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Research Article

Existence of Resonance Stability of Triangular Equilibrium Points in Circular Case of the Elliptical Restricted Three-Body Problem Under Radiating and Triaxial Primaries

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Abstract. The linear stability of the triangular librations points is studied in the presence of resonance considering both the primaries as radiating and triaxial. The study is carried out for various values of radiation pressure and Triaxiality parameter of both Primaries. It is found that the parametric resonance is only possible at the resonance frequency $\omega_2 = 1/2$ in circular case.

Keywords. ER3BP; Triangular Libration Points; Resonance; MATLAB 2016

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1. Introduction

The restricted problem of three bodies is a special version of three body problem. This model has a restriction that the third body(infinitesimal mass) has no effects on the two massive bodies(Primaries). The eccentricity of the orbits plays a significant role which might be not seen in the circular case. The problem under consideration, is the elliptical restricted three body problem. The primaries moves in elliptic orbits about their common centre of mass, without being effected by the third body of infinitesimal mass. The position of the primaries are fixed and the coordinates are obtained by dividing them by variable distance between the primaries. The orbits of most celestial bodies are elliptical rather then the circular as such, the ER3BP describes the dynamical system more accurately.

The ER3BP is a generalised form of circular restricted three body problem. it contain some of the useful property of CR3BP. The elliptical restricted three body problem has been described in detail by [3], [2], [7], [17] and [18]. The problem was further generalised by taking certain specific characteristics of celestial bodies like oblateness and triaxiality. The influence of eccentricity of orbits of the primaries with or without radiation pressure(s) and the triaxiality of the primaries on the existence and stability of the equilibrium points were studied by [1], [4], [5], [6], [9], [10], [15], [16], [19] and [21]. The author has analyzed stability of triangular equilibrium points L_4 and L_5 in ER3BP under the assumption that both primaries are triaxial and radiating [13]. Also the characteristic exponents of triangular solutions in ER3BP has been analyzed [14].

The present study is an attempt which is devoted to the analysis of the existence of resonance and the stability of triangular points in particular case when $e = 0$, assuming that the bigger and smaller primaries are triaxial and the source of radiation as well. this study will contribute to understand the effects of radiation, eccentricity and triaxiality on the celestial dynamical systems. We have adopted the method due to Markeev [7] and [8] in which the Hamiltonian function pertaining to the problem is made independent of time using several canonical transformations. The existence of resonance and the stability of infinitesimal near the resonance frequency has been analyzed. The problem is discussed earlier by Narayan and Singh [12], Usha and Narayan [20], and Narayann and Shrivastava [11] with different conditions.

2. Equations of Motion

The differential equations of the motion of the infinitesimal mass in elliptical restricted three body problem under radiating and triaxial primaries in pulsating system is given by [13] and [14] as

$$x'' - 2y'' = \frac{1}{1+e\cos v}\left(\frac{\partial\Omega}{\partial x}\right), \qquad y'' + 2x'' = \frac{1}{1+e\cos v}\left(\frac{\partial\Omega}{\partial y}\right), \tag{2.1}$$

where $'$ denotes differentiation with respect to v, and

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^3} + \frac{\mu(2\sigma_1{}'-\sigma_2{}')q_2}{2r_2^3} \right.$$
$$\left. - \frac{3(1-\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^5} - \frac{3\mu(\sigma_1{}'-\sigma_2{}')y^2q_2}{2r_2^5}\right] \tag{2.2}$$

where

$$n^2 = 1 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}\left(2\sigma_1'-\sigma_2'\right), \; r_1^2 = (x+\mu)^2+y^2, \; r_2^2 = (x-1+\mu)^2+y^2, \; \mu = \frac{m_2}{m_1+m_2}, \tag{2.3}$$

where m_1 and m_2 are masses of the primaries. q_1, q_2 are the radiation pressure. σ_1, σ_2, σ'_1 and σ'_2 are triaxial parameters, while e and v are the eccentricity of orbits and true anomaly of the primaries, respectively. There are two triangular equilibrium points in the plane of the finite bodies around the coordinates (x, y) and then the three bodies form nearly equilateral triangle. Since the equilibrium points are symmetrical to each other, the nature of motion near the two triangular equilibrium points are given by [13] and [14].

$$\begin{aligned} x &= \frac{1}{2} - \mu + \frac{(\epsilon'_2 - \epsilon'_1)}{3} + \left[\frac{-3}{8} - \epsilon'_1 - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon'_1}{2\mu}\right]\sigma_1 + \left[\frac{7}{8} + \frac{\epsilon'_1}{2} + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon'_1}{2\mu}\right]\sigma_2 \\ &\quad + \left[\frac{3}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon'_2}{8(1-\mu)} + \epsilon'_2\right]\sigma_1{}' + \left[\frac{-7}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon'_2}{8(1-\mu)} - \frac{\epsilon'_2}{2}\right]\sigma_2{}', \\ y &= \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left\{-\frac{(\epsilon'_1 + \epsilon'_2)}{3} + \left[\frac{-19}{8} - \epsilon'_1 + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon'_1}{2\mu}\right]\sigma_1\right.\right. \\ &\quad + \left[\frac{15}{8} + \frac{\epsilon'_1}{2} - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon'_1}{2\mu}\right]\sigma_2 + \left[\frac{-19}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon'_2}{8(1-\mu)} - \epsilon'_2\right]\sigma_1{}' \\ &\quad \left.\left. + \left[\frac{15}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon'_2}{8(1-\mu)} + \frac{\epsilon'_2}{2}\right]\sigma_2{}'\right\}\right] \end{aligned} \tag{2.4}$$

Thus the coordinates of the triangular equilibrium points has been obtained up to first order terms in the parameters ϵ'_1, ϵ'_2, σ_1, σ_2, σ'_1 and σ'_2 which is represented by (2.4). The system (2.1) described the motion of dynamical system with Lagrangian, which is represented as

$$\begin{aligned} L &= \frac{\dot{x}^2 + \dot{y}^2}{2} + (\dot{y}x - \dot{x}y) + \frac{1}{1 + e\cos v}\left\{\frac{x^2 + y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\sigma_1 - \sigma_2)q_1}{2r_1^3}\right.\right. \\ &\quad \left.\left. + \frac{\mu\left(2\sigma_1{}' - \sigma_2{}'\right)q_2}{2r_2^3} - \frac{3(1-\mu)(\sigma_1 - \sigma_2)y^2 q_1}{2r_1^5} - \frac{3\mu\left(\sigma_1{}' - \sigma_2{}'\right)y^2 q_2}{2r_2^5}\right]\right\}, \end{aligned} \tag{2.5}$$

where

$$P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad P_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x. \tag{2.6}$$

We formed the expression for the Hamiltonian function of the problem using the formula

$$H = -L + \frac{\partial L}{\partial \dot{x}}\,\dot{x} + \frac{\partial L}{\partial \dot{y}}\,\dot{y}. \tag{2.7}$$

Hence, we have

$$H = -L + \left(P_x^2 + P_y^2\right) + (P_x y - P_y x). \tag{2.8}$$

The perturbed Hamiltonian function of the problem can be find by use of (2.5) and (2.8), which is reduced to the following form:

$$\begin{aligned} H &= \frac{\left(P_x^2 + P_y^2\right)}{2} + \left(P_x y - P_y x\right) + \frac{x^2 + y^2}{2}\left[1 - \frac{1}{1 + e\cos v}\right] \\ &\quad - \frac{1}{(1 + e\cos v)n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\sigma_1 - \sigma_2)q_1}{2r_1^3}\right. \end{aligned}$$

$$+\frac{\mu\left(2\sigma_1{}'-\sigma_2{}'\right)q_2}{2r_2^3}-\frac{3(1-\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^5}-\frac{3\mu\left(\sigma_1{}'-\sigma_2{}'\right)y^2q_2}{2r_2^5}\Bigg\}, \tag{2.9}$$

where P_x and P_y are the generalized components of momentum.

Since the two triangular equilibrium points are symmetrical, the nature of the oscillation of infinitesimal near two points will be the same. Hence, in further calculation the motion near the equilibrium point L_4 will be considered. So, shifting the origin to L_4 by the change of variables given by:

$$x=\xi+q_1\,;\qquad y=\eta+q_2,$$
$$p_x=p_\xi+p_1;\quad p_y=p_\eta+p_2, \tag{2.10}$$

where the displacement of infinitesimal at and near the equilibrium point L_4 is represented as follows:

$$\xi=\frac{1}{2}-\mu+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left[\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1+\left[\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2$$
$$+\left[\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right]\sigma_1{}'+\left[\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right]\sigma_2{}',$$

$$\eta=\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{(\epsilon_1'+\epsilon_2')}{3}+\left[\frac{-19}{8}-\epsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1\right.\right.$$
$$+\left[\frac{15}{8}+\frac{\epsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}-\epsilon_2'\right]\sigma_1{}'$$
$$\left.\left.+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}+\frac{\epsilon_2'}{2}\right]\sigma_2{}'\right\}\right],$$

$$p_\xi=-\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{(\epsilon_1'+\epsilon_2')}{3}+\left[\frac{-19}{8}-\epsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1\right.\right.$$
$$+\left[\frac{15}{8}+\frac{\epsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}-\epsilon_2'\right]\sigma_1{}'$$
$$\left.\left.+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}+\frac{\epsilon_2'}{2}\right]\sigma_2{}'\right\}\right],$$

$$p_\eta=\frac{1}{2}-\mu+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left[\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1+\left[\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2$$
$$+\left[\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right]\sigma_1{}'+\left[\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right]\sigma_2{}'. \tag{2.11}$$

The solution (2.11) in the new variables is given by the equilibrium positions:

$$q_1=q_2=p_1=p_2=0. \tag{2.12}$$

Now, expanding the Hamiltonian function (2.9) in the power of p_i and q_i, we obtained

$$H=\sum_{K=0}^{\infty}H_K=H_0+H_1+H_2+H_3+H_4+H_5+\dots, \tag{2.13}$$

where $H_0 = H(\xi, \eta p_\xi, p_\eta) =$ constant and $H_1 = 0$.

We evaluated $H_2, H_3, \dots$ using (2.10) and the terms of the Hamiltonian (2.9) are expanded one by one, the terms are not depending upon p_i and q_i, and those of order one are neglected. Hence, we obtain

$$\text{(i)}\quad \frac{P_x^2+P_y^2}{2} = \frac{1}{2}[(p_\xi+p_1)^2+(p_\eta+p_2)] = \frac{1}{2}(p_1^2+p_2^2) \tag{2.14}$$

$$\text{(ii)}\quad (p_x y - p_y x) = (p_\xi+p_1)(\eta+q_2) - (p_\eta+p_2)(\xi+q_1) = (p_1q_2 - p_2q_1) \tag{2.15}$$

$$\text{(iii)}\quad \frac{e\cos\nu}{2(1+e\cos\nu)}(x^2+y^2) = \frac{e\cos\nu}{2(1+e\cos\nu)}\left\{(\xi+q_1)^2+(\eta+q_2)^2\right\} = \frac{e\cos\nu}{2(1+e\cos\nu)}(q_1^2+q_2^2) \tag{2.16}$$

$$\text{(iv)}\quad \frac{1}{(1+e\cos\nu)n^2} = \left\{\frac{(1-\mu)(1-\epsilon_1')}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\sigma_1-\sigma_2)(1-\epsilon_1')}{2r_1^3} + \frac{\mu(2\sigma_1'-\sigma_2')(1-\epsilon_2')}{2r_2^3} - \frac{3(1-\mu)(\sigma_1-\sigma_2)y^2(1-\epsilon_1')}{2r_1^5} - \frac{3\mu(\sigma_1'-\sigma_2')y^2(1-\epsilon_2')}{2r_2^5}\right\} \tag{2.17}$$

