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Research Article

Effects of Triaxiality and Radiation Pressure on Existence of Resonance Stability of Triangular Equilibrium Points in Elliptical Restricted Three-Body Problem

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Abstract. This paper deals with the effects of radiation and triaxiality on linear stability of the triangular librations points for small value of eccentricity 'e' in presence of resonance. The study is carried out for various values of radiation pressure and Triaxiality parameter of both Primaries. It is found that the parametric resonance is only possible at the resonance frequency $\omega_2 = 1/2$ in elliptical case.

Keywords. ER3BP; Triangular libration points; Resonance

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1. Introduction

The restricted three-body problem has a wider range of application that the general three body problem in space dynamics, celestial mechanics and analytical dynamics. The *Elliptical Restricted Three Body Problem* (ER3BP) models the motion of infinitesimal mass which moves under the influence of two massive bodies revolving around their centre of mass in an elliptical orbit, being influenced but not influencing the two primaries. The circular restricted three body problem has been generalized by the introduction of the elliptic orbit, thus improving its applicability and retaining some useful properties of the circular model suitable to the elliptic case. Various authors and researchers like Ammar [1], Bennet [2], Danby [3], Grebnikov [4], Gyorgyey [5], Kumar and Ishwar [6], Kumar and Choudhary [7], Markellos, Perdios and Labrapoulou [10], Moulton [11], Narayan and Kumar [12], Narayan and Singh [13], Narayan and Usha [14], Singh and Aishetu [19], Singh and Aishetu [20], Szebebely [21], Usha, Narayan and Ishwar [22], Zimvoschikov and Thkai [24].

The author has analyzed stability of triangular equilibrium points L_4 and L_5 in ER3BP under the assumption that both primaries are triaxial and radiating [17]. Also, the characteristic exponents of triangular solutions in ER3BP has been analyzed [18].

The present investigation is an attempt to identify the existence of resonance and the stability of triangular points in presence of small value of eccentricity 'e', assuming that the bigger and smaller primaries are triaxial and the source of radiation as well. this study will contribute to understand the effects of radiation, eccentricity and triaxiality on the celestial dynamical systems. For this study we have used method given by Markeev [8, 9] in which the Hamiltonian function pertaining to the problem is made independent of time using several canonical transformations. The existence of resonance and the stability of infinitesimal near the resonance frequency has been analyzed. The problem is discussed earlier by Narayan and Singh [16], Usha and Narayan [23], and Narayann and Shrivastava [15] with different conditions.

2. Equations of Motion

The differential equations of the motion of the infinitesimal mass in elliptical restricted three body problem under radiating and triaxial primaries in pulsating system is given by [17] and [18] as:

$$x'' - 2y'' = \frac{1}{1+e\cos v}\left(\frac{\partial \Omega}{\partial x}\right), \quad y'' + 2x'' = \frac{1}{1+e\cos v}\left(\frac{\partial \Omega}{\partial x}\right), \tag{2.1}$$

where $'$ denotes differentiation with respect to v, and

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2s_1-s_2)q_1}{2r_1^3} + \frac{\mu(2s_1'-s_2')q_2}{2r_2^3} - \frac{3(1-\mu)(s_1-s_2)y^2 q_1}{2r_1^5} - \frac{3\mu(s_1'-s_2')y^2 q_2}{2r_2^5}\right], \tag{2.2}$$

where

$$n^2 = 1 + \frac{3}{2}(2s_1 - s_2) + \frac{3}{2}(2s_1' - s_2')$$

and

$$r_1^2 = (x+\mu)^2 + y^2,$$
$$r_2^2 = (x-1+\mu)^2 + y^2, \tag{2.3}$$

$$\mu = \frac{m_2}{m_1 + m_2},$$

where m_1 and m_2 are masses of the primaries, q_1, q_2 are the radiation pressure, σ_1, σ_2, σ_1' and σ_2' are triaxial parameters, while e and v are the eccentricity of orbits and true anomaly of the primaries, respectively. There are two triangular equilibrium points in the plane of the finite bodies around the coordinates (x, y) and then the three bodies form nearly equilateral triangle. Since the equilibrium points are symmetrical to each other, the nature of motion near the two triangular equilibrium points are given by [17] and [18].

$$\begin{aligned}
x &= \frac{1}{2} - \mu + \frac{(\varepsilon_2' - \varepsilon_1')}{3} + \left[\frac{-3}{8} - \varepsilon_1' - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\varepsilon_1'}{2\mu}\right] s_1 + \left[\frac{7}{8} + \frac{\varepsilon_1'}{2} + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\varepsilon_1'}{2\mu}\right] s_2 \\
&\quad + \left[\frac{3}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\varepsilon_2'}{8(1-\mu)} + \varepsilon_2'\right] s_1' + \left[\frac{-7}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\varepsilon_2'}{8(1-\mu)} - \frac{\varepsilon_2'}{2}\right] s_2', \\
y &= \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left\{-\frac{(\varepsilon_1' + \varepsilon_2')}{3} + \left[\frac{-19}{8} - \varepsilon_1' + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\varepsilon_1'}{2\mu}\right] s_1\right.\right. \\
&\quad + \left[\frac{15}{8} + \frac{\varepsilon_1'}{2} - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\varepsilon_1'}{2\mu}\right] s_2 + \left[\frac{-19}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\varepsilon_2'}{8(1-\mu)} - \varepsilon_2'\right] s_1' \\
&\quad \left.\left. + \left[\frac{15}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\varepsilon_2'}{8(1-\mu)} + \frac{\varepsilon_2'}{2}\right] s_2'\right\}\right].
\end{aligned} \tag{2.4}$$

Thus the coordinates of the triangular equilibrium points has been obtained up to first order terms in the parameters ϵ_1', ϵ_2', s_1, s_2, s_1' and s_2' which is represented by (2.4). The system (2.1) described the motion of dynamical system with Lagrangian, which is represented as

$$\begin{aligned}
L &= \frac{\dot{x}^2 + \dot{y}^2}{2} + (\dot{y}x - \dot{x}y) + \frac{1}{1 + e\cos\varepsilon'}\left\{\frac{x^2 + y^2}{2} + \frac{1}{n}^2\left[\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2s_1 - s_2)q_1}{2r_1^3}\right.\right. \\
&\quad \left.\left. + \frac{\mu(2s_1' - s_2')q_2}{2r_2^3} - \frac{3(1-\mu)(s_1 - s_2)y^2 q_1}{2r_1^5} - \frac{3\mu(s_1' - s_2')y^2 q_2}{2r_2^5}\right]\right\}
\end{aligned} \tag{2.5}$$

where

$$P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad P_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x. \tag{2.6}$$

We formed the expression for the Hamiltonian function of the problem using the formula

$$H = -L + \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{y}}\dot{y}. \tag{2.7}$$

Hence, we have

$$H = -L + (P_x^2 + P_y^2) + (P_x y - P_y x). \tag{2.8}$$

The perturbed Hamiltonian function of the problem can be find by use of (2.5) and (2.8), which is reduced to the following form:

$$H = \frac{(P_x^2 + P_y^2)}{2} + (P_x y - P_y x) + \frac{x^2 + y^2}{2}\left[1 - \frac{1}{1 + e\cos\varepsilon}\right] - \frac{1}{(1 + e\cos\varepsilon)}n^2\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2}\right.$$

$$+\frac{(1-\mu)(2s_1-s_2)q_1}{2r_1^3}+\frac{\mu(2s_1'-s_2')q_2}{2r_2^3}-\frac{3(1-\mu)(s_1-s_2)y^2q_1}{2r_1^5}-\frac{3\mu(s_1'-s_2')y^2q_2}{2r_2^5}\Bigg\}, \tag{2.9}$$

where P_x and P_y are the generalized components of momentum.

Since the two triangular equilibrium points are symmetrical, the nature of the oscillation of infinitesimal near two points will be the same. Hence, in further calculation the motion near the equilibrium point L_4 will be considered. So, shifting the origin to L_4 by the change of variables given by:

$$\begin{aligned} &x=\xi+q_1; \qquad y=\eta+q_2, \\ &p_x=p_\xi+p_1; \;\; p_y=p_\eta+p_2, \end{aligned} \tag{2.10}$$

where the displacement of infinitesimal at and near the equilibrium point L_4 is represented as follows:

$$\begin{aligned}
\xi &= \frac{1}{2}-\mu+\frac{(\varepsilon_2'-\varepsilon_1')}{3}+\left[\frac{-3}{8}-\varepsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_1+\left[\frac{7}{8}+\frac{\varepsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_2 \\
&\quad +\left[\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\varepsilon_2'}{8(1-\mu)}+\varepsilon_2'\right]s_1'+\left[\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\varepsilon_2'}{8(1-\mu)}-\frac{\varepsilon_2'}{2}\right]s_2', \\
\eta &= \frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{(\varepsilon_1'+\varepsilon_2')}{3}+\left[\frac{-19}{8}-\varepsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_1\right.\right. \\
&\quad +\left[\frac{15}{8}+\frac{\varepsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_2+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\varepsilon_2'}{8(1-\mu)}-\varepsilon_2'\right]s_1' \\
&\quad \left.\left.+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\varepsilon_2'}{8(1-\mu)}+\frac{\varepsilon_2'}{2}\right]s_2'\right\}\right], \\
p_\xi &= -\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{(\varepsilon_1'+\varepsilon_2')}{3}+\left[\frac{-19}{8}-\varepsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_1\right.\right. \\
&\quad +\left[\frac{15}{8}+\frac{\varepsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\varepsilon_1'}{2\mu}\right]s_2+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\varepsilon_2'}{8(1-\mu)}-\varepsilon_2'\right]s_1' \\
&\quad \left.\left.+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\varepsilon_2'}{8(1-\mu)}+\frac{\varepsilon_2'}{2}\right]s_2'\right\}\right], \\
p_\eta &= \frac{1}{2}-\mu+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left[\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{\left(1-\mu\right)\epsilon_1'}{2\mu}\right]s_1+\left[\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]s_2 \\
&\quad +\left[\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right]s_1'+\left[\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right]s_2'.
\end{aligned} \tag{2.11}$$

The solution (2.11) in the new variables is given by the equilibrium positions:

$$q_1=q_2=p_1=p_2=0. \tag{2.12}$$

Now, expanding the Hamiltonian function (2.9) in the power of p_i and q_i, we obtained

$$H=\sum_{K=0}^{\infty}H_K=H_0+H_1+H_2+H_3+H_4+H_5+\dots \tag{2.13}$$

where $H_0=H(\xi,\eta,p_\xi,p_\eta)=$ constant and $H_1=0$.

We evaluated $H_2, H_3, \dots$ using (2.10) and the terms of the Hamiltonian (2.9) are expanded one by one, the terms are not depending upon p_i and q_i, and those of order one are neglected. Hence, we obtain

$$\text{(i)}\quad \frac{P_x^2+P_y^2}{2}=\frac{1}{2}[(p_\xi+p_1)^2+(p_\eta+p_2)]=\frac{1}{2}(p_1^2+p_2^2), \tag{2.14}$$

$$\text{(ii)}\quad (p_x y-p_y x)=(p_\xi+p_1)(\eta+q_2)-(p_\eta+p_2)(\xi+q_1)=(p_1q_2-p_2q_1), \tag{2.15}$$

$$\text{(iii)}\quad \frac{e\cos\nu}{2(1+e\cos\nu)}(x^2+y^2)=\frac{e\cos\nu}{2(1+e\cos\nu)}\{(\xi+q_1)^2+(\eta+q_2)^2\}=\frac{e\cos\nu}{2(1+e\cos\nu)}(q_1^2+q_2^2), \tag{2.16}$$

$$\text{(iv)}\quad \frac{1}{(1+e\cos\nu)n}{}^2=\left\{\frac{(1-\mu)(1-\epsilon_1')}{r_1}+\frac{\mu q_2}{r_2}+\frac{(1-\mu)(2\sigma_1-\sigma_2)(1-\epsilon_1')}{2r_1^3}+\frac{\mu(2\sigma_1'-\sigma_2')(1-\epsilon_2')}{2r_2^3}\right.$$
$$\left.-\frac{3(1-\mu)(\sigma_1-\sigma_2)y^2(1-\epsilon_1')}{2r_1^5}-\frac{3\mu(\sigma_1'-\sigma_2')y^2(1-\epsilon_2')}{2r_2^5}\right\}, \tag{2.17}$$

where

$$\begin{aligned}
r_1^2&=(x+\mu)^2+y^2=(\xi+q_1+\mu)^2+(\eta+q_2)^2\\
&=\left[q_1+\frac{1}{2}-\mu+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1\right.\\
&\quad+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'\\
&\quad\left.+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'+\mu\right]^2+\left[q_2+\frac{\sqrt{3}}{2}\left\{1+\frac{2}{3}\left\{-\frac{(\epsilon_1'+\epsilon_2')}{3}\right.\right.\right.\\
&\quad+\left[\frac{-19}{8}-\epsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1+\left[\frac{15}{8}+\frac{\epsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2\\
&\quad\left.\left.\left.+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}-\epsilon_2'\right]\sigma_1'+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}+\frac{\epsilon_2'}{2}\right]\sigma_2'\right\}\right\}\right]^2,
\end{aligned}$$

$$\begin{aligned}
r_1^{-1}&=f(q_1,q_2)\\
&=\left[\left[q_1+\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1\right.\right.\\
&\quad+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'\\
&\quad\left.\left.+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'+\right]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\right]^{-1/2}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
r_2^2&=(x-1+\mu)^2+y^2\\
&=(\epsilon+q_1-1+\mu)^2+(\epsilon+q_2)^2\\
&=\left[q_1-\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1\right.
\end{aligned}$$

$$+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'$$
$$+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2+\left(q_2+\frac{\Lambda}{2}\right)^2,$$

$$r_2^{-1}=g(q_1,q_2)$$
$$=\Bigg[\Bigg[q_1-\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1$$
$$+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'$$
$$+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2+\left(q_2+\frac{\epsilon}{2}\right)^2\Bigg]^{-1/2}, \tag{2.19}$$

$$r_1^{-3}=\alpha(q_1,q_2)$$
$$=\Bigg[\Bigg[q_1+\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1$$
$$+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'$$
$$+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\Bigg]^{-3/2}, \tag{2.20}$$

$$r_2^{-3}=\beta(q_1,q_2)$$
$$=\Bigg[\Bigg[q_1-\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1$$
$$+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'$$
$$+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\Bigg]^{-3/2}. \tag{2.21}$$

Similarly,

$$r_1^{-5}=a(q_1,q_2), \tag{2.22}$$

$$r_2^{-5}=b(q_1,q_2), \tag{2.23}$$

where

$$\frac{\Lambda}{2}=\frac{\sqrt{3}}{2}\Bigg[1+\frac{2}{3}\Bigg\{-\frac{(\epsilon_1'+\epsilon_2')}{3}+\left[\frac{-19}{8}-\epsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1$$
$$+\left[\frac{15}{8}+\frac{\epsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}-\epsilon_2'\right]\sigma_1'$$
$$+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}+\frac{\epsilon_2'}{2}\right]\sigma_2'\Bigg\}\Bigg]. \tag{2.24}$$

