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Research Article

The Forward Kinematics of Rolling Contact of Timelike Surfaces With Spacelike Trajectory Curves

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Abstract. In this paper, we investigate the forward kinematics of spin-rolling motion without sliding of one timelike surface on another timelike surface along the spacelike contact trajectory curves of the surfaces in Lorentzian 3-space. A Darboux frame method is adopted to develop instantaneous kinematics of spin-rolling motion, which occurs in a nonholonomic system. Then, new kinematic formulations of spin-rolling motion of timelike moving surface with regards to contravariant vectors, rolling velocity, and geometric invariants are obtained. Namely, the translational velocity formulation of an arbitrary point and the equation of the angular velocity formulation on the timelike moving surface are derived. The equation, which is represented with geometric invariants, can be easily applied to arbitrary spacelike parametric surface and spacelike contact trajectory curve and can be differentiated to any order. The influence of the relative curvatures and torsion on spin-rolling kinematics is clearly presented.

Keywords. Lorentzian 3-Space; Darboux frame; Forward kinematics; Pure-rolling; Rolling contact; Spin-rolling

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1. Introduction

Rolling contact is used in many areas of robotics and engineering such as spherical robots, single wheel robots, and multifingered robotic hands to drive from one configuration (position and orientation) to another. In mechanical systems, rolling contact without sliding engenders a non-integrable kinematic constraints on the system's velocity which are called non-holonomic constraints. This non-holonomy calls for the two contact loci have equal arc lengths in a given time interval [10].

There are two categories of kinematics of the rolling contact. The first is pure-rolling motion and the second is spin-rolling motion [8]. On the other hand, in the rolling contact, there are two geometric constraints. The first is that the unit normal vectors of the two surfaces are made to coincide at the contact point. The second is that the contact points have the same velocity. To put it another way, the two contact trajectory curves are tangent to each other and have the same rolling rate. Thus, a moving surface has spin-rolling motion or pure-rolling motion under these two geometric constraints. Further, there is another constraint for a surface to have pure-rolling motion. This constraint is explicitly demonstrated to be that the two contact trajectories have the same geodesic curvature, that is, the angular velocity ω_3 in the direction of the unit normal vector n to the surface is zero. Thus, the contact trajectories are not arbitrary [9]. Pure-rolling motion has 2 degrees of freedom (DOFs). It has instantaneous rotation axis passes through the contact point in all cases and this axis is parallel to the common tangent plane of two surfaces. Spin-rolling motion, which is also called twist-rolling motion, has 3 degrees of freedom (DOFs) consisting of three angular velocity components: ω_1, ω_2 about the axes T and g on the tangent plane, respectively, and ω_3 about the common normal axis n at the contact point. Its instantaneous rotation axis can be in any arbitrary direction which is the characteristic difference from pure-rolling motion [8].

The contact kinematics is given in two classifications. The first is forward kinematics and the second is inverse kinematics. The forward kinematics includes the problem of using kinematic equations as the inputs of the geometry of the two surfaces and the contact locus on each surface to compute the motion of the moving surface as the output. The inverse kinematics includes the problem of determining the control parameters that give the moving surface the desired motion as the inputs of the geometry of two surfaces and the desired angular velocity of the moving surface. These inputs are the angular velocity components ω_1, ω_2 and ω_3 [9,10].

Many researchers have extensively studied kinematics of a point contact between rigid bodies. Neimark and Fufaev [17] were the first to adopt the moving frame along the lines of curvature to derive the velocity equation of spin-rolling motion. Cai and Roth [4,5] investigated instantaneous time-based kinematics of rigid objects in point contact, both in planar and spatial cases, and focused on two special motions, including sliding and pure-rolling motion and they aimed to measure the relative motion at the point of contact. Montana [15] studied the kinematics of sliding-spin-rolling motion and derived a differential-geometric model of the rolling constraint between general bodies. Li and Canny [13] used Montana's contact equations to investigate the existence of an admissible path between two configurations in the case of pure rolling and, if it does, then how to find it. Sarkar *et al*. [21] extended Montana's definition but with a different approach by obtaining the acceleration equations and they demonstrated the obvious dependence on Christoffel symbols and they simplified the derivative of the metric tensor. Marigo and Bicchi [14] obtained similar equations with Montana's contact equations using a different approach that allowed an analysis of admissibility of a pure-rolling contact. Agrachev and Sachkov [1] solved the controllability problem of a pair of pure-rolling rigid bodies. Chelouah and Chitour [6] gave two procedures to analyze the motion-planning problem when one manifold was a plane and the other was a convex surface. Chitour *et al*. [7] investigated the pure-rolling of a pair of smooth convex objects, with one being over another under quantized control. Tchon [22] identified the property of repeatability of inverse-kinematics algorithms for mobile manipulators and formulated a necessary and sufficient condition under repeatability. Tchon and Jakubiak [23] designed an extended Jacobian repeatable inverse kinematics algorithm for doubly nonholonomic mobile manipulators based on the concept of endogenous configuration space. Cui and Dai [8] investigated the forward kinematics of spin-rolling motion without sliding by applying the moving-frame method and then Cui [9] studied the kinematics of sliding-rolling motion of two contact surfaces. Cui and Dai [10] also investigated the inverse kinematics of rolling contact by using polynomial formulation when the desired angular velocity and the coordinates of the contact point on each surface were given in Euclidean 3-space. Then, they obtained admissible rolling motion between two contact surfaces. For the fundamental concepts of kinematics (see [3, 12, 16]).

