$$

(because $m \geq 2$.)

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives $|R| \geq |V/R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$. Thus, R must contain atleast $\lceil \frac{p}{2} \rceil + 1$ vertices.

Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{p}{2} \right\rceil \tag{2.12}$$

(because $p = n + m$.)

From (2.11) and (2.12), we have $trb(G) = \lceil \frac{n+m}{2} \rceil$.

Case (ii): m and n are odd.

$rb_t(K_{m,n}) = 0$ when $m = n = odd$.

Therefore, the result is obviously. □

Example 2.10.

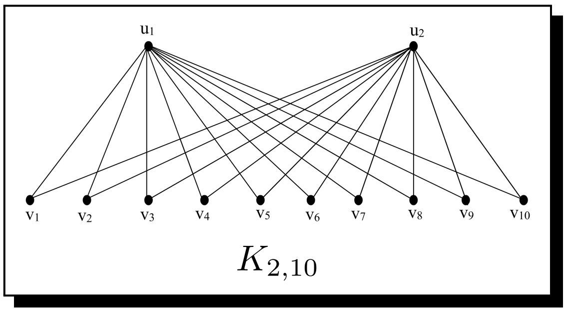

Figure 5

For the graph $K_{2,10}$ in Figure 5, the set $R = \{u_1, v_1, v_2, v_3, v_4, v_5\}$ is a trb-set and hence $trb(G) = 6$

Theorem 2.11. *For the star graph and bistar graph $trb(G) = 0$.*

Proof. Since every induced subgraph of rebellion set in star and bistar are having isolated vartices and hence total rebellion set never exist for those graphs. Hence $trb(G) = 0$. □

Example 2.12.

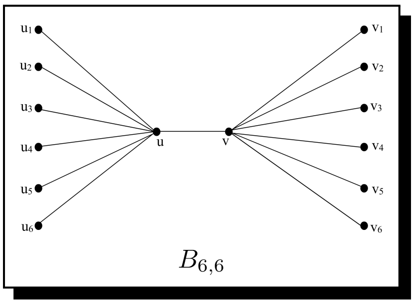

Figure 6

Theorem 2.13. *For the book graph* $B_n = 2\left[\lfloor \frac{n}{2} + 1 \rfloor\right]$, $n \geq 2$.

Proof. Let G be the book graph B_n with atleast four vertices and R be a total rebellion set in G. Let$V(G) = \{v_1, v_2, v_3, \ldots, v_{2n+2}\}$ and $E(G) = \{e_1, e_2, e_3, \ldots, e_{3n+1}\}$ be the vertex and edge set of G. The set $R = \{v_i/i = 1,2,\ldots,\lceil \frac{n}{2} \rceil\}$, $\{u_i/i = 1,2,\ldots,3\lceil \frac{n}{2} \rceil\}$ are a total rebellion set of G. Since $|R| \geq |V \setminus R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{\frac{V}{R}}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{2} \rceil$ edges. Now,

$$trb(G) \leq |R| = 2\left[\left\lfloor \frac{n}{2} + 1 \right\rfloor\right]. \tag{2.13}$$

Let R be a total rebellion set of G, such that $|R| = trb(G)$. Since R is a rebellion set gives $|R| \geq |\frac{V}{R}|$, for all $v \in R$, $N(v)$, is not a subset of R, $|N_R(v)| \leq |N_{\frac{V}{R}}(v)|$. Thus, R must contain atleast $\lceil \frac{p}{2} \rceil$ vertices (because $p = 2n+2$). Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{p}{2} \right\rceil \geq \left\lceil \frac{2n+2}{2} \right\rceil \leq 2\left[\left\lfloor \frac{n}{2} \right\rfloor\right]. \tag{2.14}$$

From (2.13) and (2.14) we have, $trb(G) = 2\left[\lfloor \frac{n}{2} + 1 \rfloor\right]$.

Example 2.14.

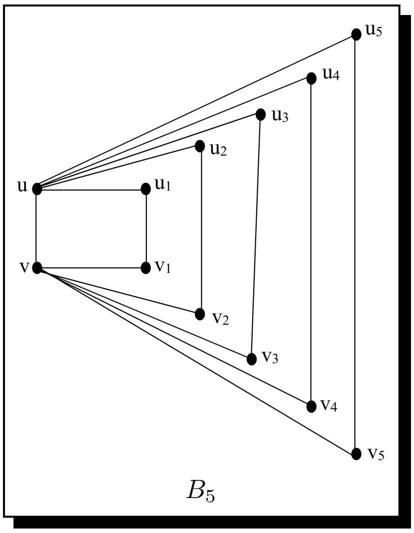

Figure 7

For the graph B_5 in Figure 7, the set $R = \{u_1, v_1, u_2, v_2, u_3, v_3\}$ is a trb-set and hence $trb(G) = 6$.

Theorem 2.15. *For the Fan graph F_n, $n \geq 3$,*

$$trb(F_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil, & n \text{ is even} \\ \frac{n+1}{2}, & n \text{ is odd.} \end{cases}$$

Proof. Let G be the fan graph F_n with atleast four vertices and R be a total rebellion set in G. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{n+1}\}$ and $E(G) = \{e_1, e_2, e_3, \dots, e_{2n-2}\}$ be the vertex and edge set of G.

Case (i): n is even.

