Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 685–698, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

$L(2,1)$-Labeling of Cartesian Product of Complete Bipartite Graph and Path

Sumonta Ghosh[1,*], Satyabrata Paul[2] and Anita Pal[1]

[1] *Department of Mathematics, NIT Durgapur, Durgapur, West Bengal 713209, India*
[2] *Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, West Bengal 721102, India*
***Corresponding author:** mesumonta@gmail.com

Abstract. An $L(2,1)$-labeling problem is a particular case of $L(h,k)$-labeling problem. An $L(2,1)$-labeling of a graph $G=(V,E)$ is a function f from the set of vertices V to the set of positive integers. For any two vertices x and y, the label difference $|f(x)-f(y)|\geq 2$ when $d(x,y)=1$ and $|f(x)-f(y)|\geq 1$ when $d(x,y)=2$ where $d(x,y)$ is the distance between the vertices x and y. In this paper we label the graph which is obtained by Cartesian product between complete bipartite graph and path by $L(2,1)$-labeling. We provide upper bound of the label in terms of number of vertices and edges. The bound is linear with respect to the order and size of the graph. This is a very good bound compare to the bound of Griggs and Yeh Conjecture.

Keywords. $L(2,1)$-labeling; Graph labeling; Cartesian product of graphs

MSC. 05C76; 05C78

Received: January 6, 2017 **Accepted:** March 15, 2017

1. Introduction

The frequency assignment problem is to assign frequency to a group of nodes like radio station or TV station in such a way so that interefering nodes are assigned different frequency under a restricted environment. This type of problem actually referred as vertex coloring problem which is introduced by Hale [10]. Further in 1988 Robert proposed the concept of "very close" node that received frequency two apart and "close" node which received frequency minimum one apart, which lead to introduction of $L(2,1)$-labeling problem. This type of frequency assignment problem can get graph theoretic structure just by considering all the nodes as vertex and two such vertices say $x, y \in V$ are said to "very close" if $d(x,y)=1$ and "close" if $d(x,y)=2$.

Definition 1. $L(2,1)$-labeling of a graph $G=(V,E)$, where V is the set vertices and E is the set of edges, is a function f whose mapping from the set of vertices V to the set of positive integer such that $|f(x)-f(y)| \geq 2$ if distance $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if distance $d(x,y)=2$. The span of $L(2,1)$-labeling f of G denoted by $\lambda_{2,1}(G)$, where $\lambda_{2,1}(G)$ is the difference between largest and smallest label used.

There exist different types of graphs $G=(V,E)$ and for each type different bound of $\lambda_{2,1}(G)$ is obtained. All bounds are obtained by using the parameter Δ, which define the maximum degree of the graph $G=(V,E)$. $\chi(G)$ and $\omega(G)$ defines the chromatic number and the size of the maximum clique of the graph $G=(V,E)$ respectively. The definite lower bound of $\lambda_{2,1}(G)$ are $\Delta+1$ and $2(\omega-1)$. Many researchers have drawn attention by improving the results of the bound day by day. In 1992 Griggs and Yeh [9] first established the bound for any graph $G=(V,E)$ is $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$. In 2003 the bound of $\lambda_{2,1}(G)$ proved by Kral and Skrekovs [19]. In 2008 it is improved by Gonclaves [8] to $\lambda_{1,2}(G) \leq \Delta^2+\Delta-2$. Griggs and Yeh [9] conjectured that for any graph $G=(V,E)$, $\lambda_{2,1}(G) \leq \Delta^2$ which was proved by Havet [12]. This above conjectured is worked efficiently for some classes of graphs such as path [9], wheels [9], cycle [9], trees [3,9,11], co-graphs [3], interval graphs [3], chordal graphs [22], permutation graphs [1,20] etc. The bound $\lambda_{2,1}(G)$ can be efficiently calculated for few classes of graphs like path, cycle, tree [3,9,11], etc. There exist some other classes of graphs like interval graphs [3], circular-arc graphs [2], chordal graphs [22] etc., for such type of graphs we are still not sure whether $\lambda_{2,1}(G)$ is satisfying polynomial time or NP- complete.