This paper is organized as follows: In Section 2, we give basic concepts in Lorentzian 3-space. In Section 3, we study the forward kinematics of spin-rolling without sliding of one timelike surface on another timelike surface by applying the moving-frame method. Initially, we give the Darboux-frame-based translational velocity formulation of an arbitrary point in Lorentzian 3-space. Then, we obtain a new equation of angular velocity with respect to the rolling speed and two sets of geometric invariants containing the geodesic curvature, the normal curvature, and the geodesic torsion, namely $\{k_g, k_n, \tau_g\}$, $\{\bar{k}_g, \bar{k}_n, \bar{\tau}_g\}$. We determine the instantaneous kinematics of a timelike moving surface by applying the translational velocity formulation and the angular velocity equation. Then, we give two examples that present spin-rolling motion and pure-rolling motion of two timelike surfaces without sliding, respectively. In Section 4, we give a conclusion.

2. Preliminaries

In this section, we give a brief summary of basic concepts for the reader who is not familiar with Lorentzian 3-space ([2, 19, 20, 24]).

Lorentzian space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3_1$ endowed with the Lorentzian inner product given by

$$\langle a, b\rangle = a_1b_1 + a_2b_2 - a_3b_3,$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3_1$.

According to this metric, an arbitrary vector $a = (a_1, a_2, a_3)$ in $\mathbb{R}^3_1$ can have one of three Lorentzian causal characters: if $\langle a, a\rangle > 0$ or $a = 0$ then a is called a spacelike vector; if $\langle a, a\rangle < 0$ then a is called a timelike vector; if $\langle a, a\rangle = 0$ and $a \neq 0$ then a is called a null (lightlike) vector [19]. We note that a timelike vector is future pointing or past pointing if the first compound of vector is positive or negative, respectively. The norm of a vector $a \in \mathbb{R}_1^3$ is given by $\|a\| = \sqrt{|\langle a,a\rangle|}$. If the vector a is a spacelike vector, then $\|a\|^2 = \langle a,a\rangle$; if a is a timelike vector, then $\|a\|^2 = -\langle a,a\rangle$ [24].

Let $a = (a_1,a_2,a_3)$ and $b = (b_1,b_2,b_3)$ be two vectors in $\mathbb{R}_1^3$. Then Lorentzian vector product of a and b can be defined by

$$a \times b = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1) \quad [24].$$

Definition 2.1 ([2,20])**.** (i) Hyperbolic angle: Let a and b be future pointing (or past pointing) timelike vectors in $\mathbb{R}_1^3$. Then there is a unique real number $\theta \geq 0$ such that $\langle a,b\rangle = -\|a\|\,\|b\|\cosh\theta$, and this number is called the hyperbolic angle between the vectors a and b.

(ii) Central angle: Let a and b be spacelike vectors in $\mathbb{R}_1^3$ and they span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle a,b\rangle = \|a\|\,\|b\|\cosh\theta$, and this number is called the central angle between the vectors a and b.

(iii) Spacelike angle: Let a and b be spacelike vectors in $\mathbb{R}_1^3$ and they span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $|\langle a,b\rangle| = \|a\|\,\|b\|\cos\theta$, and this number is called the spacelike angle between the vectors a and b.

(iv) Lorentzian timelike angle: Let a be a spacelike vector and b be a timelike vectors in $\mathbb{R}_1^3$. Then there is a unique real number $\theta \geq 0$ such that $|\langle a,b\rangle| = \|a\|\,\|b\|\sinh\theta$, and this number is called the Lorentzian timelike angle between the vectors a and b.

An arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^3$ can locally be spacelike, timelike, or null (lightlike), if all of its velocity vectors $d\alpha/ds$ are spacelike, timelike, or null (lightlike), respectively. A surface in Lorentzian space $\mathbb{R}_1^3$ is called a spacelike (timelike) surface if the normal vector of the surface is a timelike (spacelike) vector [19]. The Lorentzian and hyperbolic unit spheresare given by

$$S_1^2 = \{a = (a_1,a_2,a_3) \in \mathbb{R}_1^3 : \langle a,a\rangle = 1\} \text{ and } H_0^2 = \{a = (a_1,a_2,a_3) \in \mathbb{R}_1^3 : \langle a,a\rangle = -1\},$$

respectively. It is easy to show that the hyperbolic unit sphere is a timelike surface and Lorentzian unit sphere is a timelike surface. Let S be a timelike surface and $\alpha = \alpha(s)$ be any curvelying on the surface S. Then, the curve α is either spacelike, timelike or lightlike. When α is given as a spacelike curve, Darboux frame (T,g,n) of α is a solid perpendicular trihedron in $\mathbb{R}_1^3$ associated with each point $M \in \alpha$, where T is the unit tangent spacelike vector to the curve α, n is the unit spacelike normal vector to the timelike surface S and $g = n \times T$ (that is, g is tangential to S which is also a timelike vector) at the point M. We should note that

$$T \times g = -n, \;\; g \times n = -T, \;\; n \times T = g \;\text{ and }\; \langle T,T\rangle = 1, \;\; \langle g,g\rangle = -1, \;\; \langle n,n\rangle = 1.$$

