Research Article

Bond and CDS Pricing with Credit Events

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Abstract. We develop a calibration methodology for default probability curves to bond and *credit default swap* (CDS) pricing. We consider the *recovery of nominal* (RON) policy and *recovery of treasury* (ROT) policy for the *exposure at default* (EAD) of bond pricing. The paper presents the results of calculations of possible *profit and loss* (PnL) on bond portfolio and CDS portfolio.

Keywords. Credit default swaps; Corporate bond; Sovereign bond; Recovery of treasury; Recovery of nominal; Survival probability; Default risk; Migration risk

MSC. 62P05; 97M30

1. Introduction

A large number of new fixed-income instruments have been developed and introduced into the financial market including put bonds, zero-coupon convertibles, interest rate futures, options, and credit default swaps, and so on. The total value of the fixed income assets is about two-thirds of the market value of all outstanding securities (see [2–4]). Hence, it is crucial to study fixed-income valuation. Firstly, we provide a literature review of the studies on default-free bonds, then bonds with credit events. Next, we present the pricing formula of *credit default swap* (CDS) based on risk-neutral pricing theory. Suppose that we have a riskless (bank account) S_0 which plays the role as the "numeraire" of the market. The riskless asset S_0 admits a deterministic return $r(\cdot)$ which could depend on time. The dynamic of the riskless asset is given by the equation $\frac{dS_0(t)}{S_0(t)} = r(t)dt$ or $S_0(t) = \exp\left\{\int_0^t r(s)ds\right\}$, where $S_0(0) = 1$, $S_0(t) = \exp\left\{\int_0^t r(s)ds\right\}$, and

$r(t)$ is the instantaneous short-term rate at time t. The "Absence of Arbitrage opportunity" and the "numeraire" allows us to bring all future cash-flows back to the present. For example, an amount C_t at time t is equivalent to $C_0 = \frac{C_t}{S_0(t)} = \exp\left\{-\int_0^t r(s)ds\right\}C_t = D(0,t)C_t$, where $D(0,t) = \exp\left\{-\int_0^t r(s)ds\right\}$ is known as a discount factor (see [1]). The holder of the instrument receives a fraction R of the nominal at default by the recovery of nominal (RON). It is simpler for the nominal policy to implement but certain drawbacks. For example, the price of a long-term bond written by a very risky issuer could be higher than the one written by a risk-free issuer. Another example, we consider a zero coupon bond with zero rates 5%, time to maturity 20 years and nominal of \$1. The price of zero coupon bond without default is $\exp(-0.05(20)) = 0.38$ while the one with default risk under RON is $R*N = 0.4*1 = 0.4$. All cash flows after default (including face value and coupons) are adjusted to the fraction R which corresponds to the valuation of risk-less interest rate under the recovery of the treasury (ROT). It is more convenient in practice since the exposure at default is random and is defined by the revaluation of assets and liabilities of the firm after default. In addition, we can introduce the zero coupon spread as a discount factor between the price of a risky zero-coupon bond and the price of a non-risky zero coupon bond. We have $\frac{\tilde{B}(t,T)}{B(t,T)} = \frac{1}{((1+S(T))^T}$, where zero coupons risk-free rate is the actuarial rate of the German state in 10 years for the Europe region or US rate for outside Europe region. Zero coupons risk-free rate is the actuarial rate of the German state in 10 years for the Europe region or US rate for outside Europe region. Zero spread derived from the cash market is used in pricing sovereign bonds. The zero spread deducted from the CDS market is used in the pricing of corporate bonds. Yen Trinh presented the methodologies and results of the calibration of spread shock when the credit rating migration or default events happen. In Basel guideline, spread shock is supposed to be multiplicative. However, the author used additive spread shocks for all sovereign issuers because sovereign spreads are calibrated from bond spreads which are usually very small, and could cause an exposure of the spread shock by the multiplication (see [5], [6]). Therefore, this article presents the methodologies and results of pricing, the calibration of survival probability, and the profit and loss (PnL) when credit events happen.