In order to expand H in powers of q_1 and q_2, we required the expansion of $f(q_1,q_2)$, $g(q_1,q_2)$, $\alpha(q_1,q_2)$, $\beta(q_1,q_2)$, $a(q_1,q_2)$ and $b(q_1,q_2)$ with the help of Taylor's series, and we get

$$\begin{aligned}f(q_1,q_2) = f(0,0) &+ [q_1 f_1(0,0) + q_2 f_2(0,0)] \\ &+ \frac{1}{2}[q_1^2 f_{11}(0,0) + 2q_1q_2 f_{1\,2}(0,0) + q_2^2 f_{2\,2}(0,0)] \\ &+ \frac{1}{6}[q_1^3 f_{111}(0,0) + 3q_1^2q_2 f_{1\,1\,2}(0,0) + 3q_1q_2^2 f_{1\,2\,2}(0,0) + q_2^3 f_{2\,2\,2}(0,0)] \\ &+ \frac{1}{24}[q_1^4 f_{1111}(0,0) + 4q_1^3q_2 f_{111\,2}(0,0) + 6q_1^2q_2^2 f_{11\,2\,2}(0,0) + 4q_1q_2^3 f_{122\,2}(0,0) \\ &\quad + q_2^4 f_{2222}(0,0)] + \dots\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}g(q_1,q_2) = g(0,0) &+ [q_1 g_1(0,0) + q_2 g_2(0,0)] \\ &+ \frac{1}{2}[q_1^2 g_{11}(0,0) + 2q_1q_2 g_{1\,2}(0,0) + q_2^2 g_{2\,2}(0,0)] \\ &+ \frac{1}{6}[q_1^3 g_{111}(0,0) + 3q_1^2q_2 g_{1\,1\,2}(0,0) + 3q_1q_2^2 g_{1\,2\,2}(0,0) + q_2^3 g_{2\,2\,2}(0,0)] \\ &+ \frac{1}{24}[q_1^4 g_{1111}(0,0) + 4q_1^3q_2 g_{111\,2}(0,0) + 6q_1^2q_2^2 g_{11\,2\,2}(0,0) + 4q_1q_2^3 g_{122\,2}(0,0) \\ &\quad + q_2^4 g_{2222}(0,0)] + \dots.\end{aligned} \tag{2.26}$$

Similarly

$$\begin{aligned}\alpha(q_1,q_2) = \alpha(0,0) &+ [q_1 \alpha_1(0,0) + q_2 \alpha_2(0,0)] \\ &+ \frac{1}{2}[q_1^2 \alpha_{11}(0,0) + 2q_1q_2 \alpha_{1\,2}(0,0) + q_2^2 \alpha_{2\,2}(0,0)] \\ &+ \frac{1}{6}[q_1^3 \alpha_{111}(0,0) + 3q_1^2q_2 \alpha_{1\,1\,2}(0,0) + 3q_1q_2^2 \alpha_{1\,2\,2}(0,0) + q_2^3 \alpha_{2\,2\,2}(0,0)] \\ &+ \frac{1}{24}[q_1^4 \alpha_{1111}(0,0) + 4q_1^3q_2 \alpha_{111\,2}(0,0) + 6q_1^2q_2^2 \alpha_{11\,2\,2}(0,0) + 4q_1q_2^3 \alpha_{122\,2}(0,0) \\ &\quad + q_2^4 \alpha_{2222}(0,0)] + \dots\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\beta(q_1,q_2) = \beta(0,0) &+ [q_1 \beta_1(0,0) + q_2 \beta_2(0,0)] \\ &+ \frac{1}{2}[q_1^2 \beta_{11}(0,0) + 2q_1q_2 \beta_{1\,2}(0,0) + q_2^2 \beta_{2\,2}(0,0)] \\ &+ \frac{1}{6}[q_1^3 \beta_{111}(0,0) + 3q_1^2q_2 \beta_{1\,1\,2}(0,0) + 3q_1q_2^2 \beta_{1\,2\,2}(0,0) + q_2^3 \beta_{2\,2\,2}(0,0)] \\ &+ \frac{1}{24}[q_1^4 \beta_{1111}(0,0) + 4q_1^3q_2 \beta_{111\,2}(0,0) + 6q_1^2q_2^2 \beta_{11\,2\,2}(0,0) + 4q_1q_2^3 \beta_{122\,2}(0,0) \\ &\quad + q_2^4 \beta_{2222}(0,0)] + \dots\end{aligned} \tag{2.28}$$

similarly

$$a(q_1,q_2) = a(0,0) + [q_1 a_1(0,0) + q_2 a_2(0,0)]$$

$$+\frac{1}{2}[a_1^2 a_{11}(0,0)+2q_1q_2a_{1\,2}(0,0)+q_2^2a_{2\,2}(0,0)]$$
$$+\frac{1}{6}[q_1^3a_{111}(0,0)+3q_1^2q_2a_{1\,1\,2}(0,0)+3q_1q_2^2a_{1\,2\,2}(0,0)+q_2^3a_{2\,2\,2}(0,0)]$$
$$+\frac{1}{24}[q_1^4a_{1111}(0,0)+4q_1^3q_2a_{111\,2}(0,0)+6q_1^2q_2^2a_{11\,2\,2}(0,0)+4q_1q_2^3a_{122\,2}(0,0)$$
$$+q_2^4a_{2222}(0,0)]+\dots \tag{2.29}$$

and

$$b(q_1,q_2)=b(0,0)+[q_1b_1(0,0)+q_2b_2(0,0)]$$
$$+\frac{1}{2}[a_1^2b_{11}(0,0)+2q_1q_2b_{1\,2}(0,0)+q_2^2b_{2\,2}(0,0)]$$
$$+\frac{1}{6}[q_1^3b_{111}(0,0)+3q_1^2q_2b_{1\,1\,2}(0,0)+3q_1q_2^2b_{1\,2\,2}(0,0)+q_2^3b_{2\,2\,2}(0,0)]$$
$$+\frac{1}{24}[q_1^4b_{1111}(0,0)+4q_1^3q_2b_{111\,2}(0,0)+6q_1^2q_2^2b_{11\,2\,2}(0,0)+4q_1q_2^3b_{122\,2}(0,0)$$
$$+q_2^4b_{2222}(0,0)]+\dots. \tag{2.30}$$

Substituting values in equation (2.25), we get

$$r_1^{-1}=\Bigg[1-\frac{\epsilon_2'}{12}+\frac{\epsilon_1'}{12}+\frac{41\sigma_1}{32}-\frac{(1-\mu)\sigma_1}{8\mu}-\frac{61\sigma_2}{96}+\frac{(1-\mu)\sigma_2}{8\mu}+\frac{35\sigma_1'}{32}+\frac{9\mu\sigma_1'}{32(1-\mu)}-\frac{23\sigma_2'}{32}$$
$$+\frac{21\mu\sigma_2'}{32(1-\mu)}+q_1\left\{\frac{-1}{2}-\frac{\epsilon_2'}{12}+\frac{\epsilon_1'}{12}-\frac{27\sigma_1}{16}+\frac{(1-\mu)\sigma_1}{2\mu}+\frac{19\sigma_2}{16}-\frac{(1-\mu)\sigma_2}{2\mu}-\frac{3\sigma_1'}{4}\right.$$
$$\left.-\frac{3\mu\sigma_1'}{16(1-\mu)}+\frac{13\sigma_2'}{8}+\frac{7\mu\sigma_2'}{16(1-\mu)}\right\}-\frac{\sqrt{3}}{2}q_2\left\{1-\frac{5\epsilon_2'}{6}+\frac{\epsilon_1'}{6}+\frac{61\sigma_1}{64}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{23\sigma_2}{8}\right.$$
$$\left.-\frac{(1-\mu)\sigma_2}{3\mu}+\frac{17\sigma_1'}{12}+\frac{7\mu\sigma_1'}{8(1-\mu)}-\frac{\sigma_2'}{4}-\frac{49\mu\sigma_2'}{24(1-\mu)}\right\}+\frac{1}{2}q_1^2\left\{\frac{-1}{4}+\frac{7\epsilon_1'}{24}-\frac{7\epsilon_2'}{24}+\frac{305\sigma_1}{32}\right.$$
$$\left.-\frac{3(1-\mu)\sigma_1}{2\mu}-\frac{257\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{2\mu}+\frac{71\sigma_1'}{8}+\frac{57\mu\sigma_1'}{32(1-\mu)}-\frac{13\sigma_2'}{2}-\frac{169\mu\sigma_2'}{32(1-\mu)}\right\}$$
$$+\sqrt{3}q_1q_2\left\{\frac{3}{4}+\frac{\epsilon_1'}{12}-\frac{5\epsilon_2'}{12}+\frac{47\sigma_1}{24}+\frac{23(1-\mu)\sigma_1}{12\mu}+\frac{17\sigma_2}{12}-\frac{(1-\mu)\sigma_2}{6\mu}+\frac{41\sigma_1'}{16}\right.$$
$$\left.+\frac{115\mu\sigma_1'}{24(1-\mu)}-\frac{7\sigma_2'}{8}-\frac{119\mu\sigma_2'}{32(1-\mu)}\right\}+\frac{1}{2}q_2^2\left\{\frac{5}{4}+\frac{11\epsilon_1'}{4}-\frac{3\epsilon_2'}{4}+\frac{215\sigma_1}{24}-\frac{3(1-\mu)\sigma_1}{4\mu}\right.$$
$$\left.-\frac{337\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{137\sigma_1'}{16}+\frac{89\mu\sigma_1'}{32(1-\mu)}-\frac{21\sigma_2'}{4}-\frac{215\mu\sigma_2'}{32(1-\mu)}\right\}$$
$$+\frac{1}{6}q_1^3\left\{\frac{21}{8}+\frac{31\epsilon_1'}{16}-\frac{7\epsilon_2'}{6}+\frac{91\sigma_1}{32}+\frac{7(1-\mu)\sigma_1}{3\mu}-\frac{445\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}\right.$$
$$\left.+\frac{87\sigma_1'}{32}+\frac{195\mu\sigma_1'}{32(1-\mu)}-\frac{93\sigma_2'}{16}-\frac{273\mu\sigma_2'}{32(1-\mu)}\right\}-\frac{\sqrt{3}}{2}q_1^2q_2\left\{\frac{3}{8}-\frac{17\epsilon_1'}{24}+\frac{5\epsilon_2'}{12}-\frac{119\sigma_1}{24}\right.$$
$$\left.-\frac{47(1-\mu)\sigma_1}{12\mu}-\frac{57\sigma_2}{12}+\frac{(1-\mu)\sigma_2}{12\mu}-\frac{97\sigma_1'}{32}-\frac{267\mu\sigma_1'}{24(1-\mu)}+\frac{3\sigma_2'}{8}+\frac{337\mu\sigma_2'}{32(1-\mu)}\right\}$$

$$
\begin{aligned}
&+\frac{1}{2}q_1q_2^2\left\{-\frac{33}{8}-\frac{43\epsilon_1'}{8}+\frac{7\epsilon_2'}{12}+\frac{257\sigma_1}{24}+\frac{117(1-\mu)\sigma_1}{24\mu}-\frac{117\sigma_2}{24}+\frac{(1-\mu)\sigma_2}{6\mu}\right.\\
&\left.+\frac{167\sigma_1'}{32}+\frac{441\mu\sigma_1'}{24(1-\mu)}-\frac{7\sigma_2'}{16}-\frac{1137\mu\sigma_2'}{32(1-\mu)}\right\}+\frac{\sqrt{3}}{6}q_2^3\left\{-\frac{9}{8}-\frac{23\epsilon_1'}{8}+\frac{25\epsilon_2'}{12}+\frac{147\sigma_1}{32}\right.\\
&\left.+\frac{45(1-\mu)\sigma_1}{24\mu}-\frac{665\sigma_2}{32}+\frac{7(1-\mu)\sigma_2}{4\mu}+\frac{171\sigma_1'}{16}+\frac{111\mu\sigma_1'}{64(1-\mu)}-\frac{37\sigma_2'}{4}-\frac{487\mu\sigma_2'}{64(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{-\frac{111}{16}-\frac{85\epsilon_1'}{16}+\frac{25\epsilon_2'}{2}+\frac{475\sigma_1}{64}+\frac{13(1-\mu)\sigma_1}{4\mu}-\frac{255\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{2\mu}\right.\\
&\left.+\frac{135\sigma_1'}{2}+\frac{157\mu\sigma_1'}{32(1-\mu)}-\frac{695\sigma_2'}{8}-\frac{1171\mu\sigma_2'}{32(1-\mu)}\right\}+\frac{\sqrt{3}}{6}q_1^3q_2\left\{-\frac{75}{16}-\frac{535\epsilon_1'}{48}+\frac{20\epsilon_2'}{3}\right.\\
&+\frac{817\sigma_1}{64}+\frac{25(1-\mu)\sigma_1}{4\mu}-\frac{1577\sigma_2}{64}+\frac{13(1-\mu)\sigma_2}{4\mu}+\frac{207\sigma_1'}{4}+\frac{1323\mu\sigma_1'}{64(1-\mu)}\\
&\left.-\frac{1431\sigma_2'}{24}-\frac{1979\mu\sigma_2'}{64(1-\mu)}\right\}+\frac{1}{4}q_1^2q_2^2\left\{\frac{123}{16}+\frac{95\epsilon_1'}{24}+\frac{20\epsilon_2'}{2}+\frac{983\sigma_1}{64}+\frac{21(1-\mu)\sigma_1}{4\mu}\right.\\
&\left.-\frac{1767\sigma_2}{64}+\frac{17(1-\mu)\sigma_2}{4\mu}+\frac{337\sigma_1'}{4}+\frac{1737\mu\sigma_1'}{64(1-\mu)}-\frac{1507\sigma_2'}{24}-\frac{2319\mu\sigma_2'}{64(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1q_2^3\left\{\frac{135}{16}+\frac{235\epsilon_1'}{16}-\frac{7\epsilon_2'}{2}-\frac{889\sigma_1}{64}+\frac{17(1-\mu)\sigma_1}{4\mu}-\frac{2509\sigma_2}{64}+\frac{37(1-\mu)\sigma_2}{8\mu}\right.\\
&\left.+\frac{367\sigma_1'}{4}+\frac{1867\mu\sigma_1'}{64(1-\mu)}-\frac{1031\sigma_2'}{24}-\frac{2639\mu\sigma_2'}{64(1-\mu)}\right\}+\frac{1}{24}q_2^4\left\{\frac{9}{16}+\frac{195\epsilon_1'}{16}-\frac{45\epsilon_2'}{2}+\frac{435\sigma_1}{24}\right.\\
&\left.\left.-\frac{7(1-\mu)\sigma_1}{8\mu}-\frac{767\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{303\sigma_1'}{16}+\frac{117\mu\sigma_1'}{32(1-\mu)}-\frac{19\sigma_2'}{4}-\frac{377\mu\sigma_2'}{32(1-\mu)}\right\}\right]
\end{aligned}
\tag{2.31}
$$