Then the derivative formulae (the equations of motion) of the Darboux frame (trihedron) is given by

$$\frac{dm}{ds} = T, \quad \frac{d}{ds}\begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 0 & k_g & -k_n \\ k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ g \\ n \end{bmatrix},$$

where m is the position vector of the point M that depends on the choice of the coordinate system. Furthermore, the position vector corresponding to an arbitrary trajectory curve on a surface in $\mathbb{R}_1^3$ may have three causal characters. Hence, we can express that m is either spacelike, timelike or lightlike position vector. The components of the vector m are obtained from the measurement along the axes of the coordinate system. In these formulae, k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. It is easy to see that the geodesic curvature k_g, the normal curvature k_n and the geodesic torsion τ_g of the spacelike curve α can be given by

$$k_g = -\langle dT/ds, g\rangle, \quad k_n = -\langle dT/ds, n\rangle, \quad \tau_g = \langle dg/ds, n\rangle.$$

The Darboux instantaneous rotation vector of the Darboux trihedron is defined by

$$\omega = -\tau_g T - k_n g + k_g n \quad [24].$$

Then, for a spacelike curve $\alpha(s)$ lying on a timelike surface S, we have the following characterizations [24]: $\alpha(s)$ is

(i) geodesic $\Leftrightarrow k_g = 0$,

(ii) asymptotic $\Leftrightarrow k_n = 0$,

(iii) principal $\Leftrightarrow \tau_g = 0$.

3. The Forward Kinematics of Rolling Contact of Timelike Surfaces

In this section, we study the forward kinematics of rolling contact of timelike surfaces with spacelike trajectory curve by applying the Darboux frame method in Lorentzian 3-space. The main contribution of this section is that a new equation of the angular velocity of the spin-rolling motion of a timelike moving surface is formed. The new formulation is specified with regards to three contravariant vectors and geometric invariants, which are arc lengths of the spacelike contact trajectory curves and the induced curvatures of the two timelike surfaces.

In tensor analysis, a contravariant vector is a type $(1,0)$ tensor. While the components of a contravariant vector may change depending on the change of a coordinate system, the contravariant vector itself does not change. When the angular velocity formulation is formed in a coordinate system, change of the coordinate system results only in the change of components of contravariant vectors, and it does not change the formulation. In this context, the formulation is coordinate-invariant.

3.1 The Geometric Kinematics of Spin-rolling Motion

Firstly, we give the geometric kinematics of spin-rolling motion of two contact timelike surfaces. We note that during the rolling motion, both of the two timelike surfaces have the same spacelike unit normal vectors at the contact point. When a timelike fixed surface and a timelike moving surface relative to fixed surface undergo spin-rolling motion without sliding as in Figure 1, they maintain their timelike surface characters at every moment.

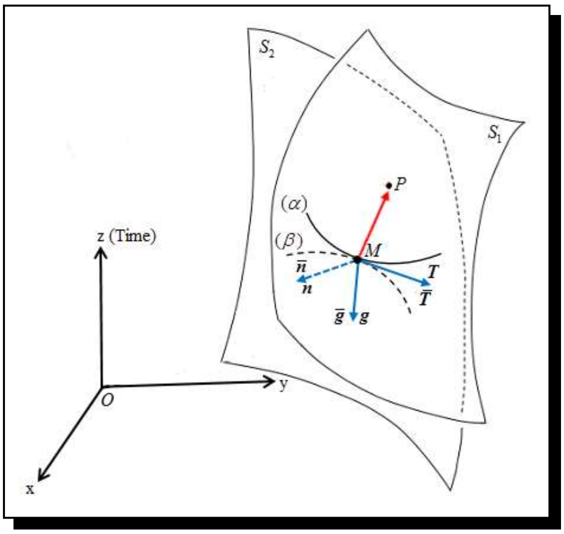

Figure 1. Timelike moving surface S_2 spin-rolling on timelike fixed surface S_1 along spacelike curves β and α

Now, let α and β be spacelike contact-trajectory curves on timelike surfaces S_1 and S_2, respectively. Let us denote the Darboux frames (the right-handed orthonormal frames)attached to the contact point M of spacelike curves α and β as (T,g,n) and $(\bar{T},\bar{g},\bar{n})$, respectively. The vectors T,g,n and $\bar{T},\bar{g},\bar{n}$ are the contravariant vectors of the timelike fixed and timelike moving surfaces, respectively and there is not any intrinsic coordinate system for these contravariant vectors. By rolling constraints,the contravariant vectors T and $\bar{T}$ are always collinear and, consequently, are n and $\bar{n}$. Therefore, the two frames can always be made to coincide, as shown in Figure 1, where n points outward of the surface S_1, and $\bar{n}$ points inward of the surface S_2. Let s and $\bar{s}$ be the arc lengths of spacelike curve α and spacelike curve β, respectively. Then the derivative formulas of the Darboux frames (T,g,n) and $(\bar{T},\bar{g},\bar{n})$ are

$$\frac{dm}{ds}=T,\quad \frac{d}{ds}\begin{bmatrix}T\\ g\\ n\end{bmatrix}=\begin{bmatrix}0 & k_g & -k_n\\ k_g & 0 & \tau_g\\ k_n & \tau_g & 0\end{bmatrix}\begin{bmatrix}T\\ g\\ n\end{bmatrix}$$

and

$$\frac{d\bar{m}}{d\bar{s}}=\bar{T},\quad \frac{d}{d\bar{s}}\begin{bmatrix}\bar{T}\\ \bar{g}\\ \bar{n}\end{bmatrix}=\begin{bmatrix}0 & \bar{k}_g & -\bar{k}_n\\ \bar{k}_g & 0 & \bar{\tau}_g\\ \bar{k}_n & \bar{\tau}_g & 0\end{bmatrix}\begin{bmatrix}\bar{T}\\ \bar{g}\\ \bar{n}\end{bmatrix},$$

where m and $\bar{m}$ are the position vectors of the point M with respect to the Darboux frames (T,g,n) and $(\bar{T},\bar{g},\bar{n})$, respectively. Both m and $\bar{m}$ have three causal characters. Now, let P denote an arbitrary point on S_2. Then we can write the (spacelike, timelike or lightlike) position vector,denoted by $\bar{p}$, of the point P in the frame $(\bar{T},\bar{g},\bar{n})$ as

$$\bar{p}=\bar{m}+\bar{\lambda}_1\bar{T}+\bar{\lambda}_2\bar{g}+\bar{\lambda}_3\bar{n}.$$