2. Bond Valuations

2.1 Bond Valuation without Credit Risk

The uncertainty of economy is modeled in a filtered probability space $(\Omega, \mathbb{F} = (F_t)_{t\geq 0}, \mathbb{P})$ satisfying usual conditions. We denote Q the neutral risk probability which is equivalent to $\mathbb{P}$. The price on a zero-coupon bond at t for \$1 maturity at T from a risk-free issuer is $B(t,T) = \mathbb{E}^{\mathbb{Q}}[1D(t,T)|\mathbb{F}_t]$. The right-hand side of the equation is the discount factor at time t. The equation is satisfied the "Absence of Arbitrage Opportunity" argument because of the following reasons. Consider a scenario in which a market participant can either invest $D(t,T)$ units of cash in a money market account for a return of \$1 at time T or buy a risk-free zero-coupon bond that has a maturity value of \$1 at time T. The bond must have a value equal to the initial investment in the money market account $D(t,T)$ due to a constant interest rate. Hence, the equation is satisfied. Equivalently, by contra-position, suppose the bond price $B(t,T)$ is less than $D(t,T)$.

In this case, the investor borrows $D(t,T)$ at the money market rate, then use $B(t,T)$ to buy zero coupon bond. At maturity, the bond pays \$1 which the investors use to repay the loan. But they still have profit $D(t,T) - B(t,T)$. Similarly, if the bond is priced higher than $D(t,T)$, investors could sell the bond and invest $D(t,T)$ in the money market account. At maturity, the investor pays \$1 to the bondholder while he gets the same \$1 on the money market account. And the investor would still get the profit at t, $B(t,T) - D(t,T)$. Therefore, arbitrage opportunities would present. The following equation gives the price of a risk-free coupon bond with nominal N, and coupon C_i at maturity T_i

$$(t = T_0 < T_1 < \ldots < T_n = T)P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} D(t,T_i)C_i + ND(t,T)|\mathbb{F}_t\right] = \sum_{i=1}^{n} C_i B(t,T_i) + N.B(t,T).$$

2.2 Bond Valuation with Credit Risk

It is natural to consider the zero coupon bond in the case of credit risk. The credit risk is the risk that the quality of the issuer or the possibility of bankruptcy for the issuer has a significant impact on the rate of return of a security. A lower-quality issuer will sell at the lower price and thus offer a higher yield than the similar security issued by a higher quality issuer. The recovery rate R is assumed nonrandom. The coupon bondholder receives the recovery R on the coupon payment date that immediately follows a default. Let us denote τ the time of default or rating migration of the issuer, then the price of a risky zero-coupon bond of \$1 face value at time t with maturity T is defined by the equation

$$\tilde{B}(t,T) = \mathbb{E}_t^{\mathbb{Q}}[ND(t,T)1_{\{\tau>T\}} + R.EAD(\tau).D(t,\tau).1_{\{\tau\leq T\}}|\mathbb{F}_t]. \tag{2.1}$$

Suppose that τ is independent of riskless rate diffusion and $EAD(\tau)$ is approximated by its value at the middle of a period, then the price of a risky zero-coupon bond could be given by the equation

$$\tilde{B}(t,T) = NB(t,T)Q(\tau > T) + R.EAD(\bar{T}_i)B(t,\bar{T}_i)Q(\tau \leq T). \tag{2.2}$$

The value of a risky bond with nominal N and coupons C_i at maturity T can be partitioned into two components: the valuation of payments without credit risk, and with credit risk. The first is equal to the value of the bond independent of the recovery value while the latter will vary with the recovery schemes RON and ROT.