Definition 2. Cartesian product of two graphs $G=(V,E)$ and $H=(V',E')$ is the Cartesian product between two set of vertices $V(G) \times V'(H)$ denoted by $G \times H$, where (u,u') and (v,v') are the order pair of the Cartesian product will be adjacent in $G \times H$ if and only if either

(1) $u=v$ and u' is adjacent with v' in H, or

(2) $u'=v'$ and u is adjacent with v in G.

The Cartesian product of two graphs are commutative.

The study of $\lambda_{2,1}(G)$ of Cartesian product between paths, cycles, complete graphs and between paths and cycles has already been done [6,7,15–18,24]. Some results are given below:

(1) (Georges et al. [7]) If $n, m \geq 2$ then

$$\lambda_{2,1}(K_n \times K_m) = \begin{cases} 4, & \text{if } n = m = 2, \\ nm - 1, & \text{otherwise.} \end{cases} \tag{1.1}$$

(2) (Whittlesey et al. [24]). If $n, m \geq 2$ then

$$\lambda_{2,1}(P_n \times P_m) = \begin{cases} 5, & \text{if } n = 2 \text{ and } m \geq 4, \\ 6, & \text{if } n, m \geq 4 \text{ or } (n \geq 3 \text{ and } m \geq 5). \end{cases} \tag{1.2}$$

(3) (Klavzar and Vesel [18]). If $n \geq 4$ and $m \geq 3$ then

$$\lambda_{2,1}(P_2 \times C_m) = \begin{cases} 5, & \text{if } m \equiv 0 \bmod 3, \\ 6, & \text{otherwise.} \end{cases} \tag{1.3}$$

$$\lambda_{2,1}(P_3 \times C_m) = \begin{cases} 7, & \text{if } m = 4 \text{ or } 5, \\ 6, & \text{otherwise.} \end{cases} \tag{1.4}$$

$$\lambda_{2,1}(P_n \times C_m) = \begin{cases} 6, & \text{if } m \equiv 0 \bmod 7, \\ 7, & \text{otherwise.} \end{cases} \tag{1.5}$$

(4) (Jha et al. [15]). If $n, m \geq 3$ then

$$\lambda_{2,1}(C_n \times C_m) = 6, \text{ if } n, m \equiv 0 \bmod 7. \tag{1.6}$$

$$\lambda_{2,1}(C_n \times C_m) \leq \begin{cases} 7, & \text{if } [n \equiv 0 \bmod 4 \text{ and } m \geq 4] \\ & \text{or } [n \equiv 0 \pmod 3) \text{ and } n \equiv 0 \pmod 6)] \end{cases} \tag{1.7}$$

(5) (Christopher and Denise [23]). If $n, m \geq 0$ then

$$\lambda_{2,1}(C_n \times C_m) \leq \begin{cases} 6, & \text{if } n, m \equiv 0 \pmod 7, \\ 8, & \text{if } n, m \in A, \\ 7, & \text{otherwise,} \end{cases} \tag{1.8}$$

where $A = \{\{3, i\} : i \geq 3, i \text{ odd or } i = 4, 10\} \cup \{\{5, i\} : i = 5, 6, 9, 10, 13, 17\} \cup \{\{6, 7\}, \{6, 11\}, \{7, 9\}, \{9, 10\}\}$.

These are the various results on Cartesian product between cycle and cycle, path and cycle and between complete graphs.

As time goes utilization of systems become very high, which experienced wider and complex network structure. Connection of different type of network model plays vital role in real life, so product of two existing network model gives a complex network structure with the facility of single integrated network. Such type of complex network may lead to high cost factor in communication but it experienced a high reliability also. In this paper, we mainly focus on Cartesian product between path and complete bipartite graph. We follow the Chang and Kuo's [3] algorithm to label the graph. We analyse all the possible occurrences of the vertex with maximum label which satisfy Griggs and Yeh conjecture for any graph $G = (V, E)$ with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.

The rest of the paper organized is as follows. Section 2 contains some preliminaries and definition, Section 3 presents Chang and Kuo's [3] algorithm, analysis of algorithm and lemma's to study Griggs and Yeh [9] conjecture followed by conclusion.

2. Preliminaries

Definition 3. A complete graph is simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph with n vertices is denoted by K_n. For this graph, for all vertices $x, y \in V$ there is an edge $(x, y) \in E$.