where $r_1^2 = (x+\mu)^2 + y^2 = (\xi+q_1+\mu)^2 + (\eta+q_2)^2$,

$$\begin{aligned} r_1^2 = \Bigg[&q_1 + \frac{1}{2} - \mu + \frac{(\epsilon_2'-\epsilon_1')}{3} + \left\{\frac{-3}{8} - \epsilon_1' - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1 + \left\{\frac{7}{8} + \frac{\epsilon_1'}{2} + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2 \\ &+ \left\{\frac{3}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon_2'}{8(1-\mu)} + \epsilon_2'\right\}\sigma_1' + \left\{\frac{-7}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon_2'}{8(1-\mu)} - \frac{\epsilon_2'}{2}\right\}\sigma_2' + \mu\Bigg]^2 \\ &+ \Bigg[q_2 + \frac{\sqrt{3}}{2}\Bigg\{1 + \frac{2}{3}\Bigg\{-\frac{(\epsilon_1'+\epsilon_2')}{3} + \left[\frac{-19}{8} - \epsilon_1' + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1 \\ &+ \left[\frac{15}{8} + \frac{\epsilon_1'}{2} - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2 + \left[\frac{-19}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon_2'}{8(1-\mu)} - \epsilon_2'\right]\sigma_1' \\ &+ \left[\frac{15}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon_2'}{8(1-\mu)} + \frac{\epsilon_2'}{2}\right]\sigma_2'\Bigg\}\Bigg\}\Bigg]^2, \end{aligned}$$

$$\begin{aligned} r_1^{-1} &= f(q_1,q_2) \\ &= \Bigg[\Bigg[q_1 + \frac{1}{2} + \frac{(\epsilon_2'-\epsilon_1')}{3} + \left\{\frac{-3}{8} - \epsilon_1' - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1 + \left\{\frac{7}{8} + \frac{\epsilon_1'}{2} + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2 \\ &\quad + \left\{\frac{3}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon_2'}{8(1-\mu)} + \epsilon_2'\right\}\sigma_1' + \left\{\frac{-7}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon_2'}{8(1-\mu)} - \frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2 \\ &\quad + \left(q_2 + \frac{\Lambda}{2}\right)^2\Bigg]^{-1/2} \end{aligned} \tag{2.18}$$

and $r_2^2 = (x-1+\mu)^2 + y^2 = (\xi+q_1-1+\mu)^2 + (\eta+q_2)^2$

$$\begin{aligned} r_2^2 = \Bigg[&q_1 - \frac{1}{2} + \frac{(\epsilon_2'-\epsilon_1')}{3} + \left\{\frac{-3}{8} - \epsilon_1' - \frac{(1-\mu)}{2\mu} + \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1 + \left\{\frac{7}{8} + \frac{\epsilon_1'}{2} + \frac{(1-\mu)}{2\mu} - \frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2 \\ &+ \left\{\frac{3}{8} - \frac{3\mu}{8(1-\mu)} + \frac{3\mu\epsilon_2'}{8(1-\mu)} + \epsilon_2'\right\}\sigma_1' + \left\{\frac{-7}{8} + \frac{7\mu}{8(1-\mu)} - \frac{7\mu\epsilon_2'}{8(1-\mu)} - \frac{\epsilon_2'}{2}\right\}\sigma_2'\Bigg]^2 + \left(q_2 + \frac{\Lambda}{2}\right)^2 \end{aligned}$$

$$r_2^{-1}=g(q_1,q_2)$$
$$=\left[\left[q_1-\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2\right.\right.$$
$$\left.\left.+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\right]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\right]^{-1/2} \tag{2.19}$$

$$r_1^{-3}=\alpha(q_1,q_2)$$
$$=\left[\left[q_1+\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2\right.\right.$$
$$\left.\left.+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\right]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\right]^{-3/2} \tag{2.20}$$

$$r_2^{-3}=\beta(q_1,q_2)$$
$$=\left[\left[q_1-\frac{1}{2}+\frac{(\epsilon_2'-\epsilon_1')}{3}+\left\{\frac{-3}{8}-\epsilon_1'-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_1+\left\{\frac{7}{8}+\frac{\epsilon_1'}{2}+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right\}\sigma_2\right.\right.$$
$$\left.\left.+\left\{\frac{3}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}+\epsilon_2'\right\}\sigma_1'+\left\{\frac{-7}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}-\frac{\epsilon_2'}{2}\right\}\sigma_2'\right]^2+\left(q_2+\frac{\Lambda}{2}\right)^2\right]^{-3/2} \tag{2.21}$$

Similarly

$$r_1^{-5}=a(q_1,q_2) \tag{2.22}$$
$$r_2^{-5}=b(q_1,q_2) \tag{2.23}$$

where

$$\frac{\Lambda}{2}=\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{(\epsilon_1'+\epsilon_2')}{3}+\left[\frac{-19}{8}-\epsilon_1'+\frac{(1-\mu)}{2\mu}-\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_1+\left[\frac{15}{8}+\frac{\epsilon_1'}{2}-\frac{(1-\mu)}{2\mu}+\frac{(1-\mu)\epsilon_1'}{2\mu}\right]\sigma_2\right.\right.$$
$$\left.\left.+\left[\frac{-19}{8}-\frac{3\mu}{8(1-\mu)}+\frac{3\mu\epsilon_2'}{8(1-\mu)}-\epsilon_2'\right]\sigma_1'+\left[\frac{15}{8}+\frac{7\mu}{8(1-\mu)}-\frac{7\mu\epsilon_2'}{8(1-\mu)}+\frac{\epsilon_2'}{2}\right]\sigma_2'\right\}\right] \tag{2.24}$$

In order to expand H in powers of q_1 and q_2, we required the expansion of $f(q_1,q_2)$, $g(q_1,q_2)$, $\alpha(q_1,q_2)$, $\beta(q_1,q_2)$, $a(q_1,q_2)$ and $b(q_1,q_2)$ with the help of Taylor's series, and we get

$$f(q_1,q_2)$$
$$=f(0,0)+[q_1f_1(0,0)+q_2f_2(0,0)]+\frac{1}{2}[q_1^2f_{11}(0,0)+2q_1q_2f_{12}(0,0)+q_2^2f_{22}(0,0)]$$
$$+\frac{1}{6}[q_1^3f_{111}(0,0)+3q_1^2q_2f_{112}(0,0)+3q_1q_2^2f_{122}(0,0)+q_2^3f_{222}(0,0)]$$
$$+\frac{1}{24}\left[q_1^4f_{1111}(0,0)+4q_1^3q_2f_{1112}(0,0)+6q_1^2q_2^2f_{1122}(0,0)+4q_1q_2^3f_{1222}(0,0)+q_2^4f_{2222}(0,0)\right]+\ldots \tag{2.25}$$

and

$$g(q_1,q_2)$$
$$= g(0,0)+[q_1g_1(0,0)+q_2g_2(0,0)]+\frac{1}{2}\left[q_1^2g_{11}(0,0)+2q_1q_2g_{1\,2}(0,0)+q_2^2g_{2\,2}(0,0)\right]$$
$$+\frac{1}{6}\left[q_1^3g_{111}(0,0)+3q_1^2q_2g_{1\,1\,2}(0,0)+3q_1q_2^2g_{1\,2\,2}(0,0)+q_2^3g_{2\,2\,2}(0,0)\right]$$
$$+\frac{1}{24}\left[q_1^4g_{1111}(0,0)+4q_1^3q_2g_{111\,2}(0,0)+6q_1^2q_2^2g_{11\,2\,2}(0,0)+4q_1q_2^3g_{122\,2}(0,0)+q_2^4g_{2222}(0,0)\right]+\dots \tag{2.26}$$

similarly

$$\alpha(q_1,q_2)$$
$$= \alpha(0,0)+[q_1\alpha_1(0,0)+q_2\alpha_2(0,0)]+\frac{1}{2}\left[q_1^2\alpha_{11}(0,0)+2q_1q_2\alpha_{1\,2}(0,0)+q_2^2\alpha_{2\,2}(0,0)\right]$$
$$+\frac{1}{6}\left[q_1^3\alpha_{111}(0,0)+3q_1^2q_2\alpha_{1\,1\,2}(0,0)+3q_1q_2^2\alpha_{1\,2\,2}(0,0)+q_2^3\alpha_{2\,2\,2}(0,0)\right]$$
$$+\frac{1}{24}\left[q_1^4\alpha_{1111}(0,0)+4q_1^3q_2\alpha_{111\,2}(0,0)+6q_1^2q_2^2\alpha_{11\,2\,2}(0,0)+4q_1q_2^3\alpha_{122\,2}(0,0)+q_2^4\alpha_{2222}(0,0)\right]+\dots \tag{2.27}$$

and

$$\beta(q_1,q_2)$$
$$= \beta(0,0)+\left[q_1\beta_1(0,0)+q_2\beta_2(0,0)\right]+\frac{1}{2}\left[q_1^2\beta_{11}(0,0)+2q_1q_2\beta_{1\,2}(0,0)+q_2^2\beta_{2\,2}(0,0)\right]$$
$$+\frac{1}{6}\left[q_1^3\beta_{111}(0,0)+3q_1^2q_2\beta_{1\,1\,2}(0,0)+3q_1q_2^2\beta_{1\,2\,2}(0,0)+q_2^3\beta_{2\,2\,2}(0,0)\right]$$
$$+\frac{1}{24}\left[q_1^4\beta_{1111}(0,0)+4q_1^3q_2\beta_{111\,2}(0,0)+6q_1^2q_2^2\beta_{11\,2\,2}(0,0)+4q_1q_2^3\beta_{122\,2}(0,0)+q_2^4\beta_{2222}(0,0)\right]+\dots \tag{2.28}$$

similarly

$$a(q_1,q_2)$$
$$= a(0,0)+[q_1a_1(0,0)+q_2a_2(0,0)]+\frac{1}{2}\left[a_1^2a_{11}(0,0)+2q_1q_2a_{1\,2}(0,0)+q_2^2a_{2\,2}(0,0)\right]$$
$$+\frac{1}{6}\left[q_1^3a_{111}(0,0)+3q_1^2q_2a_{1\,1\,2}(0,0)+3q_1q_2^2a_{1\,2\,2}(0,0)+q_2^3a_{2\,2\,2}(0,0)\right]$$
$$+\frac{1}{24}\left[q_1^4a_{1111}(0,0)+4q_1^3q_2a_{111\,2}(0,0)+6q_1^2q_2^2a_{11\,2\,2}(0,0)+4q_1q_2^3a_{122\,2}(0,0)+q_2^4a_{2222}(0,0)\right]+\dots \tag{2.29}$$

and

$$b(q_1,q_2)$$
$$= b(0,0)+[q_1b_1(0,0)+q_2b_2(0,0)]+\frac{1}{2}\left[a_1^2b_{11}(0,0)+2q_1q_2b_{1\,2}(0,0)+q_2^2b_{2\,2}(0,0)\right]$$

$$
\begin{aligned}
&+\frac{1}{6}\left[q_1^3 b_{111}(0,0)+3q_1^2 q_2 b_{1\,1\,2}(0,0)+3q_1 q_2^2 b_{1\,2\,2}(0,0)+q_2^3 b_{2\,2\,2}(0,0)\right]\\
&+\frac{1}{24}\left[q_1^4 b_{1111}(0,0)+4q_1^3 q_2 b_{111\,2}(0,0)+6q_1^2 q_2^2 b_{11\,2\,2}(0,0)+4q_1 q_2^3 b_{122\,2}(0,0)+q_2^4 b_{2222}(0,0)\right]+\dots
\end{aligned}
\tag{2.30}
$$