The default probability of the issuer, the *exposure at default* (EAD) and the recovery rate characterize the price of risky bonds.

$$\tilde{P}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} D(t,T_i)(C_i 1_{\{\tau>T_i\}} + R.EAD(\tau)1_{\{T_{i-1}<\tau\leq T_i\}}) + ND(t,T)1_{\{\tau>T\}}|\mathbb{F}_t\right].$$

Suppose that τ is independent of risk-free rate diffusion and that $EAD(\tau)$ is the approximate value at the middle of the period. The following equation presents the risky bond's price.

$$\tilde{P}(t,T) = \sum_{i=1}^{n} B(t,T_i)[C_i Q(\tau > T_i) + R.EAD(\bar{T}_i)Q(T_{i-1} < \tau \leq T_i)] + NB(t,T)Q(\tau > T). \tag{2.3}$$

We calibrate the survival probability from data including CDS market data (CDS spreads) and bond market data (bond spreads). We use CDS market data for corporate issuers because CDS

spread is more relevant for credit risk and takes into account hedging cost. In contrast, bond market data is used for sovereign issuers because CDS index spread does not well reflect the credibility of a country. Recall the discussion about the calibration of the zero coupon cash spread which is equivalent to the probability of default in the introduction. This discussion suggests a calibration from the risk-free yield curve and the yield curve of the issuer. Regarding ROT policy, the equation (2.1) can be written by the equation

$$\tilde{B}(t,T) = NB(t,T)Q(\tau > T) + RNB(t,T)(1 - Q(\tau > T)), \tag{2.4}$$

where $B(t,\tau)B(\tau,T) = B(t,T)$.

The equation (2.4) allows calculating directly the probability of survival from the price of zero coupon bonds (risk-free and risky). The implied survival probability is $Q(\tau > T) = \frac{\frac{\tilde{B}(t,T)}{B(t,T)} - R}{1 - R}$.

Regarding to RON policy, the equation (2.2) is written by the equation

$$\tilde{B}(t,T) = NB(t,T)Q(\tau > T) + RN \sum_{i=1}^{n} B(t,\bar{T}_i)(Q(T_{i-1} < \tau \le T_i). \tag{2.5}$$

From the equation (2.5), we get the recurrence equations

$$\tilde{B}(t,T_k) = [B(t,T_k) - RB(t,\bar{T}_k)]Q(\tau > T_k) - \left[R \sum_{i=1}^{k} (B(t,\bar{T}_{i-1}) - B(t,\bar{T}_i))Q(\tau > T_{i-1})\right] - RB(t,\bar{T}_0).$$

The survival probability curve is calibrated by the bootstrap method. It reduces to solving a linear system $AQ = B$, where $Q = (Q(\tau > T_i))$, $i = 1,\ldots,n$ is the vector of calibrated survival probabilities.

$$A = \begin{pmatrix} B(t,T_1) - RB(t,\bar{T}_1) & 0 & 0 & 0 & 0 \\ R[B(t,\bar{T}_2) - B(t,\bar{T}_1)] & B(t,T_2) - RB(t,\bar{T}_2) & \vdots & 0 & 0 \\ \ldots & R[B(t,\bar{T}_3) - B(t,\bar{T}_2)] & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & B(t,T_{n-1}) - RB(t,\bar{T}_{n-1}) & 0 \\ R[B(t,\bar{T}_2) - B(t,\bar{T}_1)] & R[B(t,\bar{T}_3) - B(t,\bar{T}_2)] & \ldots & R[B(t,\bar{T}_n) - B(t,\bar{T}_{n-1})] & B(t,T_n) - RB(t,\bar{T}_n) \end{pmatrix},$$

$$B = \begin{pmatrix} \tilde{B}(t,T_1) - RB(t,\bar{T}_1).1 \\ \vdots \\ \tilde{B}(t,T) - RB(t,\bar{T}_1).1 \end{pmatrix}.$$

The vector of survival probabilities is $Q = A^{-1}B$, where A^{-1} is defined.