Substituting values in equation (2.26), we get

$$
\begin{aligned}
r_2^{-1}=&\left[1+\frac{\epsilon_1'}{12}+\frac{5\epsilon_2'}{12}+\frac{7\sigma_1}{16}-\frac{(1-\mu)\sigma_1}{2\mu}-\frac{\sigma_2}{2}+\frac{(1-\mu)\sigma_2}{2\mu}+\frac{11\sigma_1'}{8}+\frac{3\mu\sigma_1'}{8(1-\mu)}-\frac{11\sigma_2'}{8}-\frac{\mu\sigma_2'}{4(1-\mu)}\right.\\
&+q_1\left\{\frac{1}{2}+\frac{11\epsilon_1'}{24}+\frac{7\epsilon_2'}{24}+\frac{69\sigma_1}{32}-\frac{(1-\mu)\sigma_1}{4\mu}-\frac{13\sigma_2}{8}+\frac{(1-\mu)\sigma_2}{4\mu}+\frac{27\sigma_1'}{16}+\frac{3\mu\sigma_1'}{8(1-\mu)}\right.\\
&\left.-\frac{19\sigma_2'}{16}-\frac{7\mu\sigma_2'}{8(1-\mu)}\right\}-\frac{\sqrt{3}}{2}q_2\left\{1-\frac{\epsilon_1'}{12}+\frac{11\epsilon_2'}{12}-\frac{13\sigma_1}{48}-\frac{7(1-\mu)\sigma_1}{6\mu}-\frac{\sigma_2}{4}+\frac{7(1-\mu)\sigma_2}{6\mu}\right.\\
&\left.+\frac{61\sigma_1'}{24}-\frac{\mu\sigma_1'}{4(1-\mu)}-\frac{23\sigma_2'}{8}+\frac{7\mu\sigma_2'}{12(1-\mu)}\right\}+\frac{1}{2}q_1^2\left\{\frac{-1}{4}+\frac{37\epsilon_1'}{24}+\frac{19\epsilon_2'}{24}+\frac{137\sigma_1}{32}-\frac{3(1-\mu)\sigma_1}{8\mu}\right.\\
&\left.-\frac{53\sigma_2}{16}+\frac{7(1-\mu)\sigma_2}{8\mu}+\frac{119\sigma_1'}{32}+\frac{11\mu\sigma_1'}{8(1-\mu)}-\frac{57\sigma_2'}{32}-\frac{17\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\sqrt{3}q_1q_2\left\{\frac{-3}{4}-\frac{13\epsilon_1'}{24}+\frac{17\epsilon_2'}{12}+\frac{23\sigma_1}{48}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{5\sigma_2}{8}-\frac{(1-\mu)\sigma_2}{3\mu}-\frac{117\sigma_1'}{24}\right.\\
&\left.+\frac{7\mu\sigma_1'}{8(1-\mu)}+\frac{27\sigma_2'}{16}-\frac{\mu\sigma_2'}{6(1-\mu)}\right\}+\frac{1}{2}q_2^2\left\{\frac{5}{4}-\frac{117\epsilon_1'}{24}+\frac{97\epsilon_2'}{24}+\frac{257s_1}{32}-\frac{5(1-\mu)s_1}{8\mu}\right.\\
&\left.-\frac{331s_2}{32}+\frac{5(1-\mu)s_2}{8\mu}+\frac{453s_1'}{32}+\frac{37\mu s_1'}{16(1-\mu)}-\frac{107s_2'}{32}-\frac{33\mu s_2'}{16(1-\mu)}\right\}
\end{aligned}
$$

$$+\frac{1}{6}q_1^3\left\{-\frac{21}{8}+\frac{29\epsilon_1'}{24}+\frac{17\epsilon_2'}{24}+\frac{253s_1}{32}-\frac{7(1-\mu)s_1}{8\mu}+\frac{107s_2}{32}+\frac{7(1-\mu)s_2}{16\mu}+\frac{437s_1'}{64}\right.$$
$$\left.+\frac{27\mu s_1'}{16(1-\mu)}-\frac{117s_2'}{32}+\frac{43\mu s_2'}{32(1-\mu)}\right\}+\frac{\sqrt{3}}{2}q_1^2q_2\left\{\frac{-3}{8}+\frac{5\epsilon_1'}{12}+\frac{17\epsilon_2'}{12}-\frac{37s_1}{48}-\frac{(1-\mu)s_1}{3\mu}\right.$$
$$\left.-\frac{5s_2}{4}+\frac{(1-\mu)s_2}{3\mu}+\frac{167s_1'}{24}+\frac{5\mu s_1'}{8(1-\mu)}-\frac{47s_2'}{16}-\frac{11\mu s_2'}{12(1-\mu)}\right\}+\frac{1}{2}q_1q_2^2\left\{\frac{33}{8}-\frac{19\epsilon_1'}{24}-\frac{17\epsilon_2'}{24}\right.$$
$$\left.+\frac{431s_1}{32}-\frac{7(1-\mu)s_1}{8\mu}-\frac{89s_2}{16}+\frac{7(1-\mu)s_2}{8\mu}+\frac{257s_1'}{32}+\frac{29\mu s_1'}{16(1-\mu)}+\frac{107s_2'}{32}+\frac{19\mu s_2'}{16(1-\mu)}\right\}$$
$$+\frac{\sqrt{3}}{6}q_2^3\left\{\frac{-9}{8}+\frac{5\epsilon_1'}{24}-\frac{13\epsilon_2'}{24}-\frac{135s_1}{48}-\frac{11(1-\mu)s_1}{6\mu}+\frac{17s_2}{4}-\frac{5(1-\mu)s_2}{6\mu}-\frac{357s_1'}{48}\right.$$
$$\left.-\frac{35\mu s_1'}{16(1-\mu)}-\frac{137s_2'}{32}-\frac{139\mu s_2'}{24(1-\mu)}\right\}+\frac{1}{24}q_1^4\left\{\frac{-111}{16}-\frac{93\epsilon_1'}{24}-\frac{37\epsilon_2'}{24}+\frac{1285s_1}{32}\right.$$
$$\left.-\frac{33(1-\mu)s_1}{16\mu}-\frac{1019s_2}{32}+\frac{29(1-\mu)s_2}{16\mu}+\frac{2143s_1'}{64}+\frac{1179\mu s_1'}{32(1-\mu)}-\frac{3563s_2'}{64}-\frac{47\mu s_2'}{32(1-\mu)}\right\}$$
$$+\frac{\sqrt{3}}{2}q_1^3q_2\left\{\frac{75}{16}-\frac{25\epsilon_1'}{12}-\frac{107\epsilon_2'}{24}-\frac{317s_1}{48}+\frac{7(1-\mu)s_1}{6\mu}-\frac{57s_2}{8}-\frac{5(1-\mu)s_2}{6\mu}+\frac{2147s_1'}{48}\right.$$
$$\left.+\frac{35\mu s_1'}{16(1-\mu)}-\frac{3515s_2'}{32}-\frac{47\mu s_2'}{24(1-\mu)}\right\}+\frac{1}{4}q_1^2q_2^2\left\{\frac{123}{16}+\frac{167\epsilon_1'}{48}-\frac{57\epsilon_2'}{24}-\frac{1051s_1}{32}\right.$$
$$\left.-\frac{67(1-\mu)s_1}{32\mu}-\frac{5505s_2}{64}+\frac{217(1-\mu)s_2}{32\mu}+\frac{3347s_1'}{64}+\frac{2157\mu s_1'}{32(1-\mu)}-\frac{4017s_2'}{128}-\frac{197\mu s_2'}{64(1-\mu)}\right\}$$
$$+\frac{\sqrt{3}}{6}q_1q_2^3\left\{-\frac{135}{16}-\frac{45\epsilon_1'}{12}-\frac{155\epsilon_2'}{24}-\frac{2205s_1}{64}-\frac{147(1-\mu)s_1}{32\mu}-\frac{2115s_2}{64}+\frac{253(1-\mu)s_2}{32\mu}\right.$$
$$\left.-\frac{6825s_1'}{128}-\frac{879\mu s_1'}{32(1-\mu)}+\frac{7785s_2'}{128}-\frac{275\mu s_2'}{32(1-\mu)}\right\}+\frac{1}{24}q_2^4\left\{\frac{9}{16}-\frac{45\epsilon_1'}{4}-\frac{135\epsilon_2'}{24}+\frac{15425s_1}{64}\right.$$
$$\left.\left.-\frac{417(1-\mu)s_1}{32\mu}-\frac{14505s_2}{64}+\frac{357(1-\mu)s_2}{32\mu}-\frac{4975s_1'}{64}-\frac{509\mu s_1'}{32(1-\mu)}+\frac{17925s_2'}{128}-\frac{417\mu s_2'}{64(1-\mu)}\right\}\right]. \tag{2.32}$$

Substituting values in equation (2.27), we get

$$r_1^{-3}=\left[\left\{1+\frac{\epsilon_1'}{4}+\frac{5\epsilon_2'}{4}+\frac{21s_1}{16}-\frac{3(1-\mu)s_1}{2\mu}-\frac{3s_2}{2}+\frac{3(1-\mu)s_2}{2\mu}+\frac{33s_1'}{8}+\frac{3\mu s_1'}{4(1-\mu)}-\frac{33s_2'}{8}\right.\right.$$
$$\left.-\frac{3\mu s_2'}{4(1-\mu)}\right\}+q_1\left\{\frac{3}{2}+\frac{13\epsilon_1'}{8}+\frac{17\epsilon_2'}{8}+\frac{249s_1}{32}-\frac{9(1-\mu)s_1}{4\mu}-\frac{51s_2}{8}+\frac{9(1-\mu)s_2}{4\mu}+\frac{147s_1'}{16}\right.$$
$$\left.+\frac{9\mu s_1'}{8(1-\mu)}-\frac{123s_2'}{16}-\frac{21\mu s_2'}{8(1-\mu)}\right\}-\frac{3\sqrt{3}}{2}q_2\left\{1+\frac{\epsilon_1'}{2}-\frac{7\epsilon_2'}{6}+\frac{127s_1}{24}+\frac{(1-\mu)s_1}{3\mu}-\frac{45s_2}{8}\right.$$
$$\left.-\frac{(1-\mu)s_2}{3\mu}+\frac{41s_1'}{12}+\frac{13\mu s_1'}{8(1-\mu)}-\frac{5s_2'}{4}-\frac{91\mu s_2'}{24(1-\mu)}\right\}+\frac{1}{2}q_1^2\left\{\frac{3}{2}+\frac{13\epsilon_1'}{4}+\frac{31\epsilon_2'}{8}+\frac{427s_1}{32}\right.$$
$$\left.-\frac{21(1-\mu)s_1}{4\mu}-\frac{147s_2}{16}+\frac{21(1-\mu)s_2}{4\mu}+\frac{329s_1'}{32}+\frac{17\mu s_1'}{8(1-\mu)}-\frac{623s_2'}{32}-\frac{43\mu s_2'}{16(1-\mu)}\right\}$$

$$
\begin{aligned}
&-\frac{15\sqrt{3}}{4}q_1 q_2\left\{1-\frac{3\epsilon_1'}{4}+\frac{13\epsilon_2'}{12}+\frac{337s_1}{48}+\frac{5(1-\mu)s_1}{3\mu}-\frac{89s_2}{16}-\frac{5(1-\mu)s_2}{3\mu}+\frac{109s_1'}{24}\right.\\
&\left.+\frac{29\mu s_1'}{8(1-\mu)}-\frac{13s_2'}{4}-\frac{267\mu s_2'}{48(1-\mu)}\right\}+\frac{1}{2}q_2^2\left\{\frac{33}{4}+\frac{27\epsilon_1'}{8}-\frac{217\epsilon_2'}{12}+\frac{3879s_1}{48}+\frac{19(1-\mu)s_1}{6\mu}\right.\\
&\left.-\frac{1485s_2}{16}-\frac{19(1-\mu)s_2}{6\mu}+\frac{1057s_1'}{12}+\frac{429\mu s_1'}{16(1-\mu)}-\frac{135s_2'}{8}-\frac{1001\mu s_2'}{8(1-\mu)}\right\}+\frac{1}{6}q_1^3\left\{\frac{15}{4}+\frac{31\epsilon_1'}{8}\right.\\
&+\frac{47\epsilon_2'}{8}+\frac{3029s_1}{64}-\frac{43(1-\mu)s_1}{8\mu}-\frac{437s_2}{32}+\frac{81(1-\mu)s_2}{4\mu}+\frac{4255s_1'}{64}+\frac{33\mu s_1'}{8(1-\mu)}-\frac{11257s_2'}{64}\\
&\left.-\frac{169\mu s_2'}{16(1-\mu)}\right\}+\frac{15\sqrt{3}}{16}q_1^2q_2\left\{1-\frac{7\epsilon_1'}{8}+\frac{29\epsilon_2'}{12}+\frac{759s_1}{48}+\frac{13(1-\mu)s_1}{3\mu}-\frac{169s_2}{16}-\frac{7(1-\mu)s_2}{3\mu}\right.\\
&\left.+\frac{337s_1'}{24}+\frac{139\mu s_1'}{16(1-\mu)}-\frac{25s_2'}{4}-\frac{889\mu s_2'}{48(1-\mu)}\right\}+\frac{1}{2}q_1q_2^2\left\{\frac{47}{8}+\frac{33\epsilon_1'}{8}-\frac{167\epsilon_2'}{12}+\frac{3119s_1}{48}\right.\\
&\left.+\frac{37(1-\mu)s_1}{6\mu}-\frac{2177s_2}{32}-\frac{31(1-\mu)s_2}{6\mu}+\frac{2107s_1'}{24}+\frac{761\mu s_1'}{32(1-\mu)}-\frac{277s_2'}{16}-\frac{3709\mu s_2'}{16(1-\mu)}\right\}\\
&+\frac{15\sqrt{3}}{16}q_2^3\left\{1+\frac{21\epsilon_1'}{8}+\frac{91\epsilon_2'}{12}+\frac{2359s_1}{48}+\frac{65(1-\mu)s_1}{6\mu}-\frac{623s_2}{16}-\frac{65(1-\mu)s_2}{6\mu}+\frac{763s_1'}{24}\right.\\
&\left.+\frac{203\mu s_1'}{16(1-\mu)}-\frac{91s_2'}{8}-\frac{623\mu s_2'}{16(1-\mu)}\right\}+\frac{1}{24}q_1^4\left\{\frac{21}{8}+\frac{65\epsilon_1'}{8}+\frac{93\epsilon_2'}{8}+\frac{1037s_1}{32}+\frac{259(1-\mu)s_1}{12\mu}\right.\\
&\left.-\frac{107s_2}{8}-\frac{217(1-\mu)s_2}{12\mu}+\frac{177s_1'}{8}+\frac{667\mu s_1'}{16(1-\mu)}-\frac{279s_2'}{16}-\frac{4849\mu s_2'}{32(1-\mu)}\right\}\\
&-\frac{35\sqrt{3}}{16}q_1^3q_2\left\{1-\frac{15\epsilon_1'}{8}+\frac{57\epsilon_2'}{24}+\frac{2277s_1}{48}+\frac{23(1-\mu)s_1}{3\mu}-\frac{507s_2}{16}-\frac{20(1-\mu)s_2}{3\mu}\right.\\
&\left.+\frac{1011s_1'}{24}+\frac{197\mu s_1'}{16(1-\mu)}-\frac{65s_2'}{4}-\frac{2667\mu s_2'}{48(1-\mu)}\right\}+\frac{1}{4}q_1^2q_2^2\left\{\frac{185}{16}+\frac{79\epsilon_1'}{16}-\frac{301\epsilon_2'}{24}+\frac{9359s_1}{48}\right.\\
&\left.+\frac{61(1-\mu)s_1}{12\mu}-\frac{6789s_2}{32}-\frac{87(1-\mu)s_2}{12\mu}+\frac{3965s_1'}{24}+\frac{1989\mu s_1'}{32(1-\mu)}-\frac{405s_2'}{16}-\frac{8757\mu s_2'}{16(1-\mu)}\right\}\\
&+\frac{55\sqrt{3}}{32}q_1q_2^3\left\{1+\frac{35\epsilon_1'}{8}-\frac{107\epsilon_2'}{24}-\frac{3997s_1}{48}+\frac{67(1-\mu)s_1}{6\mu}-\frac{1007s_2}{16}-\frac{47(1-\mu)s_2}{6\mu}\right.\\
&\left.+\frac{3415s_1'}{24}+\frac{307\mu s_1'}{16(1-\mu)}-\frac{111s_2'}{4}-\frac{4795\mu s_2'}{48(1-\mu)}\right\}+\frac{1}{24}q_2^4\left\{\frac{45}{16}+\frac{63\epsilon_1'}{8}+\frac{181\epsilon_2'}{12}-\frac{7079s_1}{48}\right.\\
&\left.\left.+\frac{155(1-\mu)s_1}{6\mu}-\frac{1869s_2}{16}-\frac{185(1-\mu)s_2}{6\mu}+\frac{1985s_1'}{24}+\frac{1407\mu s_1'}{16(1-\mu)}-\frac{207s_2'}{8}-\frac{3977\mu s_2'}{48(1-\mu)}\right\}\right]
\end{aligned}
\tag{2.33}
$$