Substituting values in equation (2.25), we get

$$
\begin{aligned}
r_1^{-1}=\Bigg[&1-\frac{\epsilon'_2}{12}+\frac{\epsilon'_1}{12}+\frac{41\sigma_1}{32}-\frac{(1-\mu)\sigma_1}{8\mu}-\frac{61\sigma_2}{96}+\frac{(1-\mu)\sigma_2}{8\mu}+\frac{35\sigma'_1}{32}+\frac{9\mu\sigma'_1}{32(1-\mu)}-\frac{23\sigma'_2}{32}+\frac{21\mu\sigma'_2}{32(1-\mu)}\\
&+q_1\left\{\frac{-1}{2}-\frac{\epsilon'_2}{12}+\frac{\epsilon'_1}{12}-\frac{27\sigma_1}{16}+\frac{(1-\mu)\sigma_1}{2\mu}+\frac{19\sigma_2}{16}-\frac{(1-\mu)\sigma_2}{2\mu}-\frac{3\sigma'_1}{4}-\frac{3\mu\sigma'_1}{16(1-\mu)}+\frac{13\sigma'_2}{8}+\frac{7\mu\sigma'_2}{16(1-\mu)}\right\}\\
&-\frac{\sqrt{3}}{2}q_2\left\{1-\frac{5\epsilon'_2}{6}+\frac{\epsilon'_1}{6}+\frac{61\sigma_1}{64}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{23\sigma_2}{8}-\frac{(1-\mu)\sigma_2}{3\mu}+\frac{17\sigma'_1}{12}+\frac{7\mu\sigma'_1}{8(1-\mu)}-\frac{\sigma'_2}{4}-\frac{49\mu\sigma'_2}{24(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{-1}{4}+\frac{7\epsilon'_1}{24}-\frac{7\epsilon'_2}{24}+\frac{305\sigma_1}{32}-\frac{3(1-\mu)\sigma_1}{2\mu}-\frac{257\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{2\mu}+\frac{71\sigma'_1}{8}+\frac{57\mu\sigma'_1}{32(1-\mu)}-\frac{13\sigma'_2}{2}-\frac{169\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\sqrt{3}q_1q_2\left\{\frac{3}{4}+\frac{\epsilon'_1}{12}-\frac{5\epsilon'_2}{12}+\frac{47\sigma_1}{24}+\frac{23(1-\mu)\sigma_1}{12\mu}+\frac{17\sigma_2}{12}-\frac{(1-\mu)\sigma_2}{6\mu}+\frac{41\sigma'_1}{16}+\frac{115\mu\sigma'_1}{24(1-\mu)}-\frac{7\sigma'_2}{8}-\frac{119\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{\frac{5}{4}+\frac{11\epsilon'_1}{4}-\frac{3\epsilon'_2}{4}+\frac{215\sigma_1}{24}-\frac{3(1-\mu)\sigma_1}{4\mu}-\frac{337\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{137\sigma'_1}{16}+\frac{89\mu\sigma'_1}{32(1-\mu)}-\frac{21\sigma'_2}{4}-\frac{215\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{1}{6}q_1^3\left\{\frac{21}{8}+\frac{31\epsilon'_1}{16}-\frac{7\epsilon'_2}{6}+\frac{91\sigma_1}{32}+\frac{7(1-\mu)\sigma_1}{3\mu}-\frac{445\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{87\sigma'_1}{32}+\frac{195\mu\sigma'_1}{32(1-\mu)}-\frac{93\sigma'_2}{16}-\frac{273\mu\sigma'_2}{32(1-\mu)}\right\}\\
&-\frac{\sqrt{3}}{2}q_1^2q_2\left\{\frac{3}{8}-\frac{17\epsilon'_1}{24}+\frac{5\epsilon'_2}{12}-\frac{119\sigma_1}{24}-\frac{47(1-\mu)\sigma_1}{12\mu}-\frac{57\sigma_2}{12}+\frac{(1-\mu)\sigma_2}{12\mu}-\frac{97\sigma'_1}{32}-\frac{267\mu\sigma'_1}{24(1-\mu)}+\frac{3\sigma'_2}{8}+\frac{337\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{1}{2}q_1q_2^2\left\{-\frac{33}{8}-\frac{43\epsilon'_1}{8}+\frac{7\epsilon'_2}{12}+\frac{257\sigma_1}{24}+\frac{117(1-\mu)\sigma_1}{24\mu}-\frac{117\sigma_2}{24}+\frac{(1-\mu)\sigma_2}{6\mu}+\frac{167\sigma'_1}{32}+\frac{441\mu\sigma'_1}{24(1-\mu)}-\frac{7\sigma'_2}{16}-\frac{1137\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_2^3\left\{-\frac{9}{8}-\frac{23\epsilon'_1}{8}+\frac{25\epsilon'_2}{12}+\frac{147\sigma_1}{32}+\frac{45(1-\mu)\sigma_1}{24\mu}-\frac{665\sigma_2}{32}+\frac{7(1-\mu)\sigma_2}{4\mu}+\frac{171\sigma'_1}{16}+\frac{111\mu\sigma'_1}{64(1-\mu)}-\frac{37\sigma'_2}{4}-\frac{487\mu\sigma'_2}{64(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{-\frac{111}{16}-\frac{85\epsilon'_1}{16}+\frac{25\epsilon'_2}{2}+\frac{475\sigma_1}{64}+\frac{13(1-\mu)\sigma_1}{4\mu}-\frac{255\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{2\mu}+\frac{135\sigma'_1}{2}+\frac{157\mu\sigma'_1}{32(1-\mu)}-\frac{695\sigma'_2}{8}-\frac{1171\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1^3q_2\left\{-\frac{75}{16}-\frac{535\epsilon'_1}{48}+\frac{20\epsilon'_2}{3}+\frac{817\sigma_1}{64}+\frac{25(1-\mu)\sigma_1}{4\mu}-\frac{1577\sigma_2}{64}+\frac{13(1-\mu)\sigma_2}{4\mu}+\frac{207\sigma'_1}{4}+\frac{1323\mu\sigma'_1}{64(1-\mu)}-\frac{1431\sigma'_2}{24}-\frac{1979\mu\sigma'_2}{64(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{123}{16}+\frac{95\epsilon'_1}{24}+\frac{20\epsilon'_2}{2}+\frac{983\sigma_1}{64}+\frac{21(1-\mu)\sigma_1}{4\mu}-\frac{1767\sigma_2}{64}+\frac{17(1-\mu)\sigma_2}{4\mu}+\frac{337\sigma'_1}{4}+\frac{1737\mu\sigma'_1}{64(1-\mu)}-\frac{1507\sigma'_2}{24}-\frac{2319\mu\sigma'_2}{64(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1q_2^3\left\{\frac{135}{16}+\frac{235\epsilon'_1}{16}-\frac{7\epsilon'_2}{2}-\frac{889\sigma_1}{64}+\frac{17(1-\mu)\sigma_1}{4\mu}-\frac{2509\sigma_2}{64}+\frac{37(1-\mu)\sigma_2}{8\mu}+\frac{367\sigma'_1}{4}+\frac{1867\mu\sigma'_1}{64(1-\mu)}-\frac{1031\sigma'_2}{24}-\frac{2639\mu\sigma'_2}{64(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{9}{16}+\frac{195\epsilon'_1}{16}-\frac{45\epsilon'_2}{2}+\frac{435\sigma_1}{24}-\frac{7(1-\mu)\sigma_1}{8\mu}-\frac{767\sigma_2}{32}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{303\sigma'_1}{16}+\frac{117\mu\sigma'_1}{32(1-\mu)}-\frac{19\sigma'_2}{4}-\frac{377\mu\sigma'_2}{32(1-\mu)}\right\}\Bigg]
\end{aligned}
\tag{2.31}
$$

Substituting values in equation (2.26), we get

$$
\begin{aligned}
r_2^{-1}=\Bigg[&1+\frac{\epsilon'_1}{12}+\frac{5\epsilon'_2}{12}+\frac{7\sigma_1}{16}-\frac{(1-\mu)\sigma_1}{2\mu}-\frac{\sigma_2}{2}+\frac{(1-\mu)\sigma_2}{2\mu}+\frac{11\sigma'_1}{8}+\frac{3\mu\sigma'_1}{8(1-\mu)}-\frac{11\sigma'_2}{8}-\frac{\mu\sigma'_2}{4(1-\mu)}\\
&+q_1\left\{\frac{1}{2}+\frac{11\epsilon'_1}{24}+\frac{7\epsilon'_2}{24}+\frac{69\sigma_1}{32}-\frac{(1-\mu)\sigma_1}{4\mu}-\frac{13\sigma_2}{8}+\frac{(1-\mu)\sigma_2}{4\mu}+\frac{27\sigma'_1}{16}+\frac{3\mu\sigma'_1}{8(1-\mu)}-\frac{19\sigma'_2}{16}-\frac{7\mu\sigma'_2}{8(1-\mu)}\right\}\\
&-\frac{\sqrt{3}}{2}q_2\left\{1-\frac{\epsilon'_1}{12}+\frac{11\epsilon'_2}{12}-\frac{13\sigma_1}{48}-\frac{7(1-\mu)\sigma_1}{6\mu}-\frac{\sigma_2}{4}+\frac{7(1-\mu)\sigma_2}{6\mu}+\frac{61\sigma'_1}{24}-\frac{\mu\sigma'_1}{4(1-\mu)}-\frac{23\sigma'_2}{8}+\frac{7\mu\sigma'_2}{12(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{-1}{4}+\frac{37\epsilon'_1}{24}+\frac{19\epsilon'_2}{24}+\frac{137\sigma_1}{32}-\frac{3(1-\mu)\sigma_1}{8\mu}-\frac{53\sigma_2}{16}+\frac{7(1-\mu)\sigma_2}{8\mu}+\frac{119\sigma'_1}{32}+\frac{11\mu\sigma'_1}{8(1-\mu)}-\frac{57\sigma'_2}{32}-\frac{17\mu\sigma'_2}{16(1-\mu)}\right\}\\
&+\sqrt{3}q_1q_2\left\{\frac{-3}{4}-\frac{13\epsilon'_1}{24}+\frac{17\epsilon'_2}{12}+\frac{23\sigma_1}{48}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{5\sigma_2}{8}-\frac{(1-\mu)\sigma_2}{3\mu}-\frac{117\sigma'_1}{24}+\frac{7\mu\sigma'_1}{8(1-\mu)}+\frac{27\sigma'_2}{16}-\frac{\mu\sigma'_2}{6(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{\frac{5}{4}-\frac{117\epsilon'_1}{24}+\frac{97\epsilon'_2}{24}+\frac{257\sigma_1}{32}-\frac{5(1-\mu)\sigma_1}{8\mu}-\frac{331\sigma_2}{32}+\frac{5(1-\mu)\sigma_2}{8\mu}+\frac{453\sigma'_1}{32}+\frac{37\mu\sigma'_1}{16(1-\mu)}-\frac{107\sigma'_2}{32}-\frac{33\mu\sigma'_2}{16(1-\mu)}\right\}\\
&+\frac{1}{6}q_1^3\left\{-\frac{21}{8}+\frac{29\epsilon'_1}{24}+\frac{17\epsilon'_2}{24}+\frac{253\sigma_1}{32}-\frac{7(1-\mu)\sigma_1}{8\mu}+\frac{107\sigma_2}{32}+\frac{7(1-\mu)\sigma_2}{16\mu}+\frac{437\sigma'_1}{64}+\frac{27\mu\sigma'_1}{16(1-\mu)}-\frac{117\sigma'_2}{32}+\frac{43\mu\sigma'_2}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{2}q_1^2q_2\left\{\frac{-3}{8}+\frac{5\epsilon'_1}{12}+\frac{17\epsilon'_2}{12}-\frac{37\sigma_1}{48}-\frac{(1-\mu)\sigma_1}{3\mu}-\frac{5\sigma_2}{4}+\frac{(1-\mu)\sigma_2}{3\mu}+\frac{167\sigma'_1}{24}+\frac{5\mu\sigma'_1}{8(1-\mu)}-\frac{47\sigma'_2}{16}-\frac{11\mu\sigma'_2}{12(1-\mu)}\right\}\\
&+\frac{1}{2}q_1q_2^2\left\{\frac{33}{8}-\frac{19\epsilon'_1}{24}-\frac{17\epsilon'_2}{24}+\frac{431\sigma_1}{32}-\frac{7(1-\mu)\sigma_1}{8\mu}-\frac{89\sigma_2}{16}+\frac{7(1-\mu)\sigma_2}{8\mu}+\frac{257\sigma'_1}{32}+\frac{29\mu\sigma'_1}{16(1-\mu)}+\frac{107\sigma'_2}{32}+\frac{19\mu\sigma'_2}{16(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_2^3\left\{\frac{-9}{8}+\frac{5\epsilon'_1}{24}-\frac{13\epsilon'_2}{24}-\frac{135\sigma_1}{48}-\frac{11(1-\mu)\sigma_1}{6\mu}+\frac{17\sigma_2}{4}-\frac{5(1-\mu)\sigma_2}{6\mu}-\frac{357\sigma'_1}{48}-\frac{35\mu\sigma'_1}{16(1-\mu)}-\frac{137\sigma'_2}{32}-\frac{139\mu\sigma'_2}{24(1-\mu)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{24}q_1^4\left\{\frac{-111}{16}-\frac{93\epsilon_1'}{24}-\frac{37\epsilon_2'}{24}+\frac{1285\sigma_1}{32}-\frac{33(1-\mu)\sigma_1}{16\mu}-\frac{1019\sigma_2}{32}+\frac{29(1-\mu)\sigma_2}{16\mu}+\frac{2143\sigma_1'}{64}+\frac{1179\mu\sigma_1'}{32(1-\mu)}-\frac{3563\sigma_2'}{64}-\frac{47\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{2}q_1^3q_2\left\{\frac{75}{16}-\frac{25\epsilon_1'}{12}-\frac{107\epsilon_2'}{24}-\frac{317\sigma_1}{48}+\frac{7(1-\mu)\sigma_1}{6\mu}-\frac{57\sigma_2}{8}-\frac{5(1-\mu)\sigma_2}{6\mu}+\frac{2147\sigma_1'}{48}+\frac{35\mu\sigma_1'}{16(1-\mu)}-\frac{3515\sigma_2'}{32}-\frac{47\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{123}{16}+\frac{167\epsilon_1'}{48}-\frac{57\epsilon_2'}{24}-\frac{1051\sigma_1}{32}-\frac{67(1-\mu)\sigma_1}{32\mu}-\frac{5505\sigma_2}{64}+\frac{217(1-\mu)\sigma_2}{32\mu}+\frac{3347\sigma_1'}{64}+\frac{2157\mu\sigma_1'}{32(1-\mu)}-\frac{4017\sigma_2'}{128}-\frac{197\mu\sigma_2'}{64(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1q_2^3\left\{-\frac{135}{16}-\frac{45\epsilon_1'}{12}-\frac{155\epsilon_2'}{24}-\frac{2205\sigma_1}{64}-\frac{147(1-\mu)\sigma_1}{32\mu}-\frac{2115\sigma_2}{64}+\frac{253(1-\mu)\sigma_2}{32\mu}-\frac{6825\sigma_1'}{128}-\frac{879\mu\sigma_1'}{32(1-\mu)}+\frac{7785\sigma_2'}{128}-\frac{275\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{9}{16}-\frac{45\epsilon_1'}{4}-\frac{135\epsilon_2'}{24}+\frac{15425\sigma_1}{64}-\frac{417(1-\mu)\sigma_1}{32\mu}-\frac{14505\sigma_2}{64}+\frac{357(1-\mu)\sigma_2}{32\mu}-\frac{4975\sigma_1'}{64}-\frac{509\mu\sigma_1'}{32(1-\mu)}+\frac{17925\sigma_2'}{128}-\frac{417\mu\sigma_2'}{64(1-\mu)}\right\}\Bigg]\Bigg]
\end{aligned}
\tag{2.32}
$$