3. CDS Valuation

Firstly, we consider how CDS works. Premium Leg is the CDS premium payment to protection seller until maturity or credit event which occurs first. Protection Leg is the default-contingent payment made by the protection seller. The accrued premium is the portion of the CDS premium that accumulated between the last payment date of the CDS and the time of default. For example, two parties enter into a 5-year CDS on 20 March 2012 with 100 million dollars notional principal and quarterly payments of 22.5 basis points. In the case of no credit events, the buyer receives no payoff and pays $0.00225(100,000,000) = 225,000$ dollars each quarter until March 20, 2017. In case of occurrence of a credit event,the buyer has the right to sell bonds at face value (physical settlement contract). The auction indicates that the bond is worth

35 dollars per 100 dollars of face value after 3 months of default. The cash payoff would be $100 - 35 = 65$ million dollars (Cash settlement contracts). Given a CDS contract with nominal N, a spread S_i at $T_1 < T_2 < \ldots < T_n = T$, and a protection at default, then the value of premium leg is

$$PV(Premium) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} S_i D(t, T_i) 1_{\{\tau > T_i\}}\right] = \sum_{i=1}^{n} S_i B(t, T_i) Q(\tau > T_i).$$

We can solve this problem on condition that the credit event may happen on each small time interval $[s, s + ds]$ within the default intensities approach. We get the present value of the protection leg at time t in continuous time. If a credit event happens before maturity, the following equation defines the payment of the seller to the buyer.

$$PV(protection) = N(1 - R)\mathbb{E}^{\mathbb{Q}}[D(t, \tau)1_{\{\tau \le T\}}] = N(1 - R)\int_{t}^{T} B(t, s)dQ(\tau \le s),$$

where $dQ(\tau \le s)$ is the probability of default between s and $s + ds$ knowing that there was no default before times. In a model with a current hazard rate λ, this amount is written by the equation $dQ(\tau \le s) = Q(\tau > s)\lambda(s)ds$. The CDS Value to Protection Buyer is the difference between the protection leg and the premium leg (including the accrued premium).

Typically, we address the issue of the accrued premium by adding half an accrual period to the premium leg. We assume that a default will on average take place midway through the period if it occurs. The accrued premium is

$$\sum_{i=1}^{n} B(t, \bar{T}_i) Q(T_{i-1} < \tau \le T_i)\frac{S_i}{2}.$$

Marking a CDS position to market is the act of determining the present value of a CDS agreement that was entered into at some time in the past. The value of a CDS contract to a protection buyer is

$$V(t, T) + \sum_{i=1}^{n} B(t, \bar{T}_i) Q(T_{i-1} < \tau \le T_i)\frac{S_i}{2} - N(1 - R)\sum_{i=1}^{n} B(t, \bar{T}_i) Q(T_{i-1} < \tau \le T_i).$$

The survival probability is calibrated by equating the premium leg and protection leg (including the accrued premium).

$$Q(\tau > T) = \frac{\left(\begin{array}{l}\sum_{i=1}^{n-1} B(t, \bar{T}_i)(1 - R - \frac{S_i}{2})(Q(\tau > T_{i-1}) - Q(\tau > T_i)) \\ -\sum_{i=1}^{n-1} B(t, T_i)Q(\tau > T_i)S_i + B(t, \bar{T}_n)(1 - R - \frac{S_n}{2})Q(\tau > T_{n-1})\end{array}\right)}{B(t, T)S_n + B(t, \bar{T}_n)(1 - R - \frac{S_n}{2})}.$$

The survival probability $Q(\tau > T_i)$, $i = 1, \ldots, n$ can be derived from above formula. Starting with a 1-period CDS contract, it is simple to work out $Q(\tau > T_1) = \frac{B(t, \bar{T}_1)(1 - R - \frac{S_1}{2})}{B(t, T)S_1 + B(t, \bar{T}_1)(1 - R - \frac{S_1}{2})}$.