Substituting values in equation (2.28), we get

$$
\begin{aligned}
r_2^{-3}=&\left[1-\frac{\epsilon_1'}{12}-\frac{5\epsilon_2'}{12}+\frac{5\sigma_1}{8}-\frac{3(1-\mu)\sigma_1}{2\mu}-\frac{3\sigma_2}{2}+\frac{3(1-\mu)\sigma_2}{2\mu}-\frac{15\sigma_1'}{8}+\frac{15\mu\sigma_1'}{8(1-\mu)}\right.\\
&+q_1\left\{\frac{3}{2}+\frac{13\epsilon_1'}{24}+\frac{17\epsilon_2'}{24}+\frac{147\sigma_1}{16}-\frac{3(1-\mu)\sigma_1}{4\mu}-\frac{123\sigma_2}{16}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{33\sigma_1'}{16}+\frac{7\mu\sigma_1'}{8(1-\mu)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{53\sigma_2'}{16}-\frac{3\mu\sigma_2'}{8(1-\mu)}\bigg\}-\frac{3\sqrt{3}}{2}q_2\bigg\{1-\frac{5\epsilon_1'}{12}+\frac{1\epsilon_2'}{12}-\frac{127\sigma_1}{8}-\frac{5(1-\mu)\sigma_1}{12\mu}-\frac{5\sigma_2}{4}+\frac{11(1-\mu)\sigma_2}{6\mu}\\
&+\frac{107\sigma_1'}{24}-\frac{5\mu\sigma_1'}{4(1-\mu)}-\frac{47\sigma_2'}{16}+\frac{5\mu\sigma_2'}{12(1-\mu)}\bigg\}+\frac{1}{2}q_1^2\bigg\{\frac{3}{4}-\frac{23\epsilon_1'}{24}-\frac{17\epsilon_2'}{24}+\frac{187\sigma_1}{32}-\frac{5(1-\mu)\sigma_1}{8\mu}\\
&+\frac{77\sigma_2}{16}-\frac{3(1-\mu)\sigma_2}{8\mu}+\frac{207\sigma_1'}{32}+\frac{19\mu\sigma_1'}{8(1-\mu)}-\frac{111\sigma_2'}{32}-\frac{23\mu\sigma_2'}{16(1-\mu)}\bigg\}\\
&-\frac{15\sqrt{3}}{4}q_1q_2\bigg\{1-\frac{55\epsilon_1'}{24}+\frac{41\epsilon_2'}{12}-\frac{41\sigma_1}{48}-\frac{5(1-\mu)\sigma_1}{3\mu}-\frac{25\sigma_2}{8}+\frac{5(1-\mu)\sigma_2}{3\mu}+\frac{215\sigma_1'}{24}\\
&-\frac{5\mu\sigma_1'}{8(1-\mu)}-\frac{41\sigma_2'}{16}-\frac{5\mu\sigma_2'}{6(1-\mu)}\bigg\}+\frac{1}{2}q_2^2\bigg\{\frac{33}{4}-\frac{207\epsilon_1'}{24}+\frac{167\epsilon_2'}{24}+\frac{441\sigma_1}{32}-\frac{7(1-\mu)\sigma_1}{8\mu}\\
&-\frac{715\sigma_2}{32}+\frac{25(1-\mu)\sigma_2}{8\mu}-\frac{1063\sigma_1'}{32}-\frac{67\mu\sigma_1'}{16(1-\mu)}-\frac{301\sigma_2'}{32}-\frac{41\mu\sigma_2'}{16(1-\mu)}\bigg\}\\
&+\frac{1}{6}q_1^3\bigg\{-\frac{75}{8}-\frac{41\epsilon_1'}{24}-\frac{27\epsilon_2'}{48}-\frac{417\sigma_1}{32}+\frac{5(1-\mu)\sigma_1}{8\mu}+\frac{157\sigma_2}{32}+\frac{7(1-\mu)\sigma_2}{8\mu}-\frac{889\sigma_1'}{64}\\
&-\frac{91\mu\sigma_1'}{16(1-\mu)}+\frac{401\sigma_2'}{32}-\frac{61\mu\sigma_2'}{32(1-\mu)}\bigg\}+\frac{\sqrt{3}}{2}q_1^2q_2\bigg\{-\frac{48}{8}+\frac{7\epsilon_1'}{12}+\frac{35\epsilon_2'}{12}+\frac{59\sigma_1}{48}+\frac{5(1-\mu)\sigma_1}{3\mu}\\
&+\frac{11\sigma_2}{4}-\frac{7(1-\mu)\sigma_2}{3\mu}-\frac{297\sigma_1'}{24}-\frac{11\mu\sigma_1'}{8(1-\mu)}+\frac{89\sigma_2'}{16}+\frac{17\mu\sigma_2'}{12(1-\mu)}\bigg\}\\
&+\frac{1}{2}q_1q_2^2\bigg\{\frac{255}{8}-\frac{37\epsilon_1'}{24}-\frac{19\epsilon_2'}{24}+\frac{1057\sigma_1}{32}-\frac{11(1-\mu)\sigma_1}{8\mu}-\frac{267\sigma_2}{16}+\frac{35(1-\mu)\sigma_2}{8\mu}+\frac{307\sigma_1'}{32}\\
&+\frac{31\mu\sigma_1'}{16(1-\mu)}+\frac{517\sigma_2'}{32}+\frac{31\mu\sigma_2'}{16(1-\mu)}\bigg\}+\frac{\sqrt{3}}{6}q_2^3\bigg\{-\frac{135}{8}+\frac{11\epsilon_1'}{24}-\frac{17\epsilon_2'}{24}-\frac{337\sigma_1}{48}\\
&-\frac{43(1-\mu)\sigma_1}{6\mu}+\frac{27\sigma_2}{8}-\frac{17(1-\mu)\sigma_2}{6\mu}+\frac{1437\sigma_1'}{48}-\frac{53\mu\sigma_1'}{16(1-\mu)}-\frac{403\sigma_2'}{32}-\frac{477\mu\sigma_2'}{24(1-\mu)}\bigg\}\\
&+\frac{1}{24}q_1^4\bigg\{-\frac{855}{16}-\frac{227\epsilon_1'}{24}-\frac{134\epsilon_2'}{24}+\frac{3379\sigma_1}{32}-\frac{109(1-\mu)\sigma_1}{16\mu}+\frac{4133\sigma_2}{32}-\frac{67(1-\mu)\sigma_2}{16\mu}\\
&-\frac{8167\sigma_1'}{64}+\frac{4555\mu\sigma_1'}{32(1-\mu)}-\frac{11345\sigma_2'}{64}-\frac{177\mu\sigma_2'}{32(1-\mu)}\bigg\}+\frac{\sqrt{3}}{6}q_1^3q_2\bigg\{\frac{315}{16}-\frac{57\epsilon_1'}{12}-\frac{257\epsilon_2'}{24}-\frac{789\sigma_1}{48}\\
&+\frac{11(1-\mu)\sigma_1}{6\mu}-\frac{189\sigma_2}{8}-\frac{11(1-\mu)\sigma_2}{6\mu}+\frac{7345\sigma_1'}{48}+\frac{111\mu\sigma_1'}{16(1-\mu)}-\frac{8877\sigma_2'}{32}-\frac{137\mu\sigma_2'}{24(1-\mu)}\bigg\}\\
&+\frac{1}{4}q_1^2q_2^2\bigg\{\frac{1395}{16}+\frac{335\epsilon_1'}{48}-\frac{127\epsilon_2'}{24}-\frac{3119\sigma_1}{32}-\frac{189(1-\mu)\sigma_1}{32\mu}-\frac{19567\sigma_2}{64}+\frac{689(1-\mu)\sigma_2}{32\mu}\\
&+\frac{11537\sigma_1'}{64}+\frac{8329\mu\sigma_1'}{32(1-\mu)}-\frac{18469\sigma_2'}{128}-\frac{945\mu\sigma_2'}{64(1-\mu)}\bigg\}+\frac{\sqrt{3}}{6}q_1q_2^3\bigg\{-\frac{2835}{16}-\frac{187\epsilon_1'}{12}-\frac{461\epsilon_2'}{24}\\
&-\frac{9195\sigma_1}{64}-\frac{607(1-\mu)\sigma_1}{32\mu}-\frac{8585\sigma_2}{64}+\frac{785(1-\mu)\sigma_2}{32\mu}-\frac{37927\sigma_1'}{128}-\frac{4233\mu\sigma_1'}{32(1-\mu)}+\frac{33935\sigma_2'}{128}\\
&-\frac{837\mu\sigma_2'}{32(1-\mu)}\bigg\}+\frac{1}{24}q_2^4\bigg\{\frac{1665}{16}-\frac{197\epsilon_1'}{4}-\frac{337\epsilon_2'}{24}+\frac{1037\sigma_1}{64}-\frac{2527(1-\mu)\sigma_1}{32\mu}-\frac{67577\sigma_2}{64}
\end{aligned}
$$

$$+\frac{2307(1-\mu)\sigma_2}{32\mu}-\frac{28957\sigma_1'}{64}-\frac{4337\mu\sigma_1'}{32(1-\mu)}+\frac{141727\sigma_2'}{128}-\frac{3237\mu\sigma_2'}{64(1-\mu)}\Bigg\}\Bigg]. \tag{2.34}$$