Substituting values in equation (2.27), we get

$$
\begin{aligned}
r_1^{-3}=\Bigg[\Bigg\{&1+\frac{\epsilon_1'}{4}+\frac{5\epsilon_2'}{4}+\frac{21\sigma_1}{16}-\frac{3(1-\mu)\sigma_1}{2\mu}-\frac{3\sigma_2}{2}+\frac{3(1-\mu)\sigma_2}{2\mu}+\frac{33\sigma_1'}{8}+\frac{3\mu\sigma_1'}{4(1-\mu)}-\frac{33\sigma_2'}{8}-\frac{3\mu\sigma_2'}{4(1-\mu)}\Bigg\}\\
&+q_1\left\{\frac{3}{2}+\frac{13\epsilon_1'}{8}+\frac{17\epsilon_2'}{8}+\frac{249\sigma_1}{32}-\frac{9(1-\mu)\sigma_1}{4\mu}-\frac{51\sigma_2}{8}+\frac{9(1-\mu)\sigma_2}{4\mu}+\frac{147\sigma_1'}{16}+\frac{9\mu\sigma_1'}{8(1-\mu)}-\frac{123\sigma_2'}{16}-\frac{21\mu\sigma_2'}{8(1-\mu)}\right\}\\
&-\frac{3\sqrt{3}}{2}q_2\left\{1+\frac{\epsilon_1'}{2}-\frac{7\epsilon_2'}{6}+\frac{127\sigma_1}{24}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{45\sigma_2}{8}-\frac{(1-\mu)\sigma_2}{3\mu}+\frac{41\sigma_1'}{12}+\frac{13\mu\sigma_1'}{8(1-\mu)}-\frac{5\sigma_2'}{4}-\frac{91\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{3}{2}+\frac{13\epsilon_1'}{4}+\frac{31\epsilon_2'}{8}+\frac{427\sigma_1}{32}-\frac{21(1-\mu)\sigma_1}{4\mu}-\frac{147\sigma_2}{16}+\frac{21(1-\mu)\sigma_2}{4\mu}+\frac{329\sigma_1'}{32}+\frac{17\mu\sigma_1'}{8(1-\mu)}-\frac{623\sigma_2'}{32}-\frac{43\mu\sigma_2'}{16(1-\mu)}\right\}\\
&-\frac{15\sqrt{3}}{4}q_1q_2\left\{1-\frac{3\epsilon_1'}{4}+\frac{13\epsilon_2'}{12}+\frac{337\sigma_1}{48}+\frac{5(1-\mu)\sigma_1}{3\mu}-\frac{89\sigma_2}{16}-\frac{5(1-\mu)\sigma_2}{3\mu}+\frac{109\sigma_1'}{24}+\frac{29\mu\sigma_1'}{8(1-\mu)}-\frac{13\sigma_2'}{4}-\frac{267\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{\frac{33}{4}+\frac{27\epsilon_1'}{8}-\frac{217\epsilon_2'}{12}+\frac{3879\sigma_1}{48}+\frac{19(1-\mu)\sigma_1}{6\mu}-\frac{1485\sigma_2}{16}-\frac{19(1-\mu)\sigma_2}{6\mu}+\frac{1057\sigma_1'}{12}+\frac{429\mu\sigma_1'}{16(1-\mu)}-\frac{135\sigma_2'}{8}-\frac{1001\mu\sigma_2'}{8(1-\mu)}\right\}\\
&+\frac{1}{6}q_1^3\left\{\frac{15}{4}+\frac{31\epsilon_1'}{8}+\frac{47\epsilon_2'}{8}+\frac{3029\sigma_1}{64}-\frac{43(1-\mu)\sigma_1}{8\mu}-\frac{437\sigma_2}{32}+\frac{81(1-\mu)\sigma_2}{4\mu}+\frac{4255\sigma_1'}{64}+\frac{33\mu\sigma_1'}{8(1-\mu)}-\frac{11257\sigma_2'}{64}-\frac{169\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{15\sqrt{3}}{16}q_1^2q_2\left\{1-\frac{7\epsilon_1'}{8}+\frac{29\epsilon_2'}{12}+\frac{759\sigma_1}{48}+\frac{13(1-\mu)\sigma_1}{3\mu}-\frac{169\sigma_2}{16}-\frac{7(1-\mu)\sigma_2}{3\mu}+\frac{337\sigma_1'}{24}+\frac{139\mu\sigma_1'}{16(1-\mu)}-\frac{25\sigma_2'}{4}-\frac{889\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{2}q_1q_2^2\left\{\frac{47}{8}+\frac{33\epsilon_1'}{8}-\frac{167\epsilon_2'}{12}+\frac{3119\sigma_1}{48}+\frac{37(1-\mu)\sigma_1}{6\mu}-\frac{2177\sigma_2}{32}-\frac{31(1-\mu)\sigma_2}{6\mu}+\frac{2107\sigma_1'}{24}+\frac{761\mu\sigma_1'}{32(1-\mu)}-\frac{277\sigma_2'}{16}-\frac{3709\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{15\sqrt{3}}{16}q_2^3\left\{1+\frac{21\epsilon_1'}{8}+\frac{91\epsilon_2'}{12}+\frac{2359\sigma_1}{48}+\frac{65(1-\mu)\sigma_1}{6\mu}-\frac{623\sigma_2}{16}-\frac{65(1-\mu)\sigma_2}{6\mu}+\frac{763\sigma_1'}{24}+\frac{203\mu\sigma_1'}{16(1-\mu)}-\frac{91\sigma_2'}{8}-\frac{623\mu\mu_2'}{16(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{\frac{21}{8}+\frac{65\epsilon_1'}{8}+\frac{93\epsilon_2'}{8}+\frac{1037\sigma_1}{32}+\frac{259(1-\mu)\sigma_1}{12\mu}-\frac{107\sigma_2}{8}-\frac{217(1-\mu)\sigma_2}{12\mu}+\frac{177\sigma_1'}{8}+\frac{667\mu\sigma_1'}{16(1-\mu)}-\frac{279\sigma_2'}{16}-\frac{4849\mu\sigma_2'}{32(1-\mu)}\right\}\\
&-\frac{35\sqrt{3}}{16}q_1^3q_2\left\{1-\frac{15\epsilon_1'}{8}+\frac{57\epsilon_2'}{24}+\frac{2277\sigma_1}{48}+\frac{23(1-\mu)\sigma_1}{3\mu}-\frac{507\sigma_2}{16}-\frac{20(1-\mu)\sigma_2}{3\mu}+\frac{1011\sigma_1'}{24}+\frac{197\mu\sigma_1'}{16(1-\mu)}-\frac{65\sigma_2'}{4}-\frac{2667\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{185}{16}+\frac{79\epsilon_1'}{16}-\frac{301\epsilon_2'}{24}+\frac{9359\sigma_1}{48}+\frac{61(1-\mu)\sigma_1}{12\mu}-\frac{6789\sigma_2}{32}-\frac{87(1-\mu)\sigma_2}{12\mu}+\frac{3965\sigma_1'}{24}+\frac{1989\mu\sigma_1'}{32(1-\mu)}-\frac{405\sigma_2'}{16}-\frac{8757\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{55\sqrt{3}}{32}q_1q_2^3\left\{1+\frac{35\epsilon_1'}{8}-\frac{107\epsilon_2'}{24}-\frac{3997\sigma_1}{48}+\frac{67(1-\mu)\sigma_1}{6\mu}-\frac{1007\sigma_2}{16}-\frac{47(1-\mu)\sigma_2}{6\mu}+\frac{3415\sigma_1'}{24}+\frac{307\mu\sigma_1'}{16(1-\mu)}-\frac{111\sigma_2'}{4}-\frac{4795\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{45}{16}+\frac{63\epsilon_1'}{8}+\frac{181\epsilon_2'}{12}-\frac{7079\sigma_1}{48}+\frac{155(1-\mu)\sigma_1}{6\mu}-\frac{1869\sigma_2}{16}-\frac{185(1-\mu)\sigma_2}{6\mu}+\frac{1985\sigma_1'}{24}+\frac{1407\mu\sigma_1'}{16(1-\mu)}-\frac{207\sigma_2'}{8}-\frac{3977\mu\sigma_2'}{48(1-\mu)}\right\}\Bigg]
\end{aligned}
\tag{2.33}
$$