Differentiating $\bar{p}$ with respect to $\bar{s}$ gives

$$\frac{d\bar{p}}{d\bar{s}}=\left(1+\frac{d\bar{\lambda}_1}{d\bar{s}}+\bar{\lambda}_2\bar{k}_g+\bar{\lambda}_3\bar{k}_n\right)\bar{T}+\left(\frac{d\bar{\lambda}_2}{d\bar{s}}+\bar{\lambda}_1\bar{k}_g+\bar{\lambda}_3\bar{\tau}_g\right)\bar{g}+\left(\frac{d\bar{\lambda}_3}{d\bar{s}}-\bar{\lambda}_1\bar{k}_n+\bar{\lambda}_2\bar{\tau}_g\right)\bar{n}, \tag{1}$$

where $\bar{k}_g$, $\bar{k}_n$ and $\bar{\tau}_g$ are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of β, respectively. Since P is a fixed point of S_2, then

$$\frac{d\bar{p}}{d\bar{s}} = 0.$$

Not that $\frac{d\bar{p}}{d\bar{s}}$ is a spacelike vector. Putting this into (1) gives

$$\frac{d\bar{\lambda}_1}{d\bar{s}} = -\bar{\lambda}_2\bar{k}_g - \bar{\lambda}_3\bar{k}_n - 1, \quad \frac{d\bar{\lambda}_2}{d\bar{s}} = -\bar{\lambda}_1\bar{k}_g - \bar{\lambda}_3\bar{\tau}_g, \quad \frac{d\bar{\lambda}_3}{d\bar{s}} = \bar{\lambda}_1\bar{k}_n - \bar{\lambda}_2\bar{\tau}_g.$$

We can also write the (spacelike, timelike or lightlike) position vector, denoted by p, of the point P in the frame (T,g,n) as

$$p = m + \lambda_1 T + \lambda_2 g + \lambda_3 n.$$

Differentiating p with respect to s gives

$$\frac{dp}{ds} = \left(1 + \frac{d\lambda_1}{ds} + \lambda_2 k_g + \lambda_3 k_n\right)T + \left(\frac{d\lambda_2}{ds} + \lambda_1 k_g + \lambda_3 \tau_g\right)g + \left(\frac{d\lambda_3}{ds} - \lambda_1 k_n + \lambda_2 \tau_g\right)n, \tag{2}$$

where k_g, k_n and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of α, respectively. The vector p has three causal characters and, therefore, $\frac{dp}{ds}$ has three causal characters. By the constraints for rolling contact, two spacelike contact trajectory curves have the same arc lengths at the contact point. Since the Darboux frames (T,g,n) and $(\bar{T},\bar{g},\bar{n})$ are made to coincide at any moment, it follows that

$$\lambda_1 = \bar{\lambda}_1, \quad \lambda_2 = \bar{\lambda}_2, \quad \lambda_3 = \bar{\lambda}_3$$

and consequently

$$\frac{d\lambda_1}{ds} = \frac{d\bar{\lambda}_1}{d\bar{s}}, \quad \frac{d\lambda_2}{ds} = \frac{d\bar{\lambda}_2}{d\bar{s}}, \quad \frac{d\lambda_3}{ds} = \frac{d\bar{\lambda}_3}{d\bar{s}}. \tag{3}$$

Substituting (1) and (3) into (2) gives

$$\frac{dp}{ds} = (-\lambda_2 k_g^* - \lambda_3 k_n^*)T + (-\lambda_1 k_g^* - \lambda_3 \tau_g^*)g + (\lambda_1 k_n^* - \lambda_2 \tau_g^*)n, \tag{4}$$

where

$$k_g^* = \bar{k}_g - k_g, \quad k_n^* = \bar{k}_n - k_n, \quad \tau_g^* = \bar{\tau}_g - \tau_g.$$

The scalars k_g^*, k_n^* and τ_g^* are called induced geodesic curvature, induced normal curvature, and induced geodesic torsion, respectively. In Euclidean 3-space, for the Darboux trihedron and the induced curvatures (see [9, 11]).