Substituting values in equation (2.29), we get

$$\begin{aligned}
r_1^{-5} = \Bigg[& 1+\frac{5\epsilon_1'}{12}+\frac{25\epsilon_2'}{12}+\frac{35\sigma_1}{16}-\frac{5(1-\mu)\sigma_1}{2\mu}-\frac{5\sigma_2}{2}+\frac{5(1-\mu)\sigma_2}{2\mu}+\frac{55\sigma_1'}{8}-\frac{55\sigma_2'}{8} \\
& +q_1\Bigg\{\frac{5}{2}+\frac{25\epsilon_1'}{8}+\frac{45\epsilon_2'}{8}+\frac{485\sigma_1}{32}-\frac{25(1-\mu)\sigma_1}{4\mu}-\frac{105\sigma_2}{8}+\frac{25(1-\mu)\sigma_2}{4\mu}+\frac{355\sigma_1'}{16} \\
& -\frac{3\mu\sigma_1'}{8(1-\mu)}-\frac{315\sigma_2'}{16}+\frac{7\mu\sigma_2'}{8(1-\mu)}\Bigg\}-\frac{5\sqrt{3}}{2}q_2\Bigg\{1+\frac{5\epsilon_1'}{6}-\frac{3\epsilon_2'}{2}+\frac{193\sigma_1}{24}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{67\sigma_2}{8} \\
& -\frac{(1-\mu)\sigma_2}{3\mu}+\frac{65\sigma_1'}{12}+\frac{19\mu\sigma_1'}{8(1-\mu)}-\frac{9\sigma_2'}{4}-\frac{133\mu\sigma_2'}{24(1-\mu)}\Bigg\}+\frac{1}{2}q_1^2\Bigg\{\frac{15}{4}+\frac{43\epsilon_1'}{8}+\frac{63\epsilon_2'}{8}+\frac{777\sigma_1}{32} \\
& -\frac{59(1-\mu)\sigma_1}{8\mu}-\frac{615\sigma_2}{8}+\frac{77(1-\mu)\sigma_2}{8\mu}+\frac{879\sigma_1'}{16}-\frac{11\mu\sigma_1'}{8(1-\mu)}-\frac{757\sigma_2'}{16}+\frac{13\mu\sigma_2'}{8(1-\mu)}\Bigg\} \\
& -\frac{35\sqrt{3}}{4}q_1q_2\Bigg\{1-\frac{7\epsilon_1'}{4}+\frac{21\epsilon_2'}{12}+\frac{1067\sigma_1}{48}+\frac{25(1-\mu)\sigma_1}{3\mu}-\frac{227\sigma_2}{16}-\frac{28(1-\mu)\sigma_2}{3\mu}+\frac{529\sigma_1'}{24} \\
& +\frac{43\mu\sigma_1'}{8(1-\mu)}-\frac{37\sigma_2'}{4}-\frac{723\mu\sigma_2'}{48(1-\mu)}\Bigg\}+\frac{1}{2}q_2^2\Bigg\{\frac{85}{4}+\frac{189\epsilon_1'}{16}-\frac{1519\epsilon_2'}{24}+\frac{27153\sigma_1}{48}+\frac{133(1-\mu)\sigma_1}{12\mu} \\
& -\frac{10395\sigma_2}{32}-\frac{133(1-\mu)\sigma_2}{12\mu}+\frac{7399\sigma_1'}{12}+\frac{10003\mu\sigma_1'}{32(1-\mu)}-\frac{945\sigma_2'}{8}-\frac{6789\mu\sigma_2'}{8(1-\mu)}\Bigg\} \\
& +\frac{1}{6}q_1^3\Bigg\{\frac{105}{8}+\frac{217\epsilon_1'}{16}+\frac{329\epsilon_2'}{16}+\frac{21203\sigma_1}{128}-\frac{301(1-\mu)\sigma_1}{16\mu}-\frac{3059\sigma_2}{64}+\frac{567(1-\mu)\sigma_2}{8\mu} \\
& +\frac{29785\sigma_1'}{128}+\frac{231\mu\sigma_1'}{16(1-\mu)}-\frac{78799\sigma_2'}{128}-\frac{1183\mu\sigma_2'}{32(1-\mu)}\Bigg\}+\frac{105\sqrt{3}}{16}q_1^2q_2\Bigg\{1-\frac{13\epsilon_1'}{8}+\frac{47\epsilon_2'}{12} \\
& +\frac{2277\sigma_1}{96}+\frac{39(1-\mu)\sigma_1}{6\mu}-\frac{507\sigma_2}{32}-\frac{21(1-\mu)\sigma_2}{6\mu}+\frac{1011\sigma_1'}{48}+\frac{417\mu\sigma_1'}{32(1-\mu)}-\frac{75\sigma_2'}{8}-\frac{2667\mu\sigma_2'}{96(1-\mu)}\Bigg\} \\
& +\frac{1}{2}q_1q_2^2\Bigg\{\frac{141}{16}+\frac{99\epsilon_1'}{16}-\frac{501\epsilon_2'}{24}+\frac{9357\sigma_1}{48}+\frac{111(1-\mu)\sigma_1}{12\mu}-\frac{6531\sigma_2}{32}-\frac{93(1-\mu)\sigma_2}{12\mu} \\
& +\frac{6321\sigma_1'}{24}+\frac{2283\mu\sigma_1'}{64(1-\mu)}-\frac{831\sigma_2'}{16}-\frac{11127\mu\sigma_2'}{32(1-\mu)}\Bigg\}+\frac{45\sqrt{3}}{32}q_2^3\Bigg\{1+\frac{63\epsilon_1'}{8}+\frac{91\epsilon_2'}{4}+\frac{2359\sigma_1}{16} \\
& +\frac{65(1-\mu)\sigma_1}{2\mu}-\frac{1869\sigma_2}{16}-\frac{65(1-\mu)\sigma_2}{2\mu}+\frac{763\sigma_1'}{8}+\frac{609\mu\sigma_1'}{16(1-\mu)}-\frac{273\sigma_2'}{8}-\frac{1869\mu\sigma_2'}{32(1-\mu)}\Bigg\} \\
& +\frac{1}{24}q_1^4\Bigg\{\frac{147}{16}+\frac{455\epsilon_1'}{16}+\frac{651\epsilon_2'}{16}+\frac{7259\sigma_1}{64}+\frac{1813(1-\mu)\sigma_1}{24\mu}-\frac{749\sigma_2}{16}-\frac{1519(1-\mu)\sigma_2}{24\mu} \\
& +\frac{1239\sigma_1'}{16}+\frac{4669\mu\sigma_1'}{32(1-\mu)}-\frac{1953\sigma_2'}{32}-\frac{33943\mu\sigma_2'}{64(1-\mu)}\Bigg\}-\frac{245\sqrt{3}}{32}q_1^3q_2\Bigg\{1-\frac{33\epsilon_1'}{8}+\frac{47\epsilon_2'}{24}+\frac{6831\sigma_1}{48} \\
& +\frac{161(1-\mu)\sigma_1}{3\mu}-\frac{1521\sigma_2}{16}-\frac{26(1-\mu)\sigma_2}{3\mu}+\frac{1399\sigma_1'}{24}+\frac{591\mu\sigma_1'}{16(1-\mu)}-\frac{455\sigma_2'}{4}-\frac{4937\mu\sigma_2'}{48(1-\mu)}\Bigg\} \\
& +\frac{1}{4}q_1^2q_2^2\Bigg\{\frac{1295}{32}+\frac{277\epsilon_1'}{16}-\frac{1053\epsilon_2'}{24}+\frac{32757\sigma_1}{48}+\frac{197(1-\mu)\sigma_1}{12\mu}-\frac{23761\sigma_2}{32}-\frac{271(1-\mu)\sigma_2}{12\mu}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{12291\sigma_1'}{32}+\frac{6921\mu\sigma_1'}{32(1-\mu)}-\frac{1417\sigma_2'}{16}-\frac{30561\mu\sigma_2'}{16(1-\mu)}\bigg\}+\frac{385\sqrt{3}}{64}q_1q_2^3\bigg\{1+\frac{67\epsilon_1'}{8}-\frac{219\epsilon_2'}{24}\\
&-\frac{5719\sigma_1}{48}+\frac{101(1-\mu)\sigma_1}{6\mu}-\frac{2717\sigma_2}{16}-\frac{71(1-\mu)\sigma_2}{6\mu}+\frac{6787\sigma_1'}{24}+\frac{461\mu\sigma_1'}{16(1-\mu)}-\frac{155\sigma_2'}{4}\\
&-\frac{7193\mu\sigma_2'}{48(1-\mu)}\bigg\}+\frac{1}{24}q_2^4\bigg\{\frac{315}{32}+\frac{441\epsilon_1'}{8}+\frac{1267\epsilon_2'}{12}-\frac{49553\sigma_1}{48}+\frac{1085(1-\mu)\sigma_1}{6\mu}-\frac{13083\sigma_2}{16}\\
&-\frac{1295(1-\mu)\sigma_2}{6\mu}+\frac{13895\sigma_1'}{24}+\frac{9849\mu\sigma_1'}{16(1-\mu)}-\frac{1449\sigma_2'}{8}-\frac{27839\mu\sigma_2'}{48(1-\mu)}\bigg\}\bigg]. \qquad (2.35)
\end{aligned}
$$

Substituting values in equation (2.30), we get

$$
\begin{aligned}
r_2^{-5}=\bigg[&\frac{3}{4}-\frac{5\epsilon_1'}{24}-\frac{7\epsilon_2'}{12}+\frac{19\sigma_1}{16}-\frac{5(1-\mu)\sigma_1}{2\mu}-\frac{5\sigma_2}{2}+\frac{7(1-\mu)\sigma_2}{2\mu}-\frac{31\sigma_1'}{8}+\frac{31\sigma_2'}{8}\\
&+q_1\bigg\{\frac{5}{2}+\frac{19\epsilon_1'}{24}+\frac{41\epsilon_2'}{24}+\frac{195\sigma_1}{16}-\frac{11(1-\mu)\sigma_1}{4\mu}-\frac{517\sigma_2}{16}+\frac{11(1-\mu)\sigma_2}{4\mu}+\frac{67\sigma_1'}{16}+\frac{29\mu\sigma_1'}{16(1-\mu)}\\
&-\frac{119\sigma_2'}{16}-\frac{9\mu\sigma_2'}{8(1-\mu)}\bigg\}-\frac{5\sqrt{3}}{2}q_2\bigg\{1-\frac{13\epsilon_1'}{12}+\frac{5\epsilon_2'}{12}-\frac{419\sigma_1}{8}-\frac{17(1-\mu)\sigma_1}{12\mu}-\frac{15\sigma_2}{4}\\
&+\frac{29(1-\mu)\sigma_2}{6\mu}+\frac{567\sigma_1'}{24}-\frac{13\mu\sigma_1'}{4(1-\mu)}-\frac{177\sigma_2'}{16}+\frac{11\mu\sigma_2'}{12(1-\mu)}\bigg\}+\frac{1}{2}q_1^2\bigg\{\frac{45}{16}-\frac{47\epsilon_1'}{24}-\frac{35\epsilon_2'}{24}\\
&+\frac{417\sigma_1}{32}-\frac{11(1-\mu)\sigma_1}{8\mu}+\frac{219\sigma_2}{16}-\frac{7(1-\mu)\sigma_2}{8\mu}+\frac{795\sigma_1'}{32}+\frac{77\mu\sigma_1'}{8(1-\mu)}-\frac{667\sigma_2'}{32}-\frac{87\mu\sigma_2'}{16(1-\mu)}\bigg\}\\
&-\frac{35\sqrt{3}}{4}q_1q_2\bigg\{1-\frac{115\epsilon_1'}{24}+\frac{71\epsilon_2'}{12}-\frac{67\sigma_1}{68}-\frac{13(1-\mu)\sigma_1}{3\mu}-\frac{35\sigma_2}{8}+\frac{13(1-\mu)\sigma_2}{3\mu}+\frac{417\sigma_1'}{24}\\
&-\frac{13\mu\sigma_1'}{8(1-\mu)}-\frac{67\sigma_2'}{16}-\frac{13\mu\sigma_2'}{6(1-\mu)}\bigg\}+\frac{1}{2}q_2^2\bigg\{-\frac{35}{8}+\frac{317\epsilon_1'}{24}-\frac{495\epsilon_2'}{24}-\frac{1069\sigma_1}{32}-\frac{13(1-\mu)\sigma_1}{8\mu}\\
&-\frac{2023\sigma_2}{32}+\frac{13(1-\mu)\sigma_2}{8\mu}-\frac{3379\sigma_1'}{32}-\frac{219\mu\sigma_1'}{16(1-\mu)}-\frac{519\sigma_2'}{32}-\frac{179\mu\sigma_2'}{16(1-\mu)}\bigg\}+\frac{1}{6}q_1^3\bigg\{\frac{105}{8}-\frac{93\epsilon_1'}{24}\\
&-\frac{109\epsilon_2'}{48}-\frac{1569\sigma_1}{32}+\frac{13(1-\mu)\sigma_1}{8\mu}+\frac{397\sigma_2}{32}+\frac{11(1-\mu)\sigma_2}{8\mu}-\frac{2679\sigma_1'}{64}-\frac{417\mu\sigma_1'}{16(1-\mu)}\\
&+\frac{1337\sigma_2'}{32}-\frac{237\mu\sigma_2'}{32(1-\mu)}\bigg\}-\frac{175\sqrt{3}}{16}q_1^2q_2\bigg\{1+\frac{13\epsilon_1'}{24}+\frac{23\epsilon_2'}{12}+\frac{47\sigma_1}{48}+\frac{11(1-\mu)\sigma_1}{3\mu}+\frac{17\sigma_2}{4}\\
&-\frac{11(1-\mu)\sigma_2}{3\mu}-\frac{403\sigma_1'}{24}-\frac{17\mu\sigma_1'}{8(1-\mu)}+\frac{107\sigma_2'}{16}+\frac{23\mu\sigma_2'}{12(1-\mu)}\bigg\}+\frac{1}{2}q_1q_2^2\bigg\{-\frac{805}{8}+\frac{141\epsilon_1'}{24}+\frac{79\epsilon_2'}{24}\\
&-\frac{3169\sigma_1}{32}-\frac{17(1-\mu)\sigma_1}{8\mu}-\frac{597\sigma_2}{16}+\frac{105(1-\mu)\sigma_2}{8\mu}+\frac{1469\sigma_1'}{32}+\frac{89\mu\sigma_1'}{16(1-\mu)}+\frac{2395\sigma_2'}{32}\\
&+\frac{89\mu\sigma_2'}{16(1-\mu)}\bigg\}+\frac{\sqrt{3}}{6}q_2^3\bigg\{-\frac{525}{8}+\frac{41\epsilon_1'}{24}-\frac{65\epsilon_2'}{24}-\frac{1283\sigma_1}{48}-\frac{163(1-\mu)\sigma_1}{6\mu}+\frac{103\sigma_2}{8}\\
&-\frac{65(1-\mu)\sigma_2}{6\mu}+\frac{5461\sigma_1'}{48}-\frac{201\mu\sigma_1'}{16(1-\mu)}-\frac{1531\sigma_2'}{32}-\frac{1813\mu\sigma_2'}{24(1-\mu)}\bigg\}+\frac{1}{24}q_1^4\bigg\{-\frac{2415}{16}-\frac{641\epsilon_1'}{24}\\
&-\frac{389\epsilon_2'}{24}+\frac{19543\sigma_1}{32}-\frac{307(1-\mu)\sigma_1}{16\mu}+\frac{11673\sigma_2}{32}-\frac{189(1-\mu)\sigma_2}{16\mu}-\frac{23067\sigma_1'}{64}+\frac{12865\mu\sigma_1'}{32(1-\mu)}
\end{aligned}
$$

$$-\frac{32043\sigma_2'}{64}-\frac{999\mu\sigma_2'}{32(1-\mu)}\Big\}+\frac{\sqrt{3}}{6}q_1^3q_2\Big\{\frac{105}{64}-\frac{217\epsilon_1'}{12}-\frac{983\epsilon_2'}{24}-\frac{3017\sigma_1}{48}+\frac{43(1-\mu)\sigma_1}{6\mu}$$
$$-\frac{723\sigma_2}{8}-\frac{43(1-\mu)\sigma_2}{6\mu}+\frac{28085\sigma_1'}{48}+\frac{425\mu\sigma_1'}{16(1-\mu)}-\frac{33943\sigma_2'}{32}-\frac{523\mu\sigma_2'}{24(1-\mu)}\Big\}$$
$$+\frac{1}{4}q_1^2q_2^2\Big\{\frac{9825}{64}+\frac{589\epsilon_1'}{48}-\frac{223\epsilon_2'}{24}-\frac{5485\sigma_1}{32}-\frac{333(1-\mu)\sigma_1}{32\mu}-\frac{34405\sigma_2}{64}+\frac{1211(1-\mu)\sigma_2}{32\mu}$$
$$+\frac{20285\sigma_1'}{64}+\frac{14645\mu\sigma_1'}{32(1-\mu)}-\frac{32475\sigma_2'}{128}-\frac{1661\mu\sigma_2'}{64(1-\mu)}\Big\}+\frac{\sqrt{3}}{6}q_1q_2^3\Big\{-\frac{3885}{64}-\frac{403\epsilon_1'}{12}-\frac{995\epsilon_2'}{24}$$
$$-\frac{19841\sigma_1}{64}-\frac{1309(1-\mu)\sigma_1}{32\mu}-\frac{18525\sigma_2}{64}+\frac{1693(1-\mu)\sigma_2}{32\mu}-\frac{81843\sigma_1'}{128}-\frac{9135\mu\sigma_1'}{32(1-\mu)}$$
$$+\frac{73227\sigma_2'}{128}-\frac{1807\mu\sigma_2'}{32(1-\mu)}\Big\}+\frac{1}{24}q_2^4\Big\{\frac{10185}{16}-\frac{1205\epsilon_1'}{4}-\frac{2061\epsilon_2'}{24}+\frac{634547\sigma_1}{64}$$
$$-\frac{15457(1-\mu)\sigma_1}{32\mu}-\frac{413375\sigma_2}{64}+\frac{14112(1-\mu)\sigma_2}{32\mu}-\frac{177133\sigma_1'}{64}-\frac{26529\mu\sigma_1'}{32(1-\mu)}$$
$$+\frac{866959\sigma_2'}{128}-\frac{19801\mu\sigma_2'}{64(1-\mu)}\Big\}\Big]. \tag{2.36}$$