In this case, the set $R = \{u\} \cup \{v_{n-2i+2}/i = 1, 2, \dots, \lceil \frac{n}{4} \rceil\} \cup \{v_{n-2i+2/i=1,2,\dots,\lfloor \frac{n}{4} \rfloor}\}$ is a total rebellion set of G, since$|R| \geq |V/R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V/R}(v)|$ and $\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) = 1 + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n}{4} \right\rceil \leq 2 \left\lceil \frac{n}{4} \right\rceil \leq 2 \left\lceil \frac{n+1}{2} \right\rceil. \tag{2.15}$$

Let R be a total rebellion set of G, such that$|R| = trb(G)$. Since R is a rebellion set gives$|R| \geq |V \setminus R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V \setminus R}|$. Thus, R must contain atleast $\lceil \frac{p}{2} \rceil$, vertices (because $p = n + 1$).

Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{p}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil. \tag{2.16}$$

From (2.15) and (2.16), we have $trb(G) = \lceil \frac{n+1}{2} \rceil$.

Case (ii): n is odd

In this case, the set $R = \{u\} \cup \{v_{2i-1}/i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ is a total rebellion set of G, Since $|R| \geq |V \setminus R|$, for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V \setminus R}(v)|$ and$\langle R \rangle$ has $\lceil \frac{n}{4} \rceil$ edges. Now,

$$trb(G) = |R| = 1 + \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n+1}{2} \leq \left\lceil \frac{n+1}{2} \right\rceil. \tag{2.17}$$

Let R be a total rebellion set of G, such that$|R| = trb(G)$.

Since R is a rebellion set gives such that, $|R| \geq |V \setminus R|$ and for all $v \in R$, $N(v)$ is not a subset of R, $|N_R(v)| \leq |N_{V \setminus R}(v)|$. Thus, R must contain atleast $\lceil \frac{n+1}{2} \rceil$ vertices (because $p = n + 1$.)

Therefore,

$$trb(G) = |R| \geq \left\lceil \frac{n+1}{2} \right\rceil. \tag{2.18}$$

From (2.17) and (2.18), we have $trb(G) = \lceil \frac{n+1}{2} \rceil$.

Example 2.16.

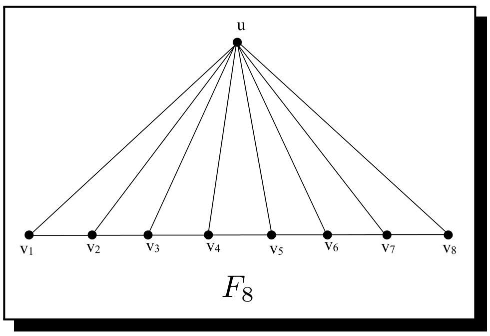

Figure 8

For the graph F_8 in Figure 8, the set $R = \{u, v_1, v_3, v_6, v_8\}$ is a trb-set and hence $trb(G) = 5$.

Theorem 2.17. *Let G be a (p,q) graph without isolated vertices. Then $trb(G) = p - m$ if and only if $G = mK_3$. where m is the number of components greater than or equal to one.*

Proof. Let m be the number of components in G. For the necessary part assume that $trb(G) = p - m$. When, $m = 1$ the result is obvious. Suppose that $G \neq mK_3$, $m > 1$, then for the existence of trb-set for each component there exist a vertex v such that it is adjacent to at least two vertices say u and w. The vertex set $V - U$ is a total rebellion set which contradict the value of $trb(G)$ and hence the result. The sufficiency is obvious.

Theorem 2.18. *A $rb(G)$-set R is a total rebellion set of G if and only if for every vertex in G there exist an adjacent vertex in R.*

Proof. Let R be a total rebellion set of G. Suppose that there exist a vertex $v \in V(G)$ such that v is not adjacent to any vertex in R. If $v \in R$ then in $\langle R \rangle$, v is an isolated vertex, which contradict R is a total rebellion set else R is not a $rb(G)$-set of G, which is also a contradiction and hence the result. For the converse, suppose that the $rb(G)$-set R is not a total rebellion set, then there exist an isolated vertex w in $\langle R \rangle$ such that $w$is not adjacent to any vertex in R. Which contradict every vertex in G there exist an adjacent vertex in R and hence the result. □

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] F. Harary, *Graph Theory*, Addison Wesley, Reading (1969).

[2] P. Kristiansen, S.K. Hedetniemi and S.T. Hedetniemi, Alliances in graphs, *J. Cobin. Ath. Cobin. Copute.* **48** (2004), 157 – 177.

[3] T.W. Haynes, S.T. Hedetniemi and A. Henning, Global defensive alliance in graphs, *Eletorn. J. Cobin.* **10**, Research paper 47, 2003.

[4] V. Mohana Selvi and P. Shyamala Anto Mary, The rebellion number in graphs, *International Journal of Scientific & Engineering Research* **7**(5) (May 2016), 56 – 62 (ISSN 2229-5518).

[5] V. Mohana Selvi and P. Shyamala Anto Mary, National conference on bridging innovative trends in pure and applied mathematics, in: *Independent Rebellion Number in Graphs*, Bannari Amman Institute of Technology, Sathyamangalam (2016).

[6] R. Kulli and B. Janakiram, The total domination number of a graph, *Indian J. Pure Appl. Math.* **27**(6) (June 1996), 537 – 542.