Definition 4. A graph G is called a complete bipartite graph if its vertices can be partitioned into two subsets V_1 and V_2 such that no edges has both end points in the same subset, and each vertex of $V_1(V_2)$ is connected with all vertices of $V_2(V_1)$. Here $V_1 = \{X_{11}, X_{12}, \dots, X_{1m}\}$ contains m vertices and $V_2 = \{Y_{11}, Y_{12}, \dots, Y_{1n}\}$ contains n vertices.

A complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

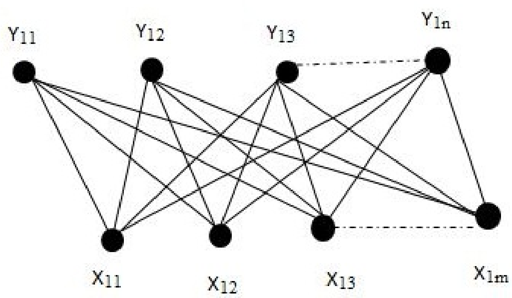

Figure 1. Complete bipartite graph $K_{m,n}$

Definition 5. A path is a trail in which all vertices (except possibly the first and last) are distinct. A trail is a walk in which all edges are distinct. A walk of length k in a graph is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \dots, v_{k-1}, e_{k-1}, v_k$ which begins and ends with vertices. If the graph is directed, then e_i is a directed arc from v_i to v_{i+1}.

Conjecture 1 (Griggs and Yeh [?]). *For any graph $G = (V, E)$ with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.*

Definition 6. For any graph $G = (V, E)$ a subset S of $V(G)$ is called an i-stable set or i-independent set if the distance between any two vertices in S is strictly greater than i. 1-stable set is known as independent set.

The Cartesian product $K_{m,n} \times P_r$ between $K_{m,n}$ and P_r can be visualized in a simple way. For this product, we draw r copies of $K_{m,n}$. Let $X_i = \{x_{i1}, x_{i2}, x_{i3}, \dots, x_{im}\}$ and $Y_i = \{y_{i1}, y_{i2}, y_{i3}, \dots, y_{in}\}$ be the set of vertices of the ith copy of the graph $K_{m,n}$. The vertices of ith copy of $K_{m,n}$ are connected with $(i+1)$th copy of $K_{m,n}$ only as per following rule:

(1) $x_{i_1 j}$ and $x_{i_2 k}$ will be connected if $j = k$ and $|i_1 - i_2| = 1$

(2) y_{i_1p} and y_{i_2q} will be connected if $p=q$ and $|i_1-i_2|=1$

Note that the set of vertices of $G=(V,E)=K_{m,n}\times P_r$ is $\cup_{i=1}^{m}X_i\cup_{j=1}^{n}Y_j$. It is clear that $(x_{ij},y_{ip})\in E$, i.e. $d(x_{ij},y_{ip})=1$ for $i=1,2,3,\dots,r$, $j=1,2,3,\dots,m$ and $p=1,2,3,\dots,n$. Again $d(x_{ij},x_{(i+1)j})=1$ for $i=1,2,3,\dots,r$, $j=1,2,3,\dots,m$ and $d(y_{ip},y_{(i+1)p})=1$ for $i=1,2,3,\dots,r$, $p=1,2,3,\dots,n$.

The number of vertices and edges of $K_{m,n}\times P_r$ are $r(m+n)$ and $rmn+(r-1)(m+n)$ respectively.

Lemma 1. *Let Δ be the degree of the graph $K_{m,n}\times P_r$, then*

$$\Delta=\begin{cases} m+1 & \text{for } m>n \text{ and } r=2\\ m+2 & \text{for } m>n \text{ and } r>2\\ m+1 & \text{for } m=n \text{ and } r=2\\ m+2 & \text{for } m=n \text{ and } r>2\end{cases} \tag{2.1}$$

Proof. Let $G=K_{m,n}\times P_2$. If $m>n$ then the maximum degree of the graph $K_{m,n}$ is m. As per definition the vertex x_{1j} connected with vertex x_{2j}, $j=1,2,3,\dots,m$ and y_{1i} is connected with y_{2i}, $i=1,2,3,\dots,n$. Therefore only one degree of each vertex will increase in $K_{m,n}\times P_2$. Hence the value of Δ is $m+1$.