Substituting values of r_1^{-1}, r_2^{-1}, r_1^{-3}, r_2^{-3}, r_1^{-5} and r_2^{-5} and $e=0$ in equation (2.9) and taking second order terms. After this we get expression for Hamiltonian, which is represented as follows:

$$H_2=\frac{p_1^2+p_2^2}{2}+(p_1q_2-p_2q_2)+\left[\left(\frac{1}{8}+A\right)q_1^2-q_1q_2(K-B)-\left(\frac{5}{8}+C\right)q_2^2\right] \tag{2.37}$$

where

$$A=-\frac{\epsilon_1'}{48}+\frac{7\epsilon_2'}{48}-\frac{437\sigma_1}{64}+\frac{349\sigma_2}{64}-\frac{77\sigma_1'}{16}+\frac{15\sigma_2'}{16}+\frac{3\sigma_1}{4\mu}-\frac{3\sigma_2}{4\mu}-\frac{3\mu\epsilon_1'}{4}-\frac{5\mu\epsilon_2'}{12}$$
$$+\frac{57\mu\sigma_1}{16}-\frac{183\mu\sigma_2}{64}-\frac{21\mu\sigma_1'}{16}+\frac{15\mu\sigma_2'}{32}, \tag{2.38}$$

$$B=\sqrt{3}\Big\{-\frac{5\epsilon_1'}{6}+\frac{5\epsilon_2'}{12}+\frac{181\sigma_1}{24}-\frac{17\sigma_2}{4}-\frac{5\sigma_1'}{16}-\frac{\sigma_2'}{4}-\frac{23\sigma_1}{12\mu}-\frac{\sigma_2}{6\mu}+\frac{11\mu\epsilon_1'}{8}-\frac{13\mu\epsilon_2'}{12}$$
$$-\frac{401\mu\sigma_1}{48}+6\mu\sigma_2\frac{91\mu\sigma_1'}{48}+\frac{131\mu\sigma_2'}{96}\Big\}, \tag{2.39}$$

$$C=2\epsilon_1'-\frac{3\epsilon_2'}{8}+\frac{17\sigma_1}{3}-\frac{341\sigma_2}{64}+\frac{77\sigma_1'}{32}-\frac{27\sigma_2'}{16}-\frac{3\sigma_1}{8\mu}+\frac{3\sigma_2}{8\mu}-\frac{71\mu\epsilon_1'}{16}+\frac{145\mu\epsilon_2'}{48}$$
$$-\frac{605\mu\sigma_1}{192}+\frac{23\mu\sigma_2}{32}+\frac{115\mu\sigma_1'}{16}-\frac{107\mu\sigma_2'}{32} \tag{2.40}$$

and

$$K=\frac{3\sqrt{3}}{4}(1-2\mu). \tag{2.41}$$

The variational equation can be written for circular case as follows:

$$\dot{p}_i=-\frac{\partial H_2}{\partial q_i}\,\dot{q}_i=-\frac{\partial H_2}{\partial p_i},\quad i=1,2, \tag{2.42}$$

where H_2 is given by the equation (2.38). Hence the canonical equation for the circular problem is given by:

$$\ddot{q}_1 - 2\dot{q}_2 = A^* q_1 + B^* q_2,$$
$$\ddot{q}_2 - 2\dot{q}_1 = B^* q_1 + C^* q_2, \tag{2.43}$$

where $A^* = \frac{3}{4} - 2A$, $B^* = K - B$ and $C^* = \frac{9}{4} + 2A$ where, A, B, C and K are defined in equations (2.38), (2.39), (2.40) and (2.41).

Now, the characteristic equation for the problem can be defined by putting

$$q_1 = Le^{\lambda t}, \quad q_2 = Me^{\lambda t},$$
$$\dot{q}_1 = L\lambda e^{\lambda t}, \quad \dot{q}_2 = M\lambda\, e^{\lambda t}$$

and

$$\ddot{q}_1 = L\lambda^2 e^{\lambda t}, \quad \ddot{q}_2 = M\lambda^2 e^{\lambda t}. \tag{2.44}$$

The characteristic equation is obtained by substituting the result obtained from equation (2.44) in equation (2.43),

$$\begin{vmatrix} \lambda^2 - A^* & -2\lambda - B^* \\ 2\lambda - B^* & \lambda^2 - C^* \end{vmatrix} = 0. \tag{2.45}$$

By solving the equation (2.45),we get

$$\lambda^4 - \lambda^2 \left(A^* + C^* - 4\right) + \left(A^* C^* - B^{*^2}\right) = 0 \tag{2.46}$$

where

$$A^* + C^* - 4 = -1 \tag{2.47}$$

and

$$\begin{aligned} A^* C^* - B^{*^2} &= \frac{27}{16} - 3A - 4A^2 - (K - B)^2 \\ &= \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\}. \end{aligned} \tag{2.48}$$

The characteristic equation (2.46) has been reduced to the following form:

$$\lambda^4 + \lambda^2 + \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\} = 0 \tag{2.49}$$

It is notified that when $\sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0$, the characteristic equation (2.49) is showing the classical restricted three body problem:

Let $\lambda_1 = i\omega_1$ and $\lambda_2 = i\omega_2$,from equation (2.49), we get

$$\omega^4 - \omega^2 + \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\} = 0 \tag{2.50}$$

3. Stability of Triangular Equilibrium Points of the Problem in Elliptical Case

The resonance stability of triangular points for circular case in elliptical restricted three body problem under the assumption that, both the primaries are radiating and triaxial has been analyzed. In this paper the resonance stability for small eccentricity e near the resonance frequency $\omega_2 = \frac{1}{2}$ will be discussed. To discuss the stability, the method given by Markeev [8,9] has been adopted. In order to investigate the stability of triangular points, the suitable Hamiltonian up to second order is obtained as given below.

Substituting values of r_1^{-1}, r_2^{-1}, r_1^{-3}, r_2^{-3}, r_1^{-5} and r_2^{-5} in equation (2.9) and taking second order terms. After this we get expression for Hamiltonian, which is represented as follows:

$$
\begin{aligned}
H_2 &= \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_2) + \frac{e\cos\nu}{2(1+e\cos\nu)}\left(q_1^2+q_2^2\right) \\
&- \frac{1}{(1+e\cos\nu)}\Bigg[\Bigg\{-\frac{1}{8}+\frac{\epsilon_1'}{48}-\frac{7\epsilon_2'}{48}+\frac{437\sigma_1}{64}-\frac{349\sigma_2}{64}+\frac{77\sigma_1'}{16}-\frac{15\sigma_2'}{16}-\frac{3\sigma_1}{4\mu}+\frac{3\sigma_2}{4\mu} \\
&+\frac{3\mu\epsilon_1'}{4}+\frac{5\mu\epsilon_2'}{12}-\frac{57\mu\sigma_1}{16}+\frac{183\mu\sigma_2}{64}+\frac{21\mu\sigma_1'}{16}-\frac{15\mu\sigma_2'}{32}\Bigg\}q_1^2 \\
&+\sqrt{3}\Bigg\{\frac{3}{4}-\frac{3\mu}{2}+\frac{5\epsilon_1'}{6}-\frac{5\epsilon_2'}{12}-\frac{181\sigma_1}{24}+\frac{17\sigma_2}{4}+\frac{5\sigma_1'}{16}+\frac{\sigma_2'}{4}+\frac{23\sigma_1}{12\mu}+\frac{\sigma_2}{6\mu}+\frac{11\mu\epsilon_1'}{8} \\
&+\frac{13\mu\epsilon_2'}{12}+\frac{401\mu\sigma_1}{48}-6\mu\sigma_2-\frac{91\mu\sigma_1'}{48}-\frac{131\mu\sigma_2'}{96}\Bigg\}q_1q_2 \\
&+\Bigg\{\frac{5}{8}+2\epsilon_1'-\frac{3\epsilon_2'}{8}+\frac{17\sigma_1}{3}-\frac{341\sigma_2}{64}+\frac{77\sigma_1'}{32}-\frac{27\sigma_2'}{16}-\frac{3\sigma_1}{8\mu}+\frac{3\sigma_2}{8\mu}-\frac{71\mu\epsilon_1'}{16}+\frac{145\mu\epsilon_2'}{48} \\
&-\frac{605\mu\sigma_1}{192}+\frac{23\mu\sigma_2}{32}+\frac{115\mu\sigma_1'}{16}-\frac{107\mu\sigma_2'}{32}\Bigg\}\Bigg]
\end{aligned}
\tag{3.1}
$$

The Hamiltonian H_2 is expanded in powers of e up to the first approximation which gives:

$$
\begin{aligned}
H_2 &= \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_2) + \left[\left(\frac{1}{8}+A\right)q_1^2 - (K-B)q_1q_2 - \left(\frac{5}{8}+C\right)q_2^2\right] \\
&+ e\cos\nu\left[\left(\frac{3}{8}-A\right)q_1^2 - (K-B)q_1q_2 + \left(\frac{9}{8}+C\right)q_2^2\right]
\end{aligned}
\tag{3.2}
$$

where A, B, C and K are defined by equations (2.38), (2.39), (2.40) and (2.41).

Now taking the canonical transformation $[q_1, q_2, p_1, p_2]$ which transform in to $[q_1', q_2', p_1', p_2']N$

where

$$
N = \begin{bmatrix} a_1 & a_1c_1 & -a_1c_1 & a_1(1-\omega_1^2 b_1) \\ a_2 & a_2c_2 & -a_2c_2 & a_2(1-\omega_2^2 b_2) \\ 0 & a_1b_1 & a_1(1-b_1) & a_1c_1 \\ 0 & -a_2b_2 & -a_2(1-b_2) & -a_2c_2 \end{bmatrix}
\tag{3.3}
$$

which provides $[q_1, q_2, p_1, p_2] = [q_1', q_2', p_1', p_2']\, N$, also

$$q_1 = a_1 q_1' + a_2 q_2'$$

$$q_2 = a_1c_1q_1' - a_2c_2q_2' + a_1(1-b_1)p_1' - a_2(1-b_2)p_2'$$
$$p_1 = -a_1c_1q_1' - a_2c_2q_2' + a_1(1-b_1)p_1' - a_2(1-b_2)p_2'$$
$$p_2 = a_1(1-\omega_1^2b_1)q_1' + a_2(1-\omega_2^2b_2)q_2' + a_1c_1p_1' - a_2c_2p_2' \tag{3.4}$$
$$a_i = \frac{1}{2}\left[\frac{2I_i}{|\omega_i^2 - \frac{1}{2}|}\right]^{1/2} \quad ; \quad b_i = \frac{2}{I_i} \; ; \quad c_i = \frac{-(K+B)}{I_i} \tag{3.5}$$

and

$$I_i = \frac{9}{4} + 2A + \omega_i^2 \tag{3.6}$$

The values of q_1^2, q_2^2, q_1, q_2, p_1^2, p_2^2, p_2q_1, p_1q_2 are calculated and are substituted in equation (3.2). Also, the Hamiltonian H_2 is represented as $H_2 = H_2^{(0)} + H_2^{(1)}$, where $H_2^{(0)}$ showing the Hamiltonian independent from eccentricity and $H_2^{(1)}$ is the Hamiltonian containing the first order approximation in 'e'. The value of $H_2^{(0)}$ is given as:

$$H_2^{(0)} = \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_2) + (q_1^2+q_2^2) + \left[\left(\frac{1}{8}+A\right)q_1^2 - (K-B)q_1q_2 - \left(\frac{5}{8}+C\right)q_2^2\right]. \tag{3.7}$$

Also, we have

$$\begin{aligned}
H_2^{(0)} = &\left\{\frac{1}{2}a_1^2(1-b_1^2)^2 + \frac{1}{2}a_1^2c_1^2 + a_1^2b_1(1-b_1) - \left(\frac{5}{8}+C\right)a_1^2b_1^2\right\}p_1'^2 \\
&+\left\{\frac{1}{2}a_1^2c_1^2 + \frac{1}{2}a_1^2(1-b_1\omega_1^2)^2 - a_1^2c_1^2(1-b_1) - a_1^2(1-b_1\omega_1^2)\right. \\
&\quad \left. - \left(\frac{1}{8}+A\right)a_1^2 - \left(\frac{5}{8}+C\right)a_1^2c_1^2 - (K-B)a_1^2c_1^2\right\}q_1'^2 \\
&+\left\{\frac{1}{2}a_2^2(1-b_2)^2 + \frac{1}{2}a_2^2c_2^2 + a_2^2b_2(1-b_2) - \left(\frac{5}{8}+C\right)a_2^2b_2^2\right\}p_2'^2 \\
&+\left\{\frac{1}{2}a_2^2(1-b_2\omega_2^2) + \frac{3}{2}a_2^2c_2^2 - a_2^2(1-b_2\omega_2^2)\right. \\
&\quad \left. + \left(\frac{1}{8}+A\right)a_2^2 - \left(\frac{5}{8}+C\right)a_2^2c_2^2 - (K-B)a_2^2c_2^2\right\}q_2'^2 \\
&+\left\{a_1a_2b_1b_2 + a_1a_2(1-b_1\omega_1^2)(1-b_2\omega_2^2) - 2a_1a_2c_1c_2 - a_1a_2(1-b_2\omega_2^2)\right. \\
&\quad \left. - a_1a_2\left(1-b_1\omega_1^2\right) + 2\left(\frac{1}{8}+A\right)a_1a_2 - 2\left(\frac{5}{8}+C\right)a_1a_2c_1c_2 - (K-B)a_1a_2(c_1+c_2)\right\}q_1'q_2' \\
&+\left\{-a_1^2c_1(1-b_1) + a_1^2c_1(1-b_1\omega_1^2) + a_1^2c_2(1-b_1) - a_1^2b_1c_1 - a_1^2c_1\right. \\
&\quad \left. -2\left(\frac{5}{8}+C\right)a_1^2b_1c_1 - (K-B)a_1^2b_1\right\}p_1'q_1' \\
&+\left\{a_1a_2c_1(1-b_1) - a_1a_2c_1(1-b_1\omega_1^2) - a_1a_2c_2(1-b_1\omega_1^2) - a_1a_2c_1(1-b_2)\right. \\
&\quad \left. + a_1a_2c_2 + 2\left(\frac{5}{8}+C\right)a_1a_2b_2c_1 + (K-B)a_1a_2b_2\right\}q_1'p_2' \\
&+\left\{-a_1a_2c_2(1-b_1 + a_1a_2c_1(1-b_2\omega_2^2) + a_1a_2c_2(1-b_2) - a_1a_2b_1c_2\right.
\end{aligned}$$

$$-2\left(\frac{5}{8}+C\right)a_1a_2b_1c_2-(K-B)a_1a_2b_1-a_1a_2c_1\Big\}p_1'q_2'$$
$$+\Big\{a_1^2c_2(1-b_2)-a_1^2c_2\left(1-b_2\omega_2^2\right)-a_2^2c_2(1-b_2)+a_2^2b_2c_2$$
$$+2\left(\frac{5}{8}+C\right)a_2^2b_2c_2+(K-B)a_2^2b_2\Big\}p_2'q_2'$$
$$+\Big\{-a_1a_2(1-b_1)(1-b_2)-a_1a_2b_2(1-b_1)$$
$$+2\left(\frac{5}{8}+C\right)a_1a_2b_1b_2-a_1a_2b_1(1-b_2)\Big\}p_1'p_2'. \tag{3.8}$$