For a 2-period CDS contract, knowing $Q(\tau > T_1)$ it is simple to work out

$$Q(\tau > T_2) = \frac{\left(\begin{array}{l}B(t, \bar{T}_1)(1 - R - \frac{S_1}{2})(Q(\tau > T_0) - Q(\tau > T_1)) \\ -B(t, T_1)Q(\tau > T_1)S_1 + B(t, \bar{T}_2)(1 - R - \frac{S_2}{2})Q(\tau > T_1)\end{array}\right)}{B(t, T)S_2 + B(t, \bar{T}_2)(1 - R - \frac{S_2}{2})}.$$

We continue to calibrate $Q(\tau > T_3)$, $Q(\tau > T_4)$, and so on.

4. Results

4.1 Bond Valuation

We test the pricing performance of two recovery schemes in the context of the sovereign and corporate bond. Bond contracts' name consist some characters. The first characters identify the bond type, the next one character is the coupon rate, and the next one character represents the month as follows: January = F, February = G, March = H, April = J, May = K, June = M, July = N, August = Q, September = U, October = V, November = X, and December = Z, and the last shows the year to maturity. We consider the present value of the bond recovery payment (at time t) under RON as a weighted sum of all the recovery payments associated with all possible default scenarios,

$$N.R \sum_{i=1}^{n} B(t, \bar{T}_i) Q(T_{i-1} < \tau \leq T_i),$$

where the weights are given by the risk-neutral probabilities of each scenario. From the equation (2.3), the price of risky coupon bond can be written by the equation

$$\tilde{P}(t,T) = \sum_{i=1}^{n} C_i B(t,T_i) Q(\tau > T_i) + NB(t,T)Q(\tau > T) + \sum_{i=1}^{n} N.RB(t,\bar{T}_i)Q(T_{i-1} < \tau \leq T_i).$$

From the equation (2.3), the price of risky coupon bond under ROT can be defined

$$\tilde{P}(t,T) = \sum_{i=1}^{n} [C_i B(t,T_i) Q(\tau > T_i)] + NB(t,T)Q(\tau > T) + R.NB(t,T)Q(\tau \leq T)$$

$$+ \sum_{i=1}^{n} \left[R \sum_{j=i}^{n} C_j B(t,T_j) Q(T_{i-1} < \tau \leq T_i) \right].$$

This section tests the pricing performance of two recovery schemes in the context of the sovereign and corporate bond. We run MATLAB code on market data: rating data, bond, and CDS portfolio.

Table 1. First 10 observations of bond valuation results

Bond code	Curve name	Bond Price without default	Price in interest rate framework	Price in RON	Price in ROT
'SPGB5.5N17'	'SOVBOND_ROYESPAGNE_SRUNSEC'	1217839	1202086	1200934	111391
'ISPIM4.125F16'	'MO_INTMI_SEN_EUR_ANY'	1122590	1116374	1116131	93597
'DBR3.25F20'	'SOVBOND_REPALLEMAG_SRUNSEC'	1188908	1123078	1119468	196877
'OAT3.5_25J26'	'SOVBOND_REPALLEMAG_SRUNSEC'	1441499	1161525	1133665	241364
'FRTR4_25J60'	'SOVBOND_TRESOR_SRUNSEC'	2789732	1362940	1023111	806381
'DBR3.25N42'	'SOVBOND_REPFINLAND_SRUNSEC'	1900792	1122548	945324	938003
'OAT4V09'	'SOVBOND_REPAUTRICH_SRUNSEC'	1964977	1238422	1071927	453166
'OAT4J55'	'SOVBOND_ROPAYSBAS_SRUNSEC'	2604631	1345859	1023111	729829
'BGB5H35'	'SOVBOND_ROYBELG_SRUNSEC'	2072325	1418715	1269408	515346
'OAT4.75J35'	'SOVBOND_REPALLEMAG_SRUNSEC'	2017111	1371785	1228721	497096

4.2 CDS Valuation

After Matlab implement, we get the CDS values to protection buyer for all positions of the portfolio. If the CDS value to protection buyer is positive, then the protection seller is paid the upfront payment by the protection buyer. If it is negative, it is the other way around.