The proof of other cases are similar. □

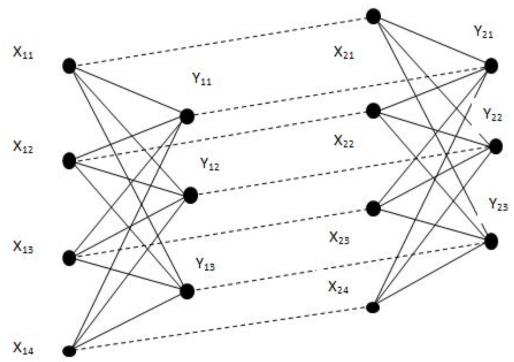

Figure 2. Cartesian product between $K_{m,n}$ and P_r for $m>n$ and $r=2$

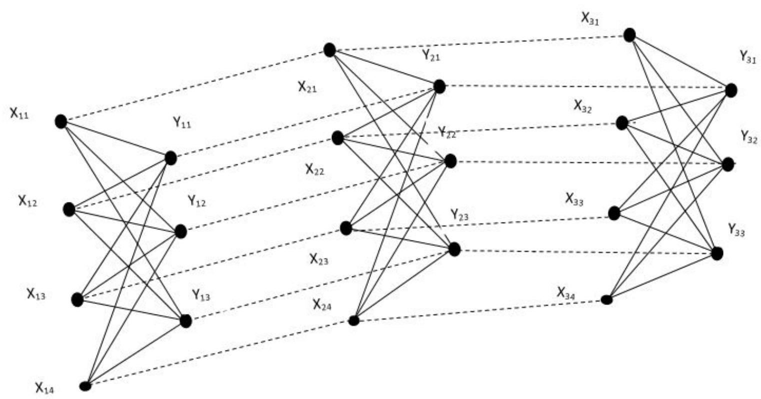

Figure 3. Cartesian product between $K_{m,n}$ and P_r for $m>n$ and $r=3$

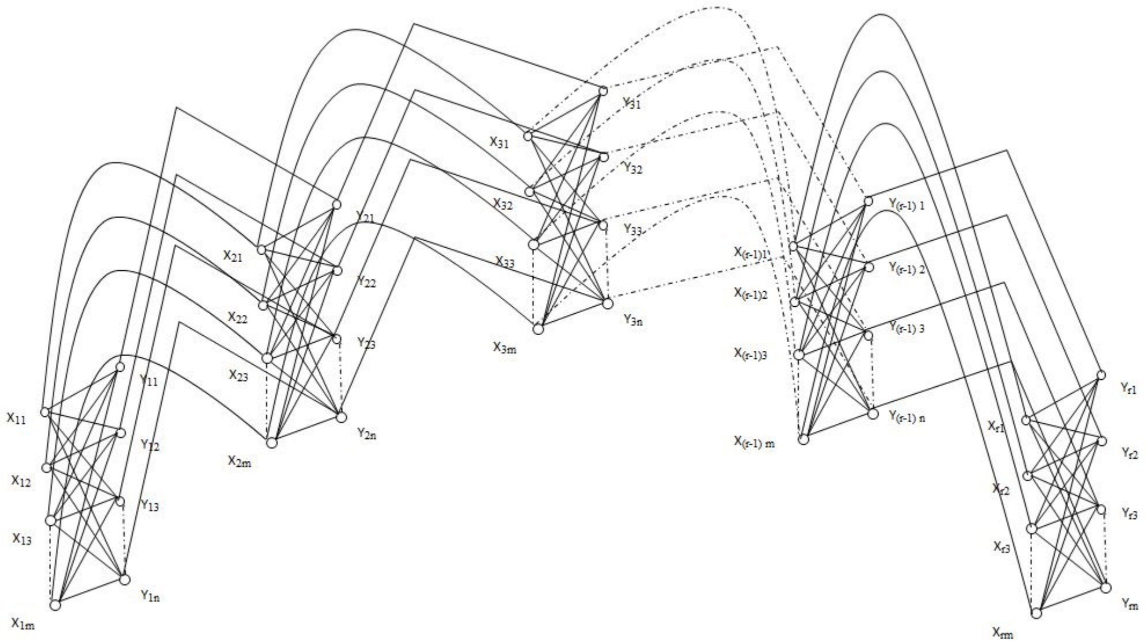

Figure 4. The graph $K_{m,n} \times P_r$

3. Labeling of Cartesian Product between Complete Bipartite Graph and Path

We already discussed different type of labeling for trivial graphs and family of intersection graphs with bounds in the form of maximum degree $\Delta(G)$. Throughout the paper, we use Δ instead of $\Delta(G)$. To analysis the labeling of Cartesian product between path and complete bipartite graph we use the concept of Chang and Kuo's [3] algorithm which is design for general graphs. Note that our algorithm is not straight forward use of Chang and Kuo's algorithm.