From equation (2.50) and (3.6) we have:

$$\omega_1^2\cdot\omega_2^2=\frac{27\mu(1-\mu)}{4}\left\{1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right\};$$
$$\omega_1^2+\omega_2^2=1$$

and

$$K^2=\frac{27}{16}-\frac{27\mu}{4}+\frac{27\mu^2}{4}$$
$$=\frac{27}{16}-\omega_1^2(1-\omega_1^2)\left\{1+\frac{\epsilon_1'}{16}-\frac{7\epsilon_2'}{16}+\frac{1311\sigma_1}{64}-\frac{1047\sigma_2}{64}+\frac{231\sigma_1'}{16}-\frac{45\sigma_2'}{16}\right\}. \tag{3.9}$$

The values of $H_2^{(1)}$ is evaluated up to sixth term as below:

$$H_2^{(1)}=e\cos v\left[\left\{\left(\frac{3}{8}+A\right)a_2^2+(K-B)a_2^2c_2+\left(\frac{9}{8}+C\right)a_2^2c_2^2\right\}q_2'^2+\left\{\left(\frac{9}{8}+C\right)a_2^2b_2^2\right\}p_2'^2\right.$$
$$+\left\{\left(\frac{3}{8}+A\right)a_1^2+(K-B)a_1^2c_1+\left(\frac{9}{8}+C\right)a_1^2c_1^2\right\}q_1'^2$$
$$+\left\{\left(\frac{3}{8}+A\right)2a_1a_2+(K-B)(a_1a_2c_1+a_1a_2c_2)+\left(\frac{9}{8}+C\right)2a_1a_2c_1c_2\right\}q_1'q_2'$$
$$+\left\{(K-B)a_1^2b_1+\left(\frac{9}{8}+C\right)2a_1^2b_1c_2\right\}p_1'q_1'$$
$$+\left\{(K-B)a_1a_2b_2-\left(\frac{9}{8}+C\right)2a_1a_2b_2c_1\right\}p_2'q_1'$$
$$+\left\{(K-B)a_1a_2b_1+\left(\frac{9}{8}+C\right)a_1a_2b_1c_2\right\}p_1'q_2'$$
$$\left.+\left(\frac{9}{8}+C\right)p_1'^2-\left\{\left(\frac{9}{8}+C\right)2a_1a_2b_1b_2\right\}p_1'p_2'\right]. \tag{3.10}$$

Hence $H_2^{(1)}=e\cos v[aq_2'^2+bp_2'^2+cq_2'p_2'+\ldots]$.

Here dash denotes the terms of second order in p_i' and q_i', which are not required in our further investigation. Putting values of $H_2^{(0)}$ and $H_2^{(1)}$ in equation (3.2),the normalised Hamiltonian function can be given as:

$$H_2=\frac{1}{2}t(p_1'^2+\omega_1^2q_1'^2)-\frac{1}{2}(p_1'^2+\omega_2^2q_2'^2)+e\cos v(aq_2'^2+bp_2'^2+cq_2'p_2'+\ldots), \tag{3.11}$$

where

$$a = \left(\frac{3}{8} - A\right)a_2^2 + (K-B)a_2^2 c_2 + \left(\frac{9}{8} + C\right)a_2^2 c_2^2 \,;$$

$$b = \left(\frac{9}{8} + C\right)a_2^2 b_2^2 \,; \tag{3.12}$$

$$c = \left\{-(K-B)a_2^2 b_2 - \left(\frac{9}{8} + A\right)2a_2^2 b_2 c_2\right\}.$$

Using transformations of the variables, given as:

$$q_1' = \frac{2\sqrt{\alpha_1}}{\omega_1}\sin\omega_1(\nu+\delta_1)\,;$$

$$q_2' = -\frac{2\sqrt{\alpha_2}}{\omega_2}\sin\omega_2(\nu-\delta_2)\,; \tag{3.13}$$

$$p_1' = 2\sqrt{\alpha_1}\,\cos\omega_1(\nu+\delta_1)\,;$$

$$p_2' = 2\sqrt{\alpha_2}\,\cos\omega_2(\nu-\delta_2)\,;$$

$$\omega_2 = \frac{1}{2} + \varepsilon\,; \quad |\varepsilon| \le 1. \tag{3.14}$$

Then the Hamiltonian given in equation (3.11), is reduces as given below:

$$H = \left[\frac{2a\alpha_2}{\omega_2^2}\sin^2\omega_2(\nu-\delta_2) + 2b\alpha_2\varepsilon\cos^2\omega_2(\nu-\delta_2) - \frac{2c\alpha_2}{\omega_2}\sin\omega_2(\nu-\delta_2)\cos\omega_2(\nu-\delta_2)\right] \tag{3.15}$$

or

$$H = \left[\frac{2a\alpha_2}{\omega_2^2}\sin^2\omega_2(\nu-\delta_2) + 2b\alpha_2\cos^2\omega_2(\nu-\delta_2) - \frac{c\alpha_2}{\omega_2}\sin 2\omega_2(\nu-\delta_2)\right].$$

From equation (3.14),

$$2\omega_2 = 1 + \varepsilon \tag{3.16}$$

Now, using equation (3.14), (3.16) and averaging the terms with finite frequencies from 0 to 2π, the perturbed Hamiltonian takes the following form:

$$H_2 = e\left[-\frac{a\alpha_2}{2\omega_2^2}\cos(2\varepsilon\nu - 2\omega_2\delta_2) + \frac{b\alpha_2}{2}\cos(2\varepsilon\nu - 2\omega_2\delta_2) + \frac{c\alpha_2}{2\omega_2^2}\sin(2\varepsilon\nu - 2\omega_2\delta_2\right]$$

or

$$H_2 = e\left[-\frac{(b-4a)}{2}\cos(2\varepsilon\nu - 2\omega_2\delta_2) + c\sin(2\varepsilon\nu - 2\omega_2\delta_2)\right]\alpha_2 \tag{3.17}$$

or

$$H_2 = e[U\cdot\cos(2\varepsilon\nu - 2\omega_2\delta_2) + V\cdot\sin(2\varepsilon\nu - 2\omega_2\delta_2)]\alpha_2$$

or

$$H_2 = e[U\cdot\cos(2\omega_2\delta_2 - 2\varepsilon\nu) - V\cdot\sin(2\omega_2\delta_2 - 2\varepsilon\nu)]\alpha_2$$

or

$$H_2 = e[U\cdot\cos 2\omega_2(\delta_2 - \varepsilon\nu) - V\cdot\sin 2\omega_2(\delta_2 - \varepsilon\nu)]\alpha_2\,, \tag{3.18}$$

where $U = \frac{b-4a}{2}$; $V = c$.

To find the rid of V from the Hamiltonian, the canonical transformation is introduced:

$$\overline{\alpha}_1 = \alpha_1 \; ; \quad \overline{\delta}_1 = \delta_1 \; ; \quad \overline{\alpha}_2 = \alpha_2 \; ; \quad \overline{\delta}_2 = \delta_2 - 2\varepsilon V . \tag{3.19}$$

Assuming $\overline{H}$ is the transformed form of Hamiltonian H in the new variable, which gives:

$$dF\,(q_i, \overline{q}_i, t) = \sum p_i dq_i + (\overline{H} - H)dt$$

or

$$dF\,\left(q_i, \overline{q}_i, V\right) = \overline{\delta}_1 d\overline{\alpha}_1 - \overline{\delta}_2 d\overline{\alpha}_2 - \delta_1\, d\alpha_1 - \delta_2 d\alpha_2 + (\overline{H} - H)\, dv$$

or

$$\begin{aligned}
&\delta_1\, d\alpha_1 + (\delta_2 - 2\varepsilon v)d\alpha_2 - \delta_1\, d\alpha_1 - \delta_2 d\alpha_2 + (\overline{H} - H)\, dv \\
&\quad = \left(\frac{\partial F}{\partial \alpha_1}\right) d\alpha_1 + \left(\frac{\partial F}{\partial \alpha_2}\right) d\alpha_2 + \left(\frac{\partial F}{\partial \overline{\alpha}_1}\right) d\overline{\alpha}_1 + \left(\frac{\partial F}{\partial \overline{\alpha}_2}\right) d\overline{\alpha}_2 + \left(\frac{\partial F}{\partial v}\right) dv - 2\varepsilon v\, d\alpha_2 + (\overline{H} - H)\, dv \\
&\quad = \left(\frac{\partial F}{\partial \alpha_1}\right) d\alpha_1 + \left(\frac{\partial F}{\partial \alpha_2}\right) d\alpha_2 + \left(\frac{\partial F}{\partial \overline{\alpha}_1}\right) d\overline{\alpha}_1 + \left(\frac{\partial F}{\partial \overline{\alpha}_2}\right) d\overline{\alpha}_2 + \left(\frac{\partial F}{\partial v}\right) dv
\end{aligned}$$

By equating coefficients of $d\alpha_2$ and dv from both sides, we get :

$$\overline{H} - H = \frac{\partial F}{\partial v} \; ; \quad -2\varepsilon v = \frac{\partial F}{\partial \alpha_2} .$$

Since

$$F = F(\alpha_2, v) \; ;$$

hence

$$\begin{aligned}
dF &= \left(\frac{\partial F}{\partial \alpha_2}\right) d\alpha_2 + \left(\frac{\partial F}{\partial v}\right) dv \; ; \\
\frac{dF}{d\alpha_2} &= \frac{\partial F}{\partial \alpha_2} = -2\varepsilon v \; ; \\
dF &= -2\varepsilon v d\alpha_2 \; ; \\
F &= -2\varepsilon v \alpha_2
\end{aligned}$$

hence

$$\overline{H} - H = \frac{\partial F}{\partial v} = -2\varepsilon v\ \alpha_2 \; ;$$

that is $\overline{H} = H + 2\varepsilon v \alpha_2$.

The non-periodic part of the perturbation is given as below:

$$\overline{H}_2 = e[U \cos 2\omega_2 \overline{\delta}_2 - V \cdot \sin 2\omega_2 \overline{\delta}_2]\overline{\alpha}_2 - 2v\overline{\alpha}_2 , \tag{3.20}$$

where $U = \frac{b-4a}{2}$; $V = c$.

The transformed Hamiltonian $\overline{H}_2$ given by equation (3.20) is now independent of .

Let

$$\overline{H}_2 = h_1 = \text{constant} \tag{3.21}$$

Then equation (3.20) becomes

$$h_1 = e[U \cos 2\omega_2 \overline{\delta}_2 - V \sin 2\omega_2 \overline{\delta}_2]\overline{\alpha}_2 - 2v\overline{\alpha}_2$$

Now, substituting

$$\cos\theta = \frac{U}{(U^2+V^2)^{\frac{1}{2}}}\ ; \quad \sin\theta = \frac{V}{(U^2+V^2)^{\frac{1}{2}}}\,. \tag{3.22}$$

Then equation (3.20) reduces to the following form:

$$h_1 = e[(U^2+V^2)^{\frac{1}{2}}\cos\theta\cos 2\omega_2\overline{\delta}_2 - (U^2+V^2)^{\frac{1}{2}}\sin\theta\sin 2\omega_2\overline{\delta}_2]\overline{\alpha}_2 - 2\nu\overline{\alpha}_2$$

or

$$h_1 = e\ (U^2+V^2)^{\frac{1}{2}}\cos(2\omega_2\overline{\delta}_2+\theta)\overline{\alpha}_2 - 2\nu\overline{\alpha}_2$$

that is,

$$\cos(2\omega_2\overline{\delta}_2+\theta) = \frac{h_1+2\nu\overline{\alpha}_2}{e\ (U^2+V^2)^{\frac{1}{2}}\ \overline{\alpha}_2}$$

or

$$\cos(2\omega_2\overline{\delta}_2+\theta) = \left[\frac{h_1}{e\ (U^2+V^2)^{\frac{1}{2}}} + \frac{2\nu\overline{\alpha}_2}{e\ (U^2+V^2)^{\frac{1}{2}}}\right]\frac{1}{\overline{\alpha}_2} \tag{3.23}$$

let $h = \frac{h_1}{e\ (U^2+V^2)^{\frac{1}{2}}}$, then,

$$\cos\left(2\omega_2\overline{\delta}_2+\theta\right) = \left[h + \frac{2\nu\overline{\alpha}_2}{e\ (U^2+V^2)^{\frac{1}{2}}}\right]\frac{1}{\overline{\alpha}_2}, \tag{3.24}$$

where θ is determined by the equation (3.22). Hence from the equation (3.24),

$$\overline{\alpha}_2 = \frac{h}{\cos(2\omega_2\overline{\delta}_2+\theta) - \frac{2\nu}{e\ (U^2+V^2)^{\frac{1}{2}}}}\,.$$

here, the parameter $\overline{\alpha}_2$ will be unbounded if:

$$\left|\frac{2\nu}{e\ (U^2+V^2)^{\frac{1}{2}}}\right| < 1\,.$$

That is,

$$|\varepsilon| < \frac{e\ (U^2+V^2)^{\frac{1}{2}}}{2}\,.$$

The inequality given in equation (3.25) shows the region of parametric resonance in the plane $(\mu - e)$, and in the neighbourhood of equilibrium points at $\omega_2 = \frac{1}{2}$ given by:

$$\mu_0 = 0.0285954 + 0.003659315722\epsilon_1' - 0.02561521005\epsilon_2' + 1.199340728\sigma_1 \\ - 0.9578258901\sigma_2 + 0.8453019319\sigma_1' - 0.1646692075\sigma_2' \tag{3.25}$$

The equation of the boundary of the region up to first approximation in 'e' taken from (3.21), that can be written in the neighbourhood of μ_0.

Let us consider $\mu = \mu_0 + h$.

Now, expanding $\omega_2(\mu)$ by using taylor's theorem, we get

$$\omega_2(\mu) = \omega_2(\mu_0 + h)$$

$$= \omega_2(\mu_0) + h\left[\frac{d\omega_2(\mu)}{d\ \mu}\right]_{\mu=\mu_0}$$

or

$$\frac{1}{2} + \varepsilon = \frac{1}{2} + (\mu - \mu_0)\left[\frac{d\omega_2(\mu)}{d\ \mu}\right]_{\mu=\mu_0},$$

i.e.,

$$\varepsilon = (\mu - \mu_0)\left[\frac{d\omega_2(\mu)}{d\ \mu}\right]_{\mu=\mu_0}.$$

Using equation (3.25), the parametric resonance can be given by

$$\left|(\mu - \mu_0)\left[\frac{d\omega_2(\mu)}{d\ \mu}\right]_{\mu=\mu_0}\right| < \frac{e\ (U^2 + V^2)^{\frac{1}{2}}}{2}.$$

That is,

$$\left|(\mu - \mu_0)\right| < \frac{e\ (U^2 + V^2)^{\frac{1}{2}}}{2\left[\frac{d\omega_2(\mu)}{d\ \mu}\right]_{\mu=\mu_0}} < eG_1, \tag{3.26}$$

where

$$G_1 = \frac{(U^2 + V^2)^{\frac{1}{2}}}{2\left[\frac{d\omega_2(\mu)}{d\mu}\right]_{\mu=\mu_0}} \tag{3.27}$$

hence, the region of existence of parametric resonance is determined by the inequality given below:

$$\mu_0 - eG_1 < \mu < \mu_0 + e\ G_1. \tag{3.28}$$

Now, calculating the value of G_1 with the help of the values of parameters obtained.