Table 2. First 10 observations of CDS valuation results

CDS name	CDS curve name	Premium	Protection	CDS value
452994	'MO_INTMI_SEN_EUR_ANY'	0.60175	5998128	5998127
482679	'MO_ALLIANZMU_SEN_EUR_ANY'	0.625212	2998432	2998432
484907	'MO_AXASAPA_SEN_ANY_ANY'	0.625212	2998432	2998432
473176	'MO_UBSAGZU_SEN_EUR_ANY'	0.599827	8997409	8997409
469033	'MO_CREDIMI_SEN_ANY_ANY'	0.617101	5998088	5998088
475795	'MO_MERLYINCNY_SEN_EUR_ANY'	0.599827	11996546	11996545
452194	'MO_MERLYINCNY_SEN_USD_ANY'	0.440798	4848780	4848779
447078	'MO_BBVMA_SUB_EUR_ANY'	0.608455	1294311	1294310
454339	'MO_HANRUECK_SUB_EUR_ANY'	0.633567	299088	299087

4.3 Profit and Loss (PnL)

Table 3. First 10 observations of PnL results

Curve name	Initial rating	Final rating	PnL
CDX_EM	BBB	D2	2004704.14
INDICE_ABBEYLN_SEC	AAA	AAA	0
INDICE_ABBEYLN_SRUNSEC	AA	AA	0
INDICE_ABNAMRO2_SEC	AAA	AA+	1329.07
INDICE_ABNAMRO2_SRUNSEC	A	A−	2745.48
INDICE_ABNAMRO2_SUB	A	A−	−4801.25
INDICE_ABNAM_SEC	AAA	AAA	0
INDICE_ACHYPHA_SRUNSEC	A	A	0
INDICE_ADIBCAPIN1_SRUNSEC	BBB	BB+	74111.46
INDICE_AKTIABK_SEC	AAA	A+	34300.38

Table 4. First 10 observations of PnL results for credit rating migration

Curvename	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC
CDX_EM	0	0	0	0	0	0	0	0	0	29905	68719	168318	360186	632504	932753	1322371	1454461
INDICE_ABBEYLN_SEC	0	756	1692	2131	2610	3132	3783	4500	5290	8072	11746	16577	21167	26759	33535	43287	69656
INDICE_ABBEYLN_SRUNSEC	-1729	-1616	0	941	1981	3131	4221	5411	6710	12433	20271	30932	43459	59903	81451	131219	333137
INDICE_ABNAMRO2_SEC	0	1329	2968	3736	4571	5479	6607	7847	9209	13964	20153	28135	35555	44382	54771	69142	104708
INDICE_ABNAMRO2_SRUNSEC	-12598	-9659	-6147	-4265	-2220	0	2745	5758	9064	19783	33556	51198	66214	84063	105256	134855	220511
INDICE_ABNAMRO2_SUB	22214	16995	10775	7457	3875	0	-4801	-10001	-15644	-33149	-54388	-79726	-99467	-121236	-144933	-173532	-238359
INDICE_ABNAM_SEC	0	4144	9294	11720	14370	17265	20881	24880	29300	44994	66034	94279	121749	156117	199199	264285	463584
INDICE_ACHYPHA_SRUNSEC	5295	5295	4270	2981	1562	0	-1564	-3274	-5141	-13132	-24017	-38760	-55461	-77068	-104947	-163539	-376395
INDICE_ADIBCAPIN1_SRUNSEC	-90599	-79702	-65614	-56921	-47679	-37913	-25980	-13337	0	35153	74111	116337	143705	172169	201617	231594	300469
INDICE_AKTIABK_SEC	0	9907	22201	27987	34300	41190	49791	59288	69775	106884	156345	222213	285687	364275	461474	605499	1024733

5. Conclusion

The article discussed the importance and methodologies of bond and CDS pricing. We estimated profit and loss as well as managed default risk and credit risk. The goal is to apply the approach to the market data.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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