3.2 Darboux-frame-based Velocity Formulation of Spin-rolling Motion

The velocity of an arbitrary point P on the timelike moving surface S_2 in terms of time t can be obtained from (4) as follows:

$$v_P = \frac{dp}{ds}\frac{ds}{dt} = \sigma(-\lambda_2 k_g^* - \lambda_3 k_n^*)T + \sigma(-\lambda_1 k_g^* - \lambda_3 \tau_g^*)g + \sigma(\lambda_1 k_n^* - \lambda_2 \tau_g^*)n, \tag{5}$$

where $\sigma = ds/dt$ is the magnitude of rolling velocity. Not that v_P has three causal characters. This equation gives the Darboux-frame-based translational-velocity formulation of an arbitrary point. Let the angular velocity of S_2 relative to timelike fixed surface S_1 be

$$\omega = \omega_x T + \omega_y g + \omega_z n. \tag{6}$$

If $r_{MP} = \lambda_1 T + \lambda_2 g + \lambda_3 n$ is also given, then the velocity of the point P can be obtained as

$$v_P = \omega \times r_{MP} = (\lambda_2\omega_z - \lambda_3\omega_y)T + (\lambda_1\omega_z - \lambda_3\omega_x)g + (\lambda_1\omega_y - \lambda_2\omega_x)n\,. \tag{7}$$

When the eq. (5) is compared with the eq. (7), we obtain that

$$\omega_x = \sigma\tau_g^*,\ \ \omega_y = \sigma k_n^*,\ \ \omega_z = -\sigma k_g^*. \tag{8}$$

From (6) and (8), the angular velocity of S_2 can be obtained as

$$\omega = \sigma(\tau_g^* T + k_n^* g - k_g^* n). \tag{9}$$

The equation (9) contains three terms. The first two terms give the pure-rolling velocity about an axis in the timelike tangent plane at the contact point and the third term gives the velocity of spin motion about the spacelike unit normal direction at the contact point in Lorentzian 3-space. Therefore, the pure-rolling velocity can be given by $\sigma\tau_g^* T + \sigma k_n^* g$ and the velocity of spin motion can be given by $-\sigma k_g^* n$. As a result, the timelike moving surface can follow the desired trajectory spacelike curve on the timelike fixed surface by the help of these three terms. We note that a pure-rolling motion does not have spin-rolling motion in the direction of the unit spacelike normal of the timelike surfaces. Then we give the following results:

(i) Let two timelike surfaces undergo pure-rolling motion in Lorentzian 3-space. Then the geodesic curvatures of the two corresponding contact-trajectory spacelike curves have to be equal, that is, $k_g = \bar{k}_g$.

(ii) Let contact-trajectory spacelike curves α and β be geodesics on timelike surfaces S_1 and S_2, respectively. Then the rolling motion consists of a pure-rolling motion in Lorentzian 3-space.

3.3 Examples

In this section, two examples are presented. The first example demonstrates the spin-rolling motion of a unit timelike cylinder on a timelike plane. The second example demonstrates the pure-rolling motion of a Lorentzian unit sphere on a timelike cylinder with radius $\frac{1}{2}$.

3.3.1 Spin-rolling Motion of a Unit Timelike Cylinder on a Timelike Plane

Let a unit timelike cylinder (surface S_2) rolls without sliding on a timelike plane (surface S_1) at a contact point M along spacelike curves α and β (see Figure 2).

Assume that spacelike curves α and β are parameterized by arc lengths s and $\bar{s}$, respectively. Let denote the Darboux frames (the right-handed orthonormal frames) attached to the contact point M of the curves α and β as (T,g,n) and $(\bar{T},\bar{g},\bar{n})$, respectively.Suppose the parametric equation of timelike plane is given by

$$x(u,v) = (-v,0,u),$$

and let α be a spacelike ellipse lying on the timelike plane parameterized as

$$\alpha(t) = x(u(t),v(t)) = (2\sinh t, 0, -1+\cosh t).$$

The derivative of s with respect to t is

$$\frac{ds}{dt} = \sqrt{\left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle} = \sqrt{1+3\cosh^2 t}\,.$$

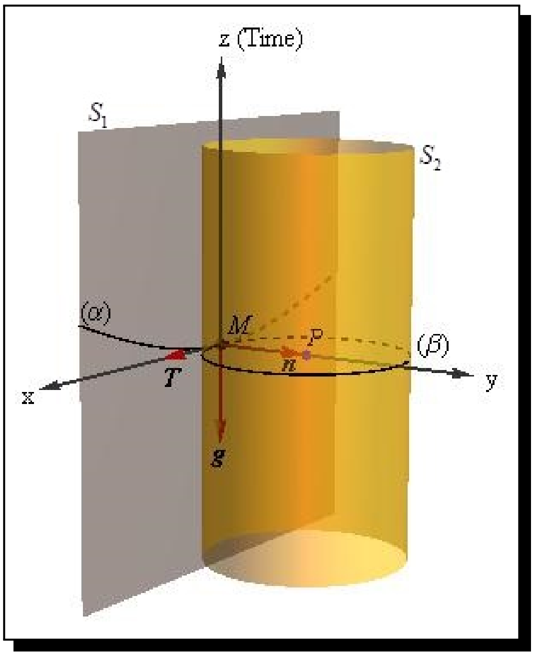

Figure 2. Spin-rolling of a unit timelike cylinder on a timelike plane along spacelike curves β and α.

The unit spacelike tangent vector T is obtained as

$$T = \frac{d\alpha}{ds} = \frac{d\alpha}{dt}\Big/\frac{ds}{dt} = \frac{1}{\sqrt{1+3\cosh^2 t}}(2\cosh t, 0, \sinh t)\ . \tag{10}$$

We assume that the unit spacelike normal vector n be outward and it is obtained as

$$n = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \begin{vmatrix} -i & -j & k \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix} \Big/ \|x_u \times x_v\| = (0,1,0) \tag{11}$$

The unit timelike vector g, which is tangential to S_1, is obtained as

$$g = n \times T = \frac{1}{\sqrt{1+3\cosh^2 t}}(-\sinh t, 0, -2\cosh t). \tag{12}$$

When the algebraic operation is applied, the geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve α is obtained as

$$\left.\begin{aligned} k_g &= -\langle dT/dt, g\rangle\Big/\tfrac{ds}{dt} = \tfrac{-2}{(1+3\cosh^2 t)^{3/2}}, \\ k_n &= -\langle dT/dt, n\rangle\Big/\tfrac{ds}{dt} = 0, \\ \tau_g &= \langle dg/dt, n\rangle\Big/\tfrac{ds}{dt} = 0, \end{aligned}\right\} \tag{13}$$

respectively. It is clear that the curve α is both principal and asymptotic.