Substituting values in equation (2.28), we get

$$
\begin{aligned}
r_2^{-3}=\Bigg[&1-\frac{\epsilon_1'}{12}-\frac{5\epsilon_2'}{12}+\frac{5\sigma_1}{8}-\frac{3(1-\mu)\sigma_1}{2\mu}-\frac{3\sigma_2}{2}+\frac{3(1-\mu)\sigma_2}{2\mu}-\frac{15\sigma_1'}{8}+\frac{15\mu\sigma_1'}{8(1-\mu)}\\
&+q_1\left\{\frac{3}{2}+\frac{13\epsilon_1'}{24}+\frac{17\epsilon_2'}{24}+\frac{147\sigma_1}{16}-\frac{3(1-\mu)\sigma_1}{4\mu}-\frac{123\sigma_2}{16}+\frac{3(1-\mu)\sigma_2}{4\mu}+\frac{33\sigma_1'}{16}+\frac{7\mu\sigma_1'}{8(1-\mu)}-\frac{53\sigma_2'}{16}-\frac{3\mu\sigma_2'}{8(1-\mu)}\right\}\\
&-\frac{3\sqrt{3}}{2}q_2\left\{1-\frac{5\epsilon_1'}{12}+\frac{1\epsilon_2'}{12}-\frac{127\sigma_1}{8}-\frac{5(1-\mu)\sigma_1}{12\mu}-\frac{5\sigma_2}{4}+\frac{11(1-\mu)\sigma_2}{6\mu}+\frac{107\sigma_1'}{24}-\frac{5\mu\sigma_1'}{4(1-\mu)}-\frac{47\sigma_2'}{16}+\frac{5\mu\sigma_2'}{12(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{3}{4}-\frac{23\epsilon_1'}{24}-\frac{17\epsilon_2'}{24}+\frac{187\sigma_1}{32}-\frac{5(1-\mu)\sigma_1}{8\mu}+\frac{77\sigma_2}{16}-\frac{3(1-\mu)\sigma_2}{8\mu}+\frac{207\sigma_1'}{32}+\frac{19\mu\sigma_1'}{8(1-\mu)}-\frac{111\sigma_2'}{32}-\frac{23\mu\sigma_2'}{16(1-\mu)}\right\}\\
&-\frac{15\sqrt{3}}{4}q_1q_2\left\{1-\frac{55\epsilon_1'}{24}+\frac{41\epsilon_2'}{12}-\frac{41\sigma_1}{48}-\frac{5(1-\mu)\sigma_1}{3\mu}-\frac{25\sigma_2}{8}+\frac{5(1-\mu)\sigma_2}{3\mu}+\frac{215\sigma_1'}{24}-\frac{5\mu\sigma_1'}{8(1-\mu)}-\frac{41\sigma_2'}{16}-\frac{5\mu\sigma_2'}{6(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{\frac{33}{4}-\frac{207\epsilon_1'}{24}+\frac{167\epsilon_2'}{24}+\frac{441\sigma_1}{32}-\frac{7(1-\mu)\sigma_1}{8\mu}-\frac{715\sigma_2}{32}+\frac{25(1-\mu)\sigma_2}{8\mu}-\frac{1063\sigma_1'}{32}-\frac{67\mu\sigma_1'}{16(1-\mu)}-\frac{301\sigma_2'}{32}-\frac{41\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{1}{6}q_1^3\left\{-\frac{75}{8}-\frac{41\epsilon_1'}{24}-\frac{27\epsilon_2'}{48}-\frac{417\sigma_1}{32}+\frac{5(1-\mu)\sigma_1}{8\mu}+\frac{157\sigma_2}{32}+\frac{7(1-\mu)\sigma_2}{8\mu}-\frac{889\sigma_1'}{64}-\frac{91\mu\sigma_1'}{16(1-\mu)}+\frac{401\sigma_2'}{32}-\frac{61\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{2}q_1^2q_2\left\{-\frac{48}{8}+\frac{7\epsilon_1'}{12}+\frac{35\epsilon_2'}{12}+\frac{59\sigma_1}{48}+\frac{5(1-\mu)\sigma_1}{3\mu}+\frac{11\sigma_2}{4}-\frac{7(1-\mu)\sigma_2}{3\mu}-\frac{297\sigma_1'}{24}-\frac{11\mu\sigma_1'}{8(1-\mu)}+\frac{89\sigma_2'}{16}+\frac{17\mu\sigma_2'}{12(1-\mu)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}q_1q_2^2\left\{\frac{255}{8}-\frac{37\epsilon_1'}{24}-\frac{19\epsilon_2'}{24}+\frac{1057\sigma_1}{32}-\frac{11(1-\mu)\sigma_1}{8\mu}-\frac{267\sigma_2}{16}+\frac{35(1-\mu)\sigma_2}{8\mu}+\frac{307\sigma_1'}{32}+\frac{31\mu\sigma_1'}{16(1-\mu)}+\frac{517\sigma_2'}{32}+\frac{31\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_2^3\left\{-\frac{135}{8}+\frac{11\epsilon_1'}{24}-\frac{17\epsilon_2'}{24}-\frac{337\sigma_1}{48}-\frac{43(1-\mu)\sigma_1}{6\mu}+\frac{27\sigma_2}{8}-\frac{17(1-\mu)\sigma_2}{6\mu}+\frac{1437\sigma_1'}{48}-\frac{53\mu\sigma_1'}{16(1-\mu)}-\frac{403\sigma_2'}{32}-\frac{477\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{-\frac{855}{16}-\frac{227\epsilon_1'}{24}-\frac{134\epsilon_2'}{24}+\frac{3379\sigma_1}{32}-\frac{109(1-\mu)\sigma_1}{16\mu}+\frac{4133\sigma_2}{32}-\frac{67(1-\mu)\sigma_2}{16\mu}-\frac{8167\sigma_1'}{64}+\frac{4555\mu\sigma_1'}{32(1-\mu)}-\frac{11345\sigma_2'}{64}-\frac{177\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1^3q_2\left\{\frac{315}{16}-\frac{57\epsilon_1'}{12}-\frac{257\epsilon_2'}{24}-\frac{789\sigma_1}{48}+\frac{11(1-\mu)\sigma_1}{6\mu}-\frac{189\sigma_2}{8}-\frac{11(1-\mu)\sigma_2}{6\mu}+\frac{7345\sigma_1'}{48}+\frac{111\mu\sigma_1'}{16(1-\mu)}-\frac{8877\sigma_2'}{32}-\frac{137\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{1395}{16}+\frac{335\epsilon_1'}{48}-\frac{127\epsilon_2'}{24}-\frac{3119\sigma_1}{32}-\frac{189(1-\mu)\sigma_1}{32\mu}-\frac{19567\sigma_2}{64}+\frac{689(1-\mu)\sigma_2}{32\mu}+\frac{11537\sigma_1'}{64}+\frac{8329\mu\sigma_1'}{32(1-\mu)}-\frac{18469\sigma_2'}{128}-\frac{945\mu\sigma_2'}{64(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1q_2^3\left\{-\frac{2835}{16}-\frac{187\epsilon_1'}{12}-\frac{461\epsilon_2'}{24}-\frac{9195\sigma_1}{64}-\frac{607(1-\mu)\sigma_1}{32\mu}-\frac{8585\sigma_2}{64}+\frac{785(1-\mu)\sigma_2}{32\mu}-\frac{37927\sigma_1'}{128}-\frac{4233\mu\sigma_1'}{32(1-\mu)}+\frac{33935\sigma_2'}{128}-\frac{837\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{1665}{16}-\frac{197\epsilon_1'}{4}-\frac{337\epsilon_2'}{24}+\frac{1037\sigma_1}{64}-\frac{2527(1-\mu)\sigma_1}{32\mu}-\frac{67577\sigma_2}{64}+\frac{2307(1-\mu)\sigma_2}{32\mu}-\frac{28957\sigma_1'}{64}-\frac{4337\mu\sigma_1'}{32(1-\mu)}+\frac{141727\sigma_2'}{128}-\frac{3237\mu\sigma_2'}{64(1-\mu)}\right\}\Bigg]
\end{aligned}
\tag{2.34}
$$

Substituting values in equation (2.29), we get

$$
\begin{aligned}
r_1^{-5}=\Bigg[&1+\frac{5\epsilon_1'}{12}+\frac{25\epsilon_2'}{12}+\frac{35\sigma_1}{16}-\frac{5(1-\mu)\sigma_1}{2\mu}-\frac{5\sigma_2}{2}+\frac{5(1-\mu)\sigma_2}{2\mu}+\frac{55\sigma_1'}{8}-\frac{55\sigma_2'}{8}\\
&+q_1\left\{\frac{5}{2}+\frac{25\epsilon_1'}{8}+\frac{45\epsilon_2'}{8}+\frac{485\sigma_1}{32}-\frac{25(1-\mu)\sigma_1}{4\mu}-\frac{105\sigma_2}{8}+\frac{25(1-\mu)\sigma_2}{4\mu}+\frac{355\sigma_1'}{16}-\frac{3\mu\sigma_1'}{8(1-\mu)}-\frac{315\sigma_2'}{16}+\frac{7\mu\sigma_2'}{8(1-\mu)}\right\}\\
&-\frac{5\sqrt{3}}{2}q_2\left\{1+\frac{5\epsilon_1'}{6}-\frac{3\epsilon_2'}{2}+\frac{193\sigma_1}{24}+\frac{(1-\mu)\sigma_1}{3\mu}-\frac{67\sigma_2}{8}-\frac{(1-\mu)\sigma_2}{3\mu}+\frac{65\sigma_1'}{12}+\frac{19\mu\sigma_1'}{8(1-\mu)}-\frac{9\sigma_2'}{4}-\frac{133\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{15}{4}+\frac{43\epsilon_1'}{8}+\frac{63\epsilon_2'}{8}+\frac{777\sigma_1}{32}-\frac{59(1-\mu)\sigma_1}{8\mu}-\frac{615\sigma_2}{8}+\frac{77(1-\mu)\sigma_2}{8\mu}+\frac{879\sigma_1'}{16}-\frac{11\mu\sigma_1'}{8(1-\mu)}-\frac{757\sigma_2'}{16}+\frac{13\mu\sigma_2'}{8(1-\mu)}\right\}\\
&-\frac{35\sqrt{3}}{4}q_1q_2\left\{1-\frac{7\epsilon_1'}{4}+\frac{21\epsilon_2'}{12}+\frac{1067\sigma_1}{48}+\frac{25(1-\mu)\sigma_1}{3\mu}-\frac{227\sigma_2}{16}-\frac{28(1-\mu)\sigma_2}{3\mu}+\frac{529\sigma_1'}{24}+\frac{43\mu\sigma_1'}{8(1-\mu)}-\frac{37\sigma_2'}{4}-\frac{723\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{\frac{85}{4}+\frac{189\epsilon_1'}{16}-\frac{1519\epsilon_2'}{24}+\frac{27153\sigma_1}{48}+\frac{133(1-\mu)\sigma_1}{12\mu}-\frac{10395\sigma_2}{32}-\frac{133(1-\mu)\sigma_2}{12\mu}+\frac{7399\sigma_1'}{12}+\frac{10003\mu\sigma_1'}{32(1-\mu)}-\frac{945\sigma_2'}{8}-\frac{6789\mu\sigma_2'}{8(1-\mu)}\right\}\\
&+\frac{1}{6}q_1^3\left\{\frac{105}{8}+\frac{217\epsilon_1'}{16}+\frac{329\epsilon_2'}{16}+\frac{21203\sigma_1}{128}-\frac{301(1-\mu)\sigma_1}{16\mu}-\frac{3059\sigma_2}{64}+\frac{567(1-\mu)\sigma_2}{8\mu}+\frac{29785\sigma_1'}{128}+\frac{231\mu\sigma_1'}{16(1-\mu)}-\frac{78799\sigma_2'}{128}-\frac{1183\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{105\sqrt{3}}{16}q_1^2q_2\left\{1-\frac{13\epsilon_1'}{8}+\frac{47\epsilon_2'}{12}+\frac{2277\sigma_1}{96}+\frac{39(1-\mu)\sigma_1}{6\mu}-\frac{507\sigma_2}{32}-\frac{21(1-\mu)\sigma_2}{6\mu}+\frac{1011\sigma_1'}{48}+\frac{417\mu\sigma_1'}{32(1-\mu)}-\frac{75\sigma_2'}{8}-\frac{2667\mu\sigma_2'}{96(1-\mu)}\right\}\\
&+\frac{1}{2}q_1q_2^2\left\{\frac{141}{16}+\frac{99\epsilon_1'}{16}-\frac{501\epsilon_2'}{24}+\frac{9357\sigma_1}{48}+\frac{111(1-\mu)\sigma_1}{12\mu}-\frac{6531\sigma_2}{32}-\frac{93(1-\mu)\sigma_2}{12\mu}+\frac{6321\sigma_1'}{24}+\frac{2283\mu\sigma_1'}{64(1-\mu)}-\frac{831\sigma_2'}{16}-\frac{11127\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{45\sqrt{3}}{32}q_2^3\left\{1+\frac{63\epsilon_1'}{8}+\frac{91\epsilon_2'}{4}+\frac{2359\sigma_1}{16}+\frac{65(1-\mu)\sigma_1}{2\mu}-\frac{1869\sigma_2}{16}-\frac{65(1-\mu)\sigma_2}{2\mu}+\frac{763\sigma_1'}{8}+\frac{609\mu\sigma_1'}{16(1-\mu)}-\frac{273\sigma_2'}{8}-\frac{1869\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{\frac{147}{16}+\frac{455\epsilon_1'}{16}+\frac{651\epsilon_2'}{16}+\frac{7259\sigma_1}{64}+\frac{1813(1-\mu)\sigma_1}{24\mu}-\frac{749\sigma_2}{16}-\frac{1519(1-\mu)\sigma_2}{24\mu}+\frac{1239\sigma_1'}{16}+\frac{4669\mu\sigma_1'}{32(1-\mu)}-\frac{1953\sigma_2'}{32}-\frac{33943\mu\sigma_2'}{64(1-\mu)}\right\}\\
&-\frac{245\sqrt{3}}{32}q_1^3q_2\left\{1-\frac{33\epsilon_1'}{8}+\frac{47\epsilon_2'}{24}+\frac{6831\sigma_1}{48}+\frac{161(1-\mu)\sigma_1}{3\mu}-\frac{1521\sigma_2}{16}-\frac{26(1-\mu)\sigma_2}{3\mu}+\frac{1399\sigma_1'}{24}+\frac{591\mu\sigma_1'}{16(1-\mu)}-\frac{455\sigma_2'}{4}-\frac{4937\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{1295}{32}+\frac{277\epsilon_1'}{16}-\frac{1053\epsilon_2'}{24}+\frac{32757\sigma_1}{48}+\frac{197(1-\mu)\sigma_1}{12\mu}-\frac{23761\sigma_2}{32}-\frac{271(1-\mu)\sigma_2}{12\mu}+\frac{12291\sigma_1'}{32}+\frac{6921\mu\sigma_1'}{32(1-\mu)}-\frac{1417\sigma_2'}{16}-\frac{30561\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{385\sqrt{3}}{64}q_1q_2^3\left\{1+\frac{67\epsilon_1'}{8}-\frac{219\epsilon_2'}{24}-\frac{5719\sigma_1}{48}+\frac{101(1-\mu)\sigma_1}{6\mu}-\frac{2717\sigma_2}{16}-\frac{71(1-\mu)\sigma_2}{6\mu}+\frac{6787\sigma_1'}{24}+\frac{461\mu\sigma_1'}{16(1-\mu)}-\frac{155\sigma_2'}{4}-\frac{7193\mu\sigma_2'}{48(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{315}{32}+\frac{441\epsilon_1'}{8}+\frac{1267\epsilon_2'}{12}-\frac{49553\sigma_1}{48}+\frac{1085(1-\mu)\sigma_1}{6\mu}-\frac{13083\sigma_2}{16}-\frac{1295(1-\mu)\sigma_2}{6\mu}+\frac{13895\sigma_1'}{24}+\frac{9849\mu\sigma_1'}{16(1-\mu)}-\frac{1449\sigma_2'}{8}-\frac{27839\mu\sigma_2'}{48(1-\mu)}\right\}\Bigg]
\end{aligned}
\tag{2.35}
$$