3.1 Algorithm L21CBP

Chang and Kuo's algorithm gives the idea of stable set. A subset U of $V(G)$ is called an i-stable set or an i-independent set if the distance between any two vertices in U is strictly greater than i. The 1-stable set is actually an independent set. An i-*stable* set is said to be maximal if U of the set $F \subseteq V$ of vertices is an i-stable subset of F such that U is not a proper subset of any other i-stable subset of G contained is F.

The technique which is used to label the Cartesian product between complete bipartite graph and path is explained below.

In the algorithm, in each step we are looking for maximal 2-stable set from the vertices that are not label and atleast two distance apart from the vertices which are labeled in the previous step. Label all the vertices which are in 2-stable set with the index i of the current step. This i is initialize from 0 and incremented by 1 in each step. The maximum label used in the algorithm is the final value of i. Let the maximum label be k.

The value of k obtained from the above algorithm represents the upper bound of $\lambda_{2,1}(G)$.

Algorithm 1 Algorithm L21CBP

Input: The graph $G = (V, E) = K_{m,n} \times P_r$.
Output: k, the value of maximum label.
Initialize: $U_{-1} = \phi$, $V = V(G)$, $i = 0$.
Start of iteration
Step 1. If $U_{-1} \neq \phi$, then set $F_i = v \in V$ such that u is unlabeled and $d(u,v) \geq 2$, $\forall\ v \in U_{i-1}$
else $F_i = V$
if $F_i \neq \phi$ then compute U_i (maximum 2-stable subset of F_i)
else set $U_i = \phi$.
Step 2. Label all the vertices of U_i by i
Step 3. Update V, $V \leftarrow V - U_i$.
Step 4. If $V \neq \phi$ then set $i \leftarrow i + 1$, and go to Step 1.
Step 5. Iteration is continued until $V = \phi$.
Step 6. Set $k = i$ (number of iteration).
Stop.

Let $V_1 \subset V$ and, be the set of vertices which are labeled by k using algorithm L21CBP. We define three sets $I_1(x)$, $I_2(x)$, $I_3(x)$ as follows.

$$I_1(x) = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) = 1, \text{ for some } y \in U_i\},$$

$$I_2(x) = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \leq 2, \text{ for some } y \in U_i\},$$

$$I_3(x) = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \geq 3, \text{ for all } y \in U_i\},$$

where U_i is the maximum 2-stable subset of F_i.

Here $I_1(x)$ represents the set of labels which are one distance apart from x, i.e $d(x,y) = 1$, for some $y \in U_i$. $I_2(x)$ and $I_3(x)$ contains the labels which are at most two and at least three distance apart from x respectively. It is clear from the algorithm L21CBP that cardinalities of $I_2(u)$ and $I_3(u)$ is k. i.e $|I_2(x)| + |I_3(x)| = k$. Again for any $i \in I_3(x)$, $x \notin F_i$ since otherwise $U_i \cup x$ would be a 2-stable subset of F_i, which contradicts the choice of U_i. That is, $d(x,y) = 1$ for some $y \in U_{i-1}$, i.e $(i-1) \in I_1(x)$. Since for every $i \in I_3(x)$, $(i-1) \in I_1(x)$. Thus $|I_3(x)| \leq |I_1(x)|$. Hence

$$\lambda_{2,1}(G) \leq k = |I_2(x)| + |I_3(x)| \leq |I_2(x)| + |I_1(x)|. \tag{3.1}$$

3.2 Analysis of Algorithm

Let $G = K_{m,n} \times P_r$ be the Cartesian product between a complete bipartite graph and a path. Let C be the set of vertices which are labeled by the largest label k. Thus, for $x \in C$, $f(x) = k$. Now, analysis the cardinality of the sets $I_1(x)$ and $I_2(x)$ as $\lambda_{2,1}(G) \leq |I_1(x) + I_2(x)|$.