Now, if we parameterize the unit timelike cylinder as

$$y(\bar{u},\bar{v}) = (\sin\bar{v}, 1+\cos\bar{v}, \bar{u}),$$

(see Figure 2). Let β be a spacelike $\bar{v}$-parametric curve (namely, a unit spacelike circle) lying on the unit timelike cylinder parameterized as

$$\beta(\bar{v}) = y(0,\bar{v}) = (\sin\bar{v}, 1+\cos\bar{v}, 0),$$

where $\bar{u} = \bar{u}_0 = 0$. Since the differentiation of arc length $\bar{s}$ of the curve β with respect to $\bar{v}$ is $\frac{d\bar{s}}{d\bar{v}} = \left\| \frac{d\beta}{d\bar{v}} \right\| = 1$, it is clear that β is a unit-speed curve.

The unit spacelike tangent vector $\bar{T}$ is given by

$$\bar{T} = (\cos\bar{v}, -\sin\bar{v}, 0). \tag{14}$$

Let the unit spacelike normal vector $\bar{n}$ be inward and it is obtained as

$$\begin{aligned}\bar{n} &= -\frac{y_{\bar{u}} \times y_{\bar{v}}}{\|y_{\bar{u}} \times y_{\bar{v}}\|} \\ &= -\begin{vmatrix} -i & -j & k \\ 0 & 0 & 1 \\ \cos\bar{v} & -\sin\bar{v} & 0 \end{vmatrix} \Big/ \|y_{\bar{u}} \times y_{\bar{v}}\| \\ &= (\sin\bar{v}, \cos\bar{v}, 0).\end{aligned} \tag{15}$$

The unit timelike vector $\bar{g}$, which is tangential to S_2, is obtained as

$$\bar{g} = \bar{n} \times \bar{T} = (0, 0, -1). \tag{16}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve β is obtained as

$$\left.\begin{aligned}\bar{k}_g &= -\langle d\bar{T}/d\bar{v}, \bar{g}\rangle \Big/ \tfrac{d\bar{s}}{d\bar{v}} = 0, \\ \bar{k}_n &= -\langle d\bar{T}/d\bar{v}, \bar{n}\rangle \Big/ \tfrac{d\bar{s}}{d\bar{v}} = 1, \\ \bar{\tau}_g &= \langle d\bar{g}/dt, \bar{n}\rangle \Big/ \tfrac{d\bar{s}}{d\bar{v}} = 0,\end{aligned}\right\} \tag{17}$$

respectively. It is clear that the curve β is both principal and geodesic. From (9), the angular velocity of the unit timelike cylinder is obtained as

$$\omega = \sigma\left(g - \frac{2}{(1+3\cosh^2 t)^{3/2}} n\right). \tag{18}$$

The coordinate of the center point P of the unit cylinder in the frame (T, g, n) at point m is origin. From Darboux-frame-based translation formulation (5) and (7), the velocity of point P is

$$v_P = \omega \times r_{MP} = \sigma\left(g - \frac{2}{(1+3\cosh^2 t)^{3/2}} n\right) \times (n) = -\sigma T.$$

After the information is used from the velocity formulation to control the timelike moving surface to follow the desired trajectory spacelike curve α lying on the fixed timelike surface, a brief discussion is provided. Moving surface has 2 DOFs. At any instant, the first term σ of (18) gives the angular velocity about the axis that is perpendicular to the unit timelike cylinder. The second term $-2\sigma/(1+3\cosh^2 t)^{3/2}$ gives the information about how fast the unit cylinder spins to follow the curve α and, in this way, yields the new tangent direction of the trajectory curve α. This information is used as the inputs of the control system to make unit cylinder follow the trajectory curve α.

3.3.2 Pure-rolling Motion of the Lorentzian Unit Sphere on a Timelike Cylinder with Radius 1/2

Let a Lorentzian unit sphere (surface S_2) S_1^2 rolls without sliding on a timelike cylinder with radius 1/2 (surface S_1) at a contact point M along spacelike curves α and β (see Figure 3).

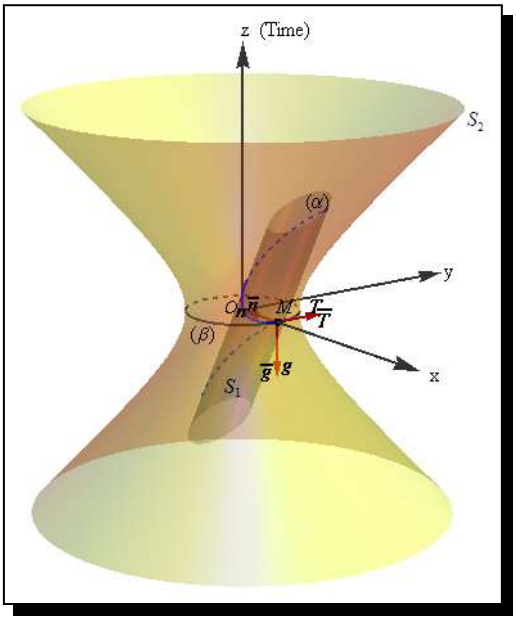

Figure 3. Pure-rolling of the Lorentzian unit sphere on a timelike cylinder along spacelike curves β and α