Substituting values in equation (2.30), we get

$$
\begin{aligned}
r_2^{-5}=\Bigg[&\frac{3}{4}-\frac{5\epsilon_1'}{24}-\frac{7\epsilon_2'}{12}+\frac{19\sigma_1}{16}-\frac{5(1-\mu)\sigma_1}{2\mu}-\frac{5\sigma_2}{2}+\frac{7(1-\mu)\sigma_2}{2\mu}-\frac{31\sigma_1'}{8}+\frac{31\sigma_2'}{8}\\
&+q_1\left\{\frac{5}{2}+\frac{19\epsilon_1'}{24}+\frac{41\epsilon_2'}{24}+\frac{195\sigma_1}{16}-\frac{11(1-\mu)\sigma_1}{4\mu}-\frac{517\sigma_2}{16}+\frac{11(1-\mu)\sigma_2}{4\mu}+\frac{67\sigma_1'}{16}+\frac{29\mu\sigma_1'}{16(1-\mu)}-\frac{119\sigma_2'}{16}-\frac{9\mu\sigma_2'}{8(1-\mu)}\right\}\\
&-\frac{5\sqrt{3}}{2}q_2\left\{1-\frac{13\epsilon_1'}{12}+\frac{5\epsilon_2'}{12}-\frac{419\sigma_1}{8}-\frac{17(1-\mu)\sigma_1}{12\mu}-\frac{15\sigma_2}{4}+\frac{29(1-\mu)\sigma_2}{6\mu}+\frac{567\sigma_1'}{24}-\frac{13\mu\sigma_1'}{4(1-\mu)}-\frac{177\sigma_2'}{16}+\frac{11\mu\sigma_2'}{12(1-\mu)}\right\}\\
&+\frac{1}{2}q_1^2\left\{\frac{45}{16}-\frac{47\epsilon_1'}{24}-\frac{35\epsilon_2'}{24}+\frac{417\sigma_1}{32}-\frac{11(1-\mu)\sigma_1}{8\mu}+\frac{219\sigma_2}{16}-\frac{7(1-\mu)\sigma_2}{8\mu}+\frac{795\sigma_1'}{32}+\frac{77\mu\sigma_1'}{8(1-\mu)}-\frac{667\sigma_2'}{32}-\frac{87\mu\sigma_2'}{16(1-\mu)}\right\}\\
&-\frac{35\sqrt{3}}{4}q_1q_2\left\{1-\frac{115\epsilon_1'}{24}+\frac{71\epsilon_2'}{12}-\frac{67\sigma_1}{68}-\frac{13(1-\mu)\sigma_1}{3\mu}-\frac{35\sigma_2}{8}+\frac{13(1-\mu)\sigma_2}{3\mu}+\frac{417\sigma_1'}{24}-\frac{13\mu\sigma_1'}{8(1-\mu)}-\frac{67\sigma_2'}{16}-\frac{13\mu\sigma_2'}{6(1-\mu)}\right\}\\
&+\frac{1}{2}q_2^2\left\{-\frac{35}{8}+\frac{317\epsilon_1'}{24}-\frac{495\epsilon_2'}{24}-\frac{1069\sigma_1}{32}-\frac{13(1-\mu)\sigma_1}{8\mu}-\frac{2023\sigma_2}{32}+\frac{13(1-\mu)\sigma_2}{8\mu}-\frac{3379\sigma_1'}{32}-\frac{219\mu\sigma_1'}{16(1-\mu)}-\frac{519\sigma_2'}{32}-\frac{179\mu\sigma_2'}{16(1-\mu)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{6}q_1^3\left\{\frac{105}{8}-\frac{93\epsilon_1'}{24}-\frac{109\epsilon_2'}{48}-\frac{1569\sigma_1}{32}+\frac{13(1-\mu)\sigma_1}{8\mu}+\frac{397\sigma_2}{32}+\frac{11(1-\mu)\sigma_2}{8\mu}-\frac{2679\sigma_1'}{64}-\frac{417\mu\sigma_1'}{16(1-\mu)}+\frac{1337\sigma_2'}{32}-\frac{237\mu\sigma_2'}{32(1-\mu)}\right\}\\
&-\frac{175\sqrt{3}}{16}q_1^2q_2\left\{1+\frac{13\epsilon_1'}{24}+\frac{23\epsilon_2'}{12}+\frac{47\sigma_1}{48}+\frac{11(1-\mu)\sigma_1}{3\mu}+\frac{17\sigma_2}{4}-\frac{11(1-\mu)\sigma_2}{3\mu}-\frac{403\sigma_1'}{24}-\frac{17\mu\sigma_1'}{8(1-\mu)}+\frac{107\sigma_2'}{16}+\frac{23\mu\sigma_2'}{12(1-\mu)}\right\}\\
&+\frac{1}{2}q_1q_2^2\left\{-\frac{805}{8}+\frac{141\epsilon_1'}{24}+\frac{79\epsilon_2'}{24}-\frac{3169\sigma_1}{32}-\frac{17(1-\mu)\sigma_1}{8\mu}-\frac{597\sigma_2}{16}+\frac{105(1-\mu)\sigma_2}{8\mu}+\frac{1469\sigma_1'}{32}+\frac{89\mu\sigma_1'}{16(1-\mu)}+\frac{2395\sigma_2'}{32}+\frac{89\mu\sigma_2'}{16(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_2^3\left\{-\frac{525}{8}+\frac{41\epsilon_1'}{24}-\frac{65\epsilon_2'}{24}-\frac{1283\sigma_1}{48}-\frac{163(1-\mu)\sigma_1}{6\mu}+\frac{103\sigma_2}{8}-\frac{65(1-\mu)\sigma_2}{6\mu}+\frac{5461\sigma_1'}{48}-\frac{201\mu\sigma_1'}{16(1-\mu)}-\frac{1531\sigma_2'}{32}-\frac{1813\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{24}q_1^4\left\{-\frac{2415}{16}-\frac{641\epsilon_1'}{24}-\frac{389\epsilon_2'}{24}+\frac{19543\sigma_1}{32}-\frac{307(1-\mu)\sigma_1}{16\mu}+\frac{11673\sigma_2}{32}-\frac{189(1-\mu)\sigma_2}{16\mu}-\frac{23067\sigma_1'}{64}+\frac{12865\mu\sigma_1'}{32(1-\mu)}-\frac{32043\sigma_2'}{64}-\frac{999\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1^3q_2\left\{\frac{105}{64}-\frac{217\epsilon_1'}{12}-\frac{983\epsilon_2'}{24}-\frac{3017\sigma_1}{48}+\frac{43(1-\mu)\sigma_1}{6\mu}-\frac{723\sigma_2}{8}-\frac{43(1-\mu)\sigma_2}{6\mu}+\frac{28085\sigma_1'}{48}+\frac{425\mu\sigma_1'}{16(1-\mu)}-\frac{33943\sigma_2'}{32}-\frac{523\mu\sigma_2'}{24(1-\mu)}\right\}\\
&+\frac{1}{4}q_1^2q_2^2\left\{\frac{9825}{64}+\frac{589\epsilon_1'}{48}-\frac{223\epsilon_2'}{24}-\frac{5485\sigma_1}{32}-\frac{333(1-\mu)\sigma_1}{32\mu}-\frac{34405\sigma_2}{64}+\frac{1211(1-\mu)\sigma_2}{32\mu}+\frac{20285\sigma_1'}{64}+\frac{14645\mu\sigma_1'}{32(1-\mu)}-\frac{32475\sigma_2'}{128}-\frac{1661\mu\sigma_2'}{64(1-\mu)}\right\}\\
&+\frac{\sqrt{3}}{6}q_1q_2^3\left\{-\frac{3885}{64}-\frac{403\epsilon_1'}{12}-\frac{995\epsilon_2'}{24}-\frac{19841\sigma_1}{64}-\frac{1309(1-\mu)\sigma_1}{32\mu}-\frac{18525\sigma_2}{64}+\frac{1693(1-\mu)\sigma_2}{32\mu}-\frac{81843\sigma_1'}{128}-\frac{9135\mu\sigma_1'}{32(1-\mu)}+\frac{73227\sigma_2'}{128}-\frac{1807\mu\sigma_2'}{32(1-\mu)}\right\}\\
&+\frac{1}{24}q_2^4\left\{\frac{10185}{16}-\frac{1205\epsilon_1'}{4}-\frac{2061\epsilon_2'}{24}+\frac{634547\sigma_1}{64}-\frac{15457(1-\mu)\sigma_1}{32\mu}-\frac{413375\sigma_2}{64}+\frac{14112(1-\mu)\sigma_2}{32\mu}-\frac{177133\sigma_1'}{64}-\frac{26529\mu\sigma_1'}{32(1-\mu)}+\frac{866959\sigma_2'}{128}-\frac{19801\mu\sigma_2'}{64(1-\mu)}\right\}\Bigg]
\end{aligned}
\tag{2.36}
$$

3. Stability of Triangular Equilibrium Points of the Problem in Circular Case

We will discusses the stability of infinitesimal in elliptical restricted three body problem under the assumption that, both the primaries are radiating and triaxial. This discussion is for particular case $e = 0$. To test the stability we will find a suitable Hamiltonian.