At $\omega_2 = \frac{1}{2}$;

$$\left|\omega_2^2 - \frac{1}{2}\right| = \frac{1}{4}$$

From equation (3.6)

$$I_2 = \frac{9}{4} + 2A + \omega_2^2$$

i.e.

$$I_2 = \frac{5}{4} + 2A,$$

$$a_2^2 = \left(\frac{1}{2}\right)^2 \frac{2I_2}{||\omega_2^2 - \frac{1}{2}||}$$

That is,

$$a_2^2 = 5 + 4\epsilon a$$

$$b_2 = \frac{2}{I_2} = \frac{2}{5 + 4A}$$

and

$$c_2 = \frac{-K + B}{I_2} = \frac{2(-K + B)}{5 + 4\ A},$$

$$2\omega_2^2 = 1-\left[1-27\mu(1-\mu)\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)\right]^{1/2}$$

hence

$$\mu^2-\mu+\frac{1}{36}\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)$$

or

$$\mu = 0.0285954+0.003659315722\epsilon_1'-0.02561521005\epsilon_2'+1.199340728\sigma_1$$
$$-0.9578258901\sigma_2+0.8453019319\sigma_1'-0.1646692075\sigma_2'$$

substituting the values of μ in equations (2.38), (2.39), (2.40) and(2.41), then we get:

$$A = -0.04227989\epsilon_1'+0.13391858\epsilon_2'+19.50174081\sigma_1$$
$$-20.85663467\sigma_2-4.774968538\sigma_1'+0.95090409\sigma_2'$$

and

$$A = -1.37527373\epsilon_1'+0.66803176\epsilon_2'-103.4455595\sigma_1+3.031116049\sigma_2$$
$$-0.4473677441\sigma_1'-0.3654266829\sigma_2'.$$

Similarly,

$$C = 1.873107913\epsilon_1'-0.28861806\epsilon_2'-7.53743598\sigma_1$$
$$+7.806425294\sigma_2+2.61177944\sigma_1'-1.783115869\sigma_2'$$

and

$$K = 1.224745-0.00950718107\epsilon_1'+0.066550267\epsilon_2'-3.115978615\sigma_1$$
$$+2.48850466\sigma_2-2.196158841\sigma_1'+0.4278231507\sigma_2'.$$

From equation (3.12), the parameters a, b and c are represented as below:

$$a = \left(\frac{3}{8}-A\right)a_2^2+(K-B)a_2^2c_2+\left(\frac{9}{8}+C\right)a_2^2c_2^2$$
$$= \left(\frac{3}{8}-A\right)(5+4A)+(K-B)(5+4A)\frac{2(-K+B)}{5+4A}+\left(\frac{9}{8}+C\right)(5+4A)\frac{4(-K+B)^2}{(5+4A)^2}.$$

That is,

$$a = 0.225+2.10217069\epsilon_1'-0.8123564784\epsilon_2'-122.9385266\sigma_1$$
$$+82.30056147\sigma_2+25.43185586\sigma_1'-0.3557546\sigma_2'$$

similarly,

$$b = \left(\frac{9}{8}+C\right)a_2^2b_2^2$$
$$= \left(\frac{9}{8}+C\right)(5+4A)\frac{16}{(5+4A)^2},$$

i.e.,

$$b = 3.6+6.115711405\epsilon_1'-1.3092633\epsilon_2'-80.28480867\sigma_1+85.0476688\sigma_2$$

$$+22.1096036\sigma_1' + 8.44457456\sigma_2'$$

and

$$c = \left\{-(K-B)a_2^2 b_2 - \left(\frac{9}{8}+A\right)2a_2^2 b_2 c_2\right\}$$
$$= -(K-B)(5+4A)\frac{4}{5+4A} - 2\left(\frac{9}{8}+A\right)(5+4A)\frac{4}{5+4A}\,\frac{2(-K+B)}{5+4A}.$$

That is,

$$c = -0.489898 + 8.044099329\epsilon_1' - 1.362921086\epsilon_2' - 138.4602504\sigma_1$$
$$+104.3787517\sigma_2 + 27.7781429\sigma_1' + 10.02515054\sigma_2'.$$

In the same way,

$$U = 1.35 - 1.146485677\epsilon_1' - 0.3557546\epsilon_2' + 205.7346489\sigma_1$$
$$-122.0772885\sigma_2 - 39.80890992\sigma_1' + 4.93379648\sigma_2'$$

and

$$V = -0.489898 + 8.044099329\epsilon_1' - 1.362921086\epsilon_2'$$
$$-138.4602504\sigma_1 + 104.3787517\sigma_2 + 27.7781429\sigma_1' + 10.02515054\sigma_2'.$$

Similarly,

$$(U^2+V^2)^{\frac{1}{2}} = 1.4361406 - 2.661112162\epsilon_1' + 0.9586919166\epsilon_2' + 167.5506307\sigma_1$$
$$-104.6978333\sigma_2 - 32.65478063\sigma_1' + 0.8481571156\sigma_2'$$

and

$$\left[\frac{d\omega_2}{d\,\mu}\right]_{\mu=\mu_0} = 12.72792206 - 0.8455317724\epsilon_1' + 5.918722407\epsilon_2' - 277.1230384\sigma_1$$
$$+221.3179415\sigma_2 - 195.3178394\sigma_1' + 38.04892976\sigma_2'$$

Now evaluating,

$$G_1 = \frac{(U^2+V^2)^{\frac{1}{2}}}{2\left[\frac{d\omega_2(\mu)}{d\mu}\right]_{\mu=\mu_0}}$$

i.e.,

$$G_1 = 0.05641693095 - 0.1007905114\epsilon_1' - 0.0114260446\epsilon_2' + 7.810367332\sigma_1$$
$$-5.093918344\sigma_2 + 2.148553648\sigma_1' - 0.1353343678\sigma_2'$$

Hence, the boundary of the region obtained by equation (3.27) of the parametric resonance about $\omega_2 = \frac{1}{2}$ in the first approximation in 'e' is given by

$$\mu = \mu_0 \pm e\,G_1$$

i.e.

$$\mu = (0.0285954 + 0.003659315722\epsilon_1' - 0.02561521005\epsilon_2' + 1.199340728\sigma_1$$

$$-0.9578258901\sigma_2 + 0.8453019319\sigma'_1 - 0.1646692075\sigma'_2) \pm e(0.05641693095$$
$$-0.1007905114\epsilon'_1 - 0.0114260446\epsilon'_2 + 7.810367332\sigma_1 - 5.0939183344\sigma_2$$
$$+2.148553648\sigma'_1 - 0.1353343678\sigma'_2). \tag{3.29}$$

The dependency of μ and e is represented in figures for different values of triaxiality and radiation pressures.

4. Conclusion

The discussion about the stability of infinitesimal around the triangular equilibrium points in elliptical restricted three body problem under the assumption that both the primaries are radiating and triaxial, for small eccentricity e near the resonance frequency $\omega_2 = \frac{1}{2}$ has been done. For the verification of stability we have constructed a suitable Hamiltonian function and investigated the stability of infinitesimal around the triangular equilibrium points and the perturbed system analytically and numerically due to triaxiality and radiation of the primaries elliptical case up to second order terms. The region of stability and instability has been found by using simulation technique. In the curves drawn between μ and e, the system is unstable in the region lying within the boundary and the system is stable in the region lying out side the boundary. We conclude that the effect of the triaxiality and radiation of primaries affects the location and resonance stability of triangular equilibrium points in elliptical restricted three body problem in the elliptical case (i.e. for small value of eccentricity 'e') at and near the resonance frequency $\omega_2 = \frac{1}{2}$, which is verified by the graphical behaviour of the triangular equilibrium points around the binary system.

Hence we have came on conclusion that the shifting of the location and the resonance stability of triangular equilibrium points in elliptical case around binary system would be possible by changing the triaxiality and radiation parameters, which are shown in Figures 1–6.

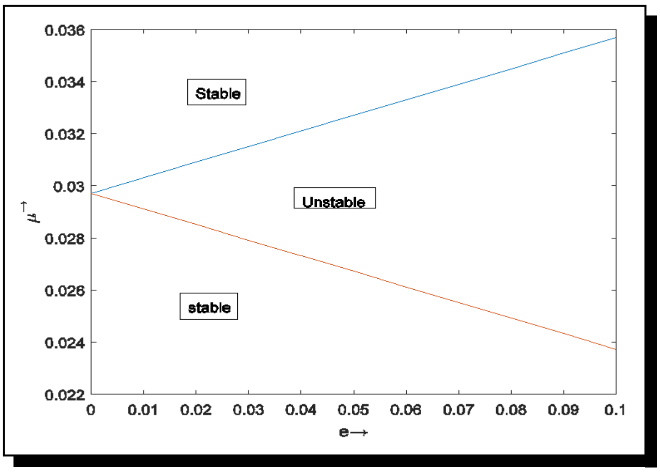

Figure 1. Correlation between μ and e for $\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma'_1 = 0.003$, $\sigma'_2 = 0.004$, $\varepsilon'_1 = 0.001$, $\varepsilon'_2 = 0.002$

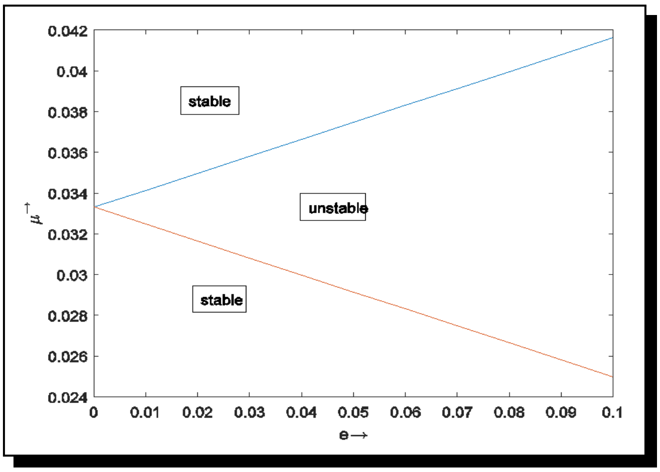

Figure 2. Correlation between μ and e for $\sigma_1 = 0.004$, $\sigma_2 = 0.002$, $\sigma_1' = 0.003$, $\sigma_2' = 0.004$, $\varepsilon_1' = 0.001$, $\varepsilon_2' = 0.002$

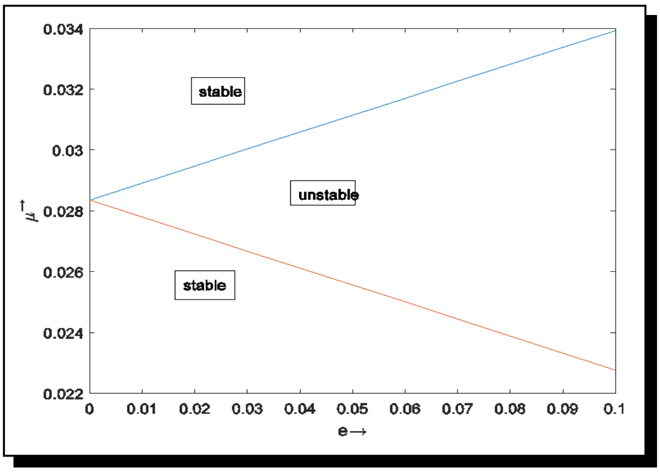

Figure 3. Correlation between μ and e for $\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.001$, $\sigma_2' = 0.002$, $\varepsilon_1' = 0.001$, $\varepsilon_2' = 0.002$

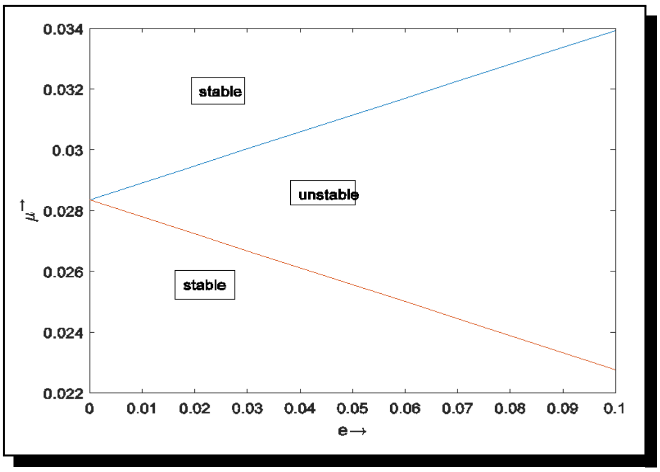

Figure 4. Correlation between μ and e for $\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.001$, $\sigma_2' = 0.004$, $\varepsilon_1' = 0.001$, $\varepsilon_2' = 0.002$

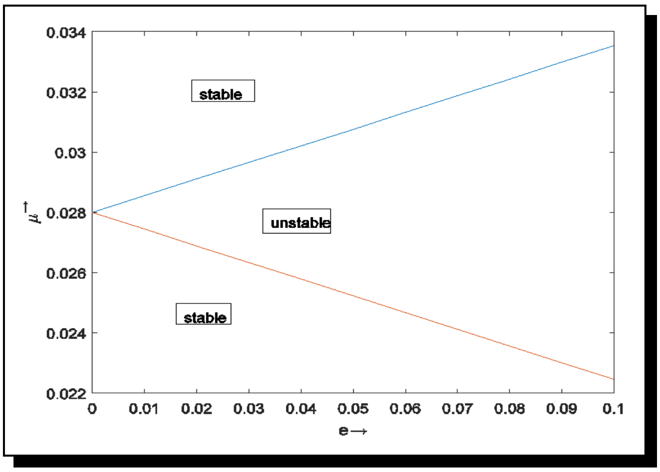

Figure 5. Correlation between μ and e for $\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.001$, $\sigma_2' = 0.004$, $\varepsilon_1' = 0.002$, $\varepsilon_2' = 0.003$

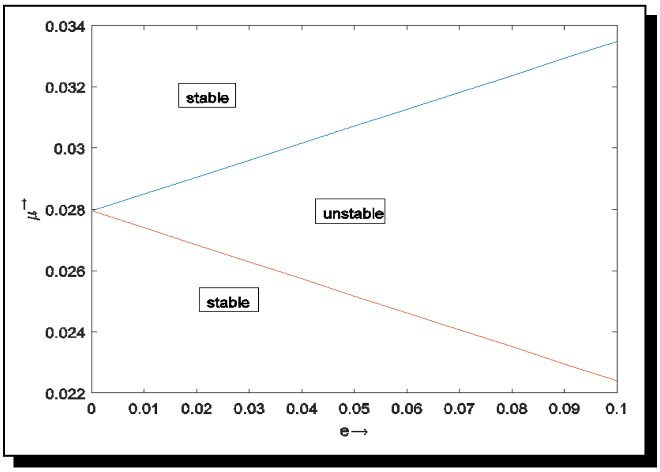

Figure 6. Correlation between μ and e for $\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.001$, $\sigma_2' = 0.004$, $\varepsilon_1' = 0.002$, $\varepsilon_2' = 0.005$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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