Assume that spacelike curves α and β are parameterized by arc lengths s and $\bar{s}$, respectively. Let denote the Darboux frames (the right-handed orthonormal frames) attached to the contact point M of the curves α and β as (T,g,n) and $(\bar{T},\bar{g},\bar{n})$, respectively. Suppose the parametric equation of the timelike cylinder with radius $\frac{1}{2}$ is given by

$$x(u,v)=\left(\frac{1+\cos v}{2},-\frac{u}{\sqrt{3}}+\frac{\sin v}{\sqrt{3}},\frac{-2u}{\sqrt{3}}+\frac{\sin v}{2\sqrt{3}}\right),$$

which is generated by rotating the surface $x_1(u,v)=\left(\frac{1+\cos v}{2},\frac{\sin v}{2},-u\right)$ around x-axis with the central angle $\operatorname{arccos}h\left(\frac{2}{\sqrt{3}}\right)$ in the negative direction, and let α be a spacelike helix curve lying on the timelike cylinder parameterized as

$$\alpha(t)=x(u(t),v(t))=x\left(\frac{t}{4},t\right)=\left(\frac{1+\cos t}{2},-\frac{t}{4\sqrt{3}}+\frac{\sin t}{\sqrt{3}},\frac{-t}{2\sqrt{3}}+\frac{\sin t}{2\sqrt{3}}\right).$$

The differentiation of s of curve α with respect to t is obtained as

$$\frac{ds}{dt}=\sqrt{\left\langle\frac{d\alpha}{dt},\frac{d\alpha}{dt}\right\rangle}=\frac{\sqrt{3}}{4}.$$

The unit spacelike tangent vector T of curve α is obtained as

$$T=T(\frac{t}{4},t)=\frac{d\alpha}{ds}=\frac{d\alpha}{dt}\Big/\frac{ds}{dt}=\frac{4}{\sqrt{3}}\left(\frac{-\sin t}{2},-\frac{1}{4\sqrt{3}}+\frac{\cos t}{\sqrt{3}},-\frac{1}{2\sqrt{3}}+\frac{\cos t}{2\sqrt{3}}\right) \tag{19}$$

Let the unit spacelike normal vector n be outward and it is obtained as

$$n=n(\frac{t}{4},t)=\left(-\cos t,-\frac{2\sin t}{\sqrt{3}},-\frac{\sin t}{\sqrt{3}}\right) \tag{20}$$

The unit timelike vector g, which is tangential to S_1, is obtained as

$$g=g(\frac{t}{4},t)=\frac{4}{\sqrt{3}}\left(-\frac{\sin t}{4},\frac{-1+\cos t}{2\sqrt{3}},\frac{-4+\cos t}{4\sqrt{3}}\right) \tag{21}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike helix curve α lying on timelike cylinder is obtained as

$$k_g = -\langle dT/dt, g\rangle \Big/ \frac{ds}{dt} = 0, \quad k_n = -\langle dT/dt, n\rangle \Big/ \frac{ds}{dt} = -\frac{8}{3}, \quad \tau_g = \langle dg/dt, n\rangle \Big/ \frac{ds}{dt} = \frac{4}{3}, \tag{22}$$

respectively. It is clear that the curve α is a geodesic.

Now, let us parameterize the Lorentzian unit sphere S_1^2 as

$$y(\bar{u},\bar{v}) = (\cos\bar{v}\cosh\bar{u}, \sin\bar{v}\cosh\bar{u}, \sinh\bar{u}).$$

Let β be a spacelike $\bar{v}$-parametric curve (namely, a unit spacelike circle) lying on S_1^2 parameterized as

$$\beta(\bar{v}) = y(0,\bar{v}) = (\cos\bar{v}, \sin\bar{v}, 0),$$

where $\bar{u} = 0$. Since the differentiation of $\bar{s}$ with respect to $\bar{v}$ is $\frac{d\bar{s}}{d\bar{v}} = \left\|\frac{d\beta}{d\bar{v}}\right\| = 1$, it is clear that β is a unit-speed curve. The unit spacelike tangent vector $\bar{T}$ of curve β is given by

$$\bar{T} = \frac{d\beta}{d\bar{v}} = (-\sin\bar{v}, \cos\bar{v}, 0). \tag{23}$$

Let the unit spacelike normal vector $\bar{n}$ of S_1^2 be inward (points origin) and it is obtained as

$$\bar{n} = -\frac{y_{\bar{u}} \times y_{\bar{v}}}{\|y_{\bar{u}} \times y_{\bar{v}}\|} = (-\cos\bar{v}, -\sin\bar{v}, 0). \tag{24}$$

The unit timelike vector $\bar{g}$, which is tangential to S_2, is obtained as

$$\bar{g} = \bar{n} \times \bar{T} = (0,0,-1) \tag{25}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve β is obtained as

$$\bar{k}_g = -\langle d\bar{T}/d\bar{v}, \bar{g}\rangle = 0, \quad \bar{k}_n = -\langle d\bar{T}/d\bar{v}, \bar{n}\rangle = -1, \quad \bar{\tau}_g = \langle d\bar{g}/d\bar{v}, \bar{n}\rangle = 0, \tag{26}$$

respectively. It is clear that the curve β is both principal and geodesic. From (9), the angular velocity of Lorentzian unit sphere is obtained as

$$\omega = \sigma\left(-\frac{4}{3}T + \frac{5}{3}g\right).$$

Consequently, we can see that this method is expressed with regards to geometric invariants that can be easily applied to arbitrary timelike parametric surfaces and spacelike curves.