Substituting values of r_1^{-1}, r_2^{-1}, r_1^{-3}, r_2^{-3}, r_1^{-5} and r_2^{-5} in equation (2.9) and taking second order terms. After this we get expression for Hamiltonian, which is represented as follows:

$$H_2 = \frac{p_1^2+p_2^2}{2} + (p_1q_2 - p_2q_2) + \left[\left(\frac{1}{8}+A\right)q_1^2 - q_1q_2(K-B) - \left(\frac{5}{8}+C\right)q_2^2\right], \tag{3.1}$$

where

$$
\begin{aligned}
A = &-\frac{\epsilon_1'}{48}+\frac{7\epsilon_2'}{48}-\frac{437\sigma_1}{64}+\frac{349\sigma_2}{64}-\frac{77\sigma_1'}{16}+\frac{15\sigma_2'}{16}+\frac{3\sigma_1}{4\mu}-\frac{3\sigma_2}{4\mu}-\frac{3\mu\epsilon_1'}{4}-\frac{5\mu\epsilon_2'}{12}+\frac{57\mu\sigma_1}{16}\\
&-\frac{183\mu\sigma_2}{64}-\frac{21\mu\sigma_1'}{16}+\frac{15\mu\sigma_2'}{32},
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
B = \sqrt{3}\Bigg\{&-\frac{5\epsilon_1'}{6}+\frac{5\epsilon_2'}{12}+\frac{181\sigma_1}{24}-\frac{17\sigma_2}{4}-\frac{5\sigma_1'}{16}-\frac{\sigma_2'}{4}-\frac{23\sigma_1}{12\mu}-\frac{\sigma_2}{6\mu}+\frac{11\mu\epsilon_1'}{8}-\frac{13\mu\epsilon_2'}{12}-\frac{401\mu\sigma_1}{48}\\
&+6\mu\sigma_2\frac{91\mu\sigma_1'}{48}+\frac{131\mu\sigma_2'}{96}\Bigg\},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
C = 2\epsilon_1' &- \frac{3\epsilon_2'}{8}+\frac{17\sigma_1}{3}-\frac{341\sigma_2}{64}+\frac{77\sigma_1'}{32}-\frac{27\sigma_2'}{16}-\frac{3\sigma_1}{8\mu}+\frac{3\sigma_2}{8\mu}-\frac{71\mu\epsilon_1'}{16}+\frac{145\mu\epsilon_2'}{48}-\frac{605\mu\sigma_1}{192}\\
&+\frac{23\mu\sigma_2}{32}+\frac{115\mu\sigma_1'}{16}-\frac{107\mu\sigma_2'}{32}
\end{aligned}
\tag{3.4}
$$

and

$$K = \frac{3\sqrt{3}}{4}(1-2\mu). \tag{3.5}$$

The variational equation can be written for circular case as follows:

$$\dot{p}_i = -\frac{\partial H_2}{\partial q_i}, \quad \dot{q}_i = -\frac{\partial H_2}{\partial p_i}, \quad i = 1,2, \tag{3.6}$$

where H_2 is given by the equation (3.1). Hence the canonical equation for the circular problem is given by:

$$\ddot{q}_1 - 2\dot{q}_2 = A^* q_1 + B^* q_2; \quad \ddot{q}_2 - 2\dot{q}_1 = B^* q_1 + C^* q_2, \tag{3.7}$$

where $A^* = \frac{3}{4} - 2A$, $B^* = K - B$ and $C^* = \frac{9}{4} + 2A$, where A, B, C and K are defined in equations (3.2), (3.3), (3.4) and (3.5).

Now the characteristic equation for the problem can be defined by putting

$$q_1 = Le^{\lambda t},\ q_2 = Me^{\lambda t}; \quad \dot{q}_1 = L\lambda\, e^{\lambda t},\ \dot{q}_2 = M\lambda\, e^{\lambda t}; \quad \ddot{q}_1 = L\lambda^2 e^{\lambda t},\ \ddot{q}_2 = M\lambda^2 e^{\lambda t}. \tag{3.8}$$

The characteristic equation is obtained by substituting the result obtained from equation (3.8) in equation (3.7),

$$\begin{vmatrix} \lambda^2 - A^* & -2\lambda - B^* \\ 2\lambda - B^* & \lambda^2 - C^* \end{vmatrix} = 0. \tag{3.9}$$

By solving the equation (3.9), we get

$$\lambda^4 - \lambda^2(A^* + C^* - 4) + (A^* C^* - {B^*}^2) = 0, \tag{3.10}$$

where

$$A^* + C^* - 4 = -1 \tag{3.11}$$

and

$$\begin{aligned} A^* C^* - {B^*}^2 &= \frac{27}{16} - 3A - 4A^2 - (K - B)^2 \\ &= \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\}. \end{aligned} \tag{3.12}$$

The characteristic equation (3.10) has been reduced to the following form:

$$\lambda^4 + \lambda^2 + \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\} = 0. \tag{3.13}$$

It is notified that when $\sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0$, the characteristic equation (3.13) is showing the classical restricted three body problem:

Let $\lambda_1 = i\omega_1$ and $\lambda_2 = i\omega_2$, from equation (3.13), we get

$$\omega^4 - \omega^2 + \frac{27\mu(1-\mu)}{4}\left\{1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right\} = 0. \tag{3.14}$$

Hence, we get

$$\omega_{1,2}^2 = \frac{1}{2}\left[1 \pm \left\{1 - 27\mu(1-\mu)\left(1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right)\right\}^{1/2}\right] \tag{3.15}$$

i.e.

$$\omega_1 = \left[\frac{1}{2}\left[1 + \left\{1 - 27\mu(1-\mu)\left(1 - \frac{\epsilon_1'}{16} + \frac{7\epsilon_2'}{16} - \frac{1311\sigma_1}{64} + \frac{1047\sigma_2}{64} - \frac{231\sigma_1'}{16} + \frac{45\sigma_2'}{16}\right)\right\}^{1/2}\right]\right]^{1/2},$$

$$\omega_2=\left[\frac{1}{2}\left[1-\left\{1-27\mu(1-\mu)\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)\right\}^{1/2}\right]\right]^{1/2}. \quad (3.16)$$

The equilibrium position is stable, if $\omega_{1,2}$ are purely imaginary. For $\omega_{1,2}$ to be purely imaginary it is necessary that $\omega_{1,2}^2$ is negative. Hence the discriminant of equation (3.14) is represented as follows:

$$1-27\mu(1-\mu)\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)=0. \quad (3.17)$$

If the equality relation holds in (3.17), we get

$$1-27\mu(1-\mu)\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)=0 \quad (3.18)$$

i.e.

$$\begin{aligned}27&\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)\mu^2\\&-27\left(1-\frac{\epsilon_1'}{16}+\frac{7\epsilon_2'}{16}-\frac{1311\sigma_1}{64}+\frac{1047\sigma_2}{64}-\frac{231\sigma_1'}{16}+\frac{45\sigma_2'}{16}\right)\mu+1=0\end{aligned} \quad (3.19)$$

that is,

$$\mu=\frac{1}{18}\left[9\pm\sqrt{69}\left(1-\frac{\epsilon_1'}{92}+\frac{7\epsilon_2'}{92}-\frac{57\sigma_1}{16}+\frac{1047\sigma_2}{368}-\frac{231\sigma_1'}{92}+\frac{45\sigma_2'}{92}\right)\right]. \quad (3.20)$$

Since $\mu=\frac{1}{2}$, the positive sign is not acceptable. Hence, the region of stability in first approximation can be written as

$$0<\mu<\frac{1}{18}\left[9\pm\sqrt{69}\left(1-\frac{\epsilon_1'}{92}+\frac{7\epsilon_2'}{92}-\frac{57\sigma_1}{16}+\frac{1047\sigma_2}{368}-\frac{231\sigma_1'}{92}+\frac{45\sigma_2'}{92}\right)\right]. \quad (3.21)$$

Thus the value of μ for stability equilibrium of equilibrium points is given by

$$\begin{aligned}\mu_{critical}&=0.0385208965+0.005016077212\epsilon_1'-0.03511254048\epsilon_2'+1.644019306\sigma_1\\&\quad-1.31295821\sigma_2+1.1587138836\sigma_1'-0.2257234745\sigma_2'.\end{aligned} \quad (3.22)$$

It is notified that, when $\epsilon_1'=\epsilon_2'=\sigma_1=\sigma_2=\sigma_1'=\sigma_2'=0$

$$\omega_1(\mu_c)=\omega_2(\mu_c)=\frac{1}{\sqrt{2}} \quad \text{and} \quad \omega_1(0)=1,\ \omega_2(0)=0. \quad (3.23)$$

Further it is observed that in the neighborhood of the value of μ the parametric resonance is possible. For which the frequencies ω_1 and ω_2 given in equation(3.16) must satisfy at least one of the following conditions:

$$\omega_1=\frac{N}{2}\omega_2=\frac{N}{2} \quad \text{and} \quad \omega_1-\omega_2=N, \quad (3.24)$$

where N is the natural number. The dependency of $\omega_1$and ω_2 on μ can be verified drawing the curves between μ and $\omega_{1,2}$ by using simulation technique. Clearly in the region (3.17), the only resonance $\omega_2=\frac{1}{2}$ is possible.

The corresponding value of μ for $\omega_2=\frac{1}{2}$ is given by

$$\begin{aligned}\mu_0&=0.0285954+0.003659315722\epsilon_1'-0.02561521005\epsilon_2'+1.199340728\sigma_1\\&\quad-0.9578258901\sigma_2+0.8453019319\sigma_1'-0.1646692075\sigma_2'.\end{aligned} \quad (3.25)$$

4. Conclusion

The discussion about the stability of infinitesimal around the triangular equilibrium points in elliptical restricted three body problem under the assumption that both the primaries are radiating and triaxial, for the circular case (i.e. $e = 0$) has been done. For the verification of stability we have constructed a suitable Hamiltonian function and investigated the stability of infinitesimal around the triangular equilibrium points and the perturbed system analytically and numerically due to triaxiality and radiation of the primaries in circular case up to second order terms. The region of stability and instability has been found by using simulation technique. In the curves drawn between μ and $\omega_{1,2}$, the system is unstable in the region lying out side the boundary and the system is stable in the region lying in side the boundary. We conclude that the effect of the triaxiality and radiation of primaries affects the location and resonance stability of triangular equilibrium points in elliptical restricted three body problem in the circular case (i.e. $e = 0$) at and near the resonance frequency $\omega_2 = \frac{1}{2}$, which is verified by the graphical behaviour of the triangular equilibrium points around the binary system.

Hence we have came on conclusion that the shifting of the location and the resonance stability of triangular equilibrium points in particular case $e = 0$ around binary system would be possible by changing the triaxiality and radiation parameters, which are shown in Figures 1-6.

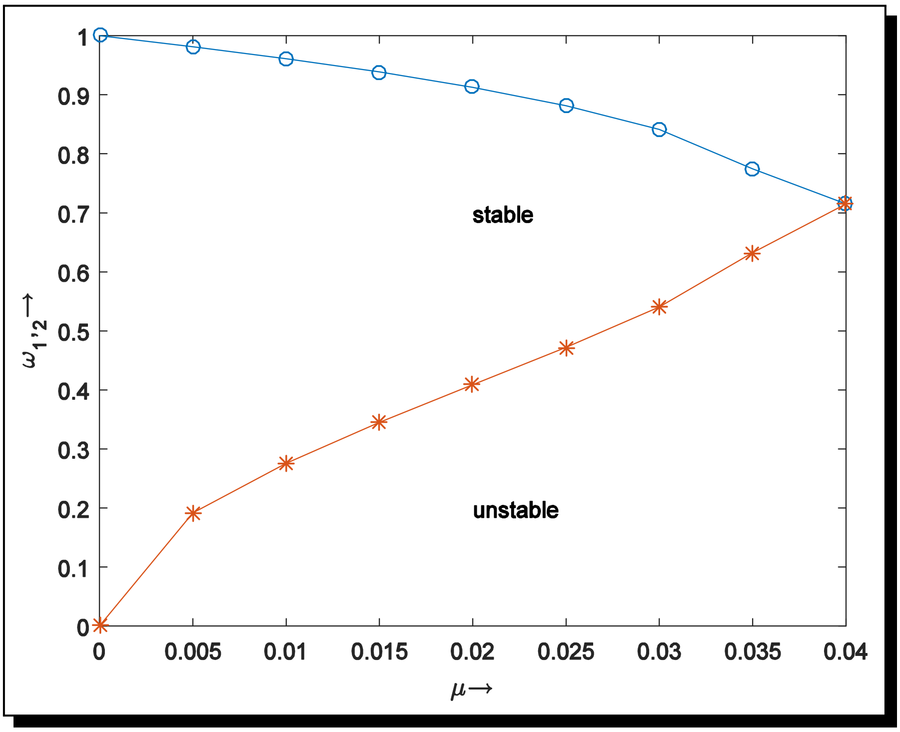

Figure 1. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma'_1 = 0.001$, $\sigma'_2 = 0.002$, $\varepsilon'_1 = 0.0001$, $\varepsilon'_2 = 0.0002$

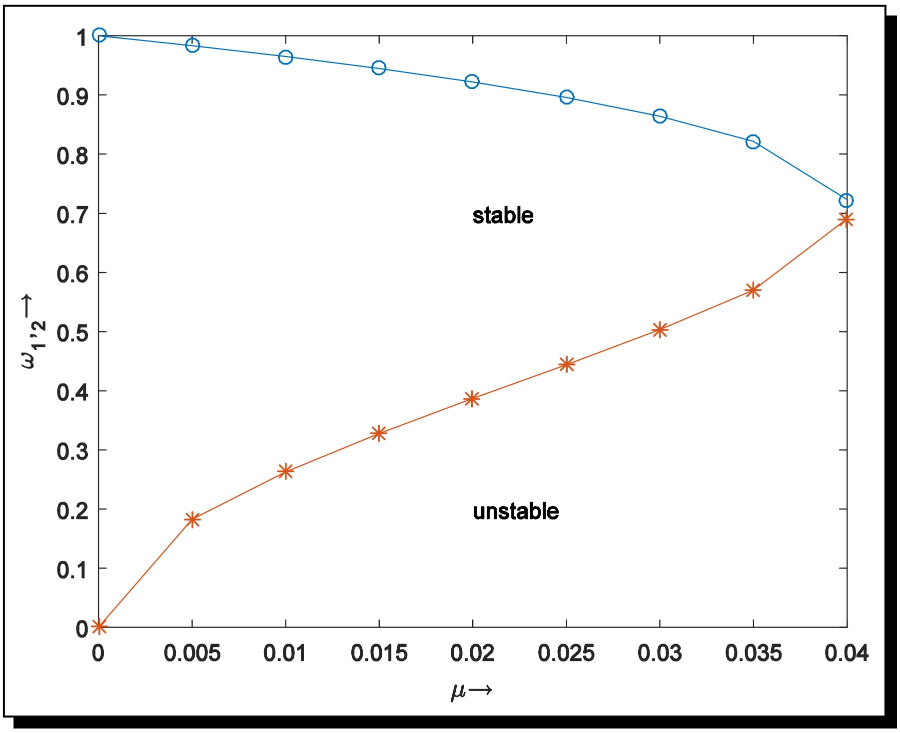

Figure 2. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.003$, $\sigma_2 = 0.002$, $\sigma_1' = 0.001$, $\sigma_2' = 0.002$, $\varepsilon_1' = 0.0001$, $\varepsilon_2' = 0.0002$

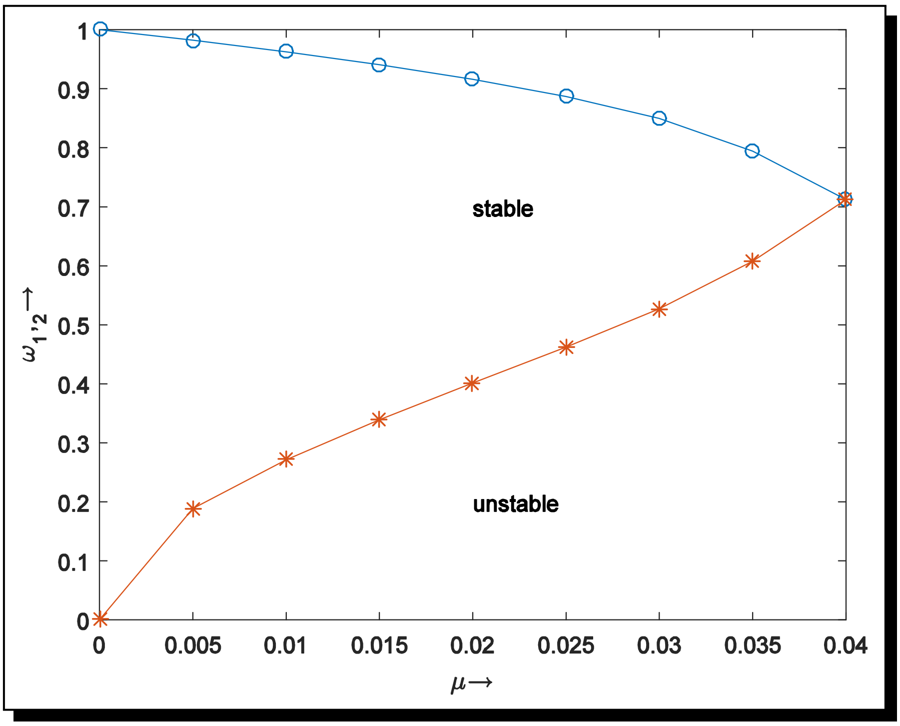

Figure 3. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma_1' = 0.003$, $\sigma_2' = 0.001$, $\varepsilon_1' = 0.001$, $\varepsilon_2' = 0.002$

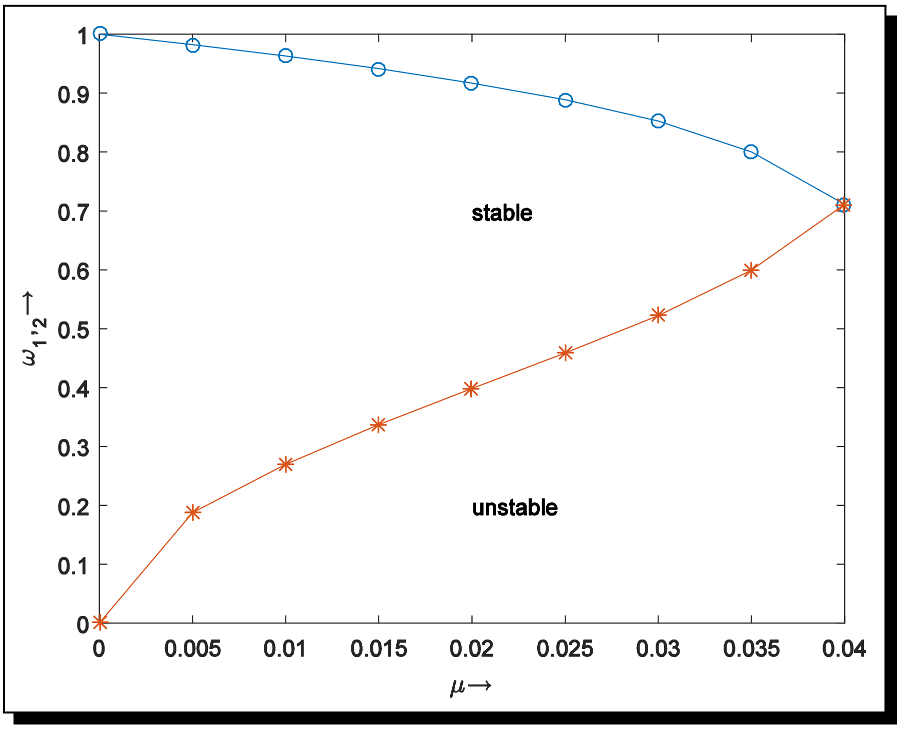

Figure 4. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma_1' = 0.004$, $\sigma_2' = 0.002$, $\varepsilon_1' = 0.001$, $\varepsilon_2' = 0.002$

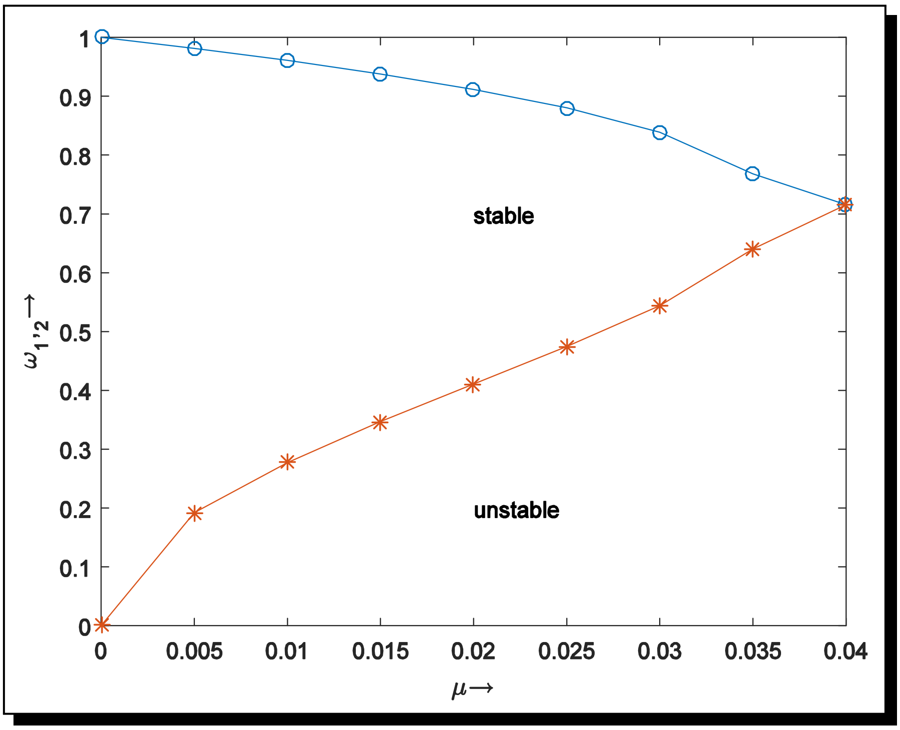

Figure 5. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma_1' = 0.001$, $\sigma_2' = 0.005$, $\varepsilon_1' = 0.00001$, $\varepsilon_2' = 0.00002$

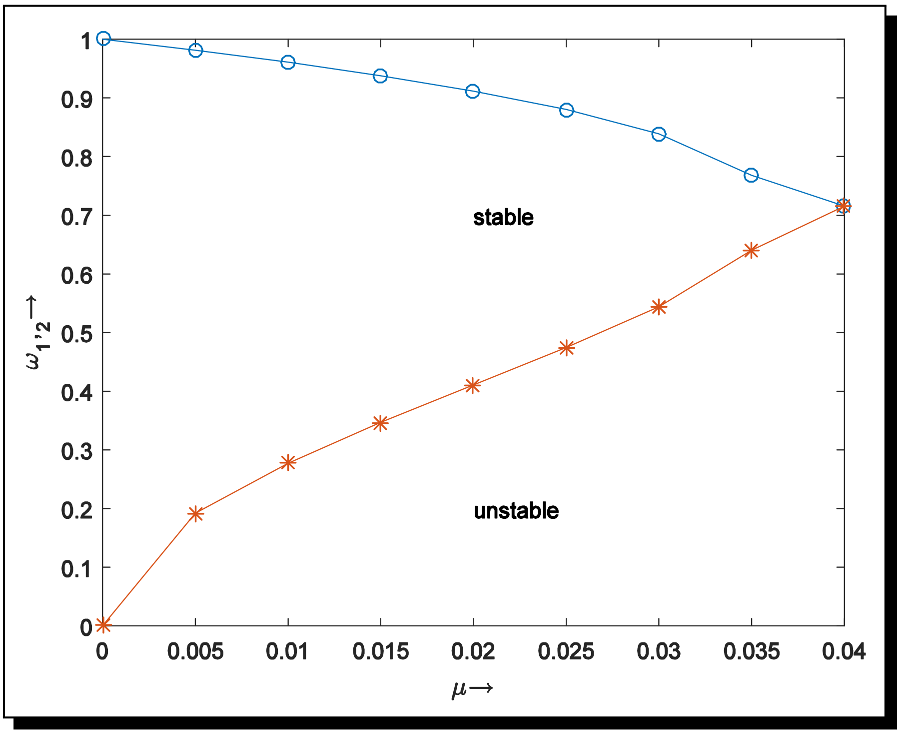

Figure 6. Correlation between μ and $\omega_{1,2}$ for $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma_1 = 0.001$, $\sigma_2' = 0.005$, $\varepsilon_1' = 0.00004$, $\varepsilon_2' = 0.00002$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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