4. Conclusion

In this paper, we adopt the Darboux frame method to investigate the forward kinematics of the instantaneous spin-rolling motion and pure-rolling motion between the timelike moving surface and the timelike fixed surface through the contact point in Lorentzian 3-space. We remark that both the fixed and moving surfaces always maintain their causal character during the rolling motion. The forward kinematics of the moving surface is determined by the magnitude of rolling velocity σ and induced curvatures k_g^*, k_n^* and τ_g^*. The result was given with regards to geometric invariants that can be easily generalized to arbitrary timelike parametric surfaces and spacelike contact curves.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. A. Agrachev and Y. L. Sachkov, An intrinsic approach to the control of rolling bodies, *in Proc. 38th IEEE Conf. Decis. Control*, 1999, 431–435, Phoenix, AZ, USA, DOI: 10.1109/CDC.1999.832815.

[2] G. S. Birman and K. Nomizu, Trigonometry in Lorentzian geometry, *Ann. Math. Month.* **91**(9) (1984), 543 – 549, DOI: 10.1080/00029890.1984.11971490.

[3] O. Bottema and B. Roth, *Theoretical Kinematics*, North-Holland Publ. Co., Amsterdam (1979), 556 p., DOI: 10.1137/1022104

[4] C. Cai and B. Roth, On the spatial motion of rigid bodies with point contact, in *Proc. IEEE Conf. Robot. Autom.*, 1987, pp. 686 – 695, DOI: 10.1109/ROBOT.1987.1087971.

[5] C. Cai and B. Roth, On the planar motion of rigid bodies with point contact, *Mech. Mach. Theory* **21** (1986), 453 – 466, DOI: 10.1016/0094-114X(86)90128-X.

[6] A. Chelouah and Y. Chitour, On the motion planning of rolling surfaces, *Forum Math.* **15**(5) (2003), 727 – 758, DOI: 10.1515/form.2003.039.

[7] Y. Chitour, A. Marigo and B. Piccoli, Quantization of the rolling-body problem with applications to motion planning, *Syst. Control Lett.* **54**(10) (2005), 999 – 1013, DOI: 10.1016/j.sysconle.2005.02.012.

[8] L. Cui and J. S. Dai, A Darboux-frame-based formulation of spin-rolling motion of rigid objects with point contact, *IEEE Trans. Rob.* **26**(2) (2010), 383 – 388, DOI: 10.1109/TRO.2010.2040201.

[9] L. Cui, *Differential Geometry Based Kinematics of Sliding-Rolling Contact and Its Use for multifingered Hands*, Ph.D Thesis, King's College London, University of London, London, UK (2010).

[10] L. Cui and J. S. Dai, A polynomial formulation of inverse kinematics of rolling contact, *ASME J. Mech. Rob.* **7**(4) (2015), 041003_041001-041009, DOI: 10.1115/1.4029498.

[11] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, New Jersey (1976).

[12] A. Karger and J. Novak, *Space Kinematics and Lie Groups*, STNL Publishers of Technical Lit., Prague, Czechoslovakia (1978).

[13] Z. X. Li and J. Canny, Motion of two rigid bodies with rolling constraint, *IEEE Trans. Robot. Autom.* **6**(1) (1990), 62 – 72, DOI: 10.1109/70.88118.

[14] A. Marigo and A. Bicchi, Rolling bodies with regular surface: Controllability theory and application, *IEEE Trans. Autom. Control* **45**(9) (2000), 1586 – 1599, DOI: 10.1109/9.880610.

[15] D. J. Montana, The kinematics of multi-fingered manipulation, *IEEE Trans. Robot. Autom.* **11**(4) (1995), 491 – 503, DOI: 10.1109/70.406933.

[16] H. R. Müller, *Kinematik Dersleri*, Ankara Üniversitesi Fen Fakültesi Yayınları (1963).

[17] J. I. Neimark and N. A. Fufaev, *Dynamics of Nonholonomic Systems*, Amer. Math. Soc., Providence, RI (1972).

[18] E. W. Nelson, C. L. Best and W. G. McLean, *Schaum's Outline of Theory and Problems of Engineering Mechanics, Statics and Dynamics*, 5th ed., McGraw-Hill, New York (1997).

[19] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London (1983).

[20] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, New York (2006), DOI: 10.1007/978-1-4757-4013-4.

[21] N. Sarkar, V. Kumar and X. Yun, Velocity and acceleration analysis of contact between three-dimensional rigid bodies, *ASME J. Appl. Mech.* **63**(4) (1996), 974 – 984, DOI: 10.1115/1.2787255.

[22] K. Tchon, Repeatability of inverse kinematics algorithms for mobile manipulators, *IEEE Trans. Autom. Control* **47**(8) (2002), 1376 – 1380, DOI: 10.1109/TAC.2002.801192.

[23] K. Tchon and J. Jakubiak, An extended Jacobian inverse kinematics algorithm for doubly nonholonomic mobile manipulators, in *Proc. IEEE Int. Conf. Robot. Autom.*, 2005, 1548 – 1553, Barcelona, Spain, DOI: 10.1109/ROBOT.2005.1570334.

[24] H. H. Uğurlu and A. Çalışkan, *Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi*, Celal Bayar Üniversitesi Yayınları, Manisa (2012).