From the definition of $I_1(x)$, it is clear that $I_1(x)$ contains at most deg(x) number of labels. So $|I_1(x)| \leq \deg(x) \leq \Delta$.

Since $I_2(x) = \{i : 0 \leq i \leq k-1\}$ and $d(x,y) \leq 2$, for some $y \in U_i$. So, obviously $|I_2(x)| \leq$ (number of 1-nbd vertices of x) + (number of 2-nbd vertices of x).

Now, our aim is to find the cardinality of the set $I_2(x)$. For different positions of the vertex $x \in C$, $I_2(x)$ is different. Here we discuss the different positions of the vertex $x \in C$ such that we can find the maximum possible cardinality of the set $I_2(x)$.

(1) Suppose $x \in X_1$ or $x \in Y_1$. Then all the 2-nbd vertices of x belong to X_1 or Y_1, X_2 or Y_2 and X_3 or Y_3 (if exist). X_3 or Y_3 exist only when $r \geq 3$.
Similarly, if $x \in X_r$ or $x \in Y_r$ then all the 2-nbd vertices of x belong to X_r or Y_r, X_{r-1} or Y_{r-1} and X_{r-2} or Y_{r-2} (if exist). X_{r-2} or Y_{r-2} exist only when $r \geq 3$.

(2) Suppose $x \in X_2$ or $x \in Y_2$. Then all the 2-nbd vertices of x belong to X_2 or Y_2, X_1 or Y_1, X_3 or Y_3 and X_4 or Y_4 (if exist). X_4 or Y_4 exist only when $r \geq 4$.
Similarly, if $x \in X_{r-1}$ or $x \in Y_{r-1}$ then all 2-nbd vertices of x belong to X_r or Y_r, X_{r-1} or Y_{r-1}, X_{r-2} or Y_{r-2} and X_{r-3} or Y_{r-3} (if exist). X_{r-3} or Y_{r-3} exist only when $r \geq 4$.

(3) Suppose $x \in X_l$ or $x \in Y_l$ such that X_{l-2} or Y_{l-2} and X_{l+2} or Y_{l+2} exist for some $l < r$. Thus the 2-nbd vertices of x belong to X_{l-2} or Y_{l-2}, X_{l-1} or Y_{l-1}, X_l or Y_l, X_{l+1} or Y_{l+1} and X_{l+2} or Y_{l+2}. This situation occur only when $r \geq 5$.

Now, we can briefly discuss the above cases:

Claim 1: If $x \in X_1 (x \in Y_1)$ or $x \in X_r (x \in Y_r)$ then

$$|I_1(x) + I_2(x)| \leq \begin{cases} 3m+n+2 & \text{if } m > n \\ 3n+m+2 & \text{if } n > m \\ 4n+2 & \text{if } m = n. \end{cases} \tag{3.2}$$

Proof. Let $x \in X_1$. Then the number of 1-nbd vertices of x is $n+1$. So, $|I_1(x|) \leq n+1$.

Now, x is distance two away from $(m-1)$ number of vertices of X_1, n number of vertices of Y_2 and 1 vertices of Y_3.

Thus $|I_2(x)| \leq (n+1)+(m-1)+n+1 = 2n+m+1$.

So, $\quad |I_1(x)+I_2(x)| \leq (n+1)+(2n+m+1) = 3n+m+2$.

Similarly, $x \in Y_1$ then $|I_1(x)| \leq m+1$

$$|I_2(x)| \leq (m+1)+(n-1)+m+1 = 2m+n+1.$$

So, $\quad |I_1(x)+I_2(x)| \leq (m+1)+(2m+n+1) = 3m+n+2$.

Clearly, if $m > n$ then $3m+n+2 > 3n+m+2$,

if $m < n$ then $3m+n+2 < 3n+m+2$,

and if $m = n$ then $3m+n+2 = 3n+m+2$.

Hence, we conclude the following result:

$$|I_1(x) + I_2(x)| \leq \begin{cases} 3m+n+2 & \text{if } m > n \\ 3n+m+2 & \text{if } n > m \\ 4n+2 & \text{if } m = n. \end{cases}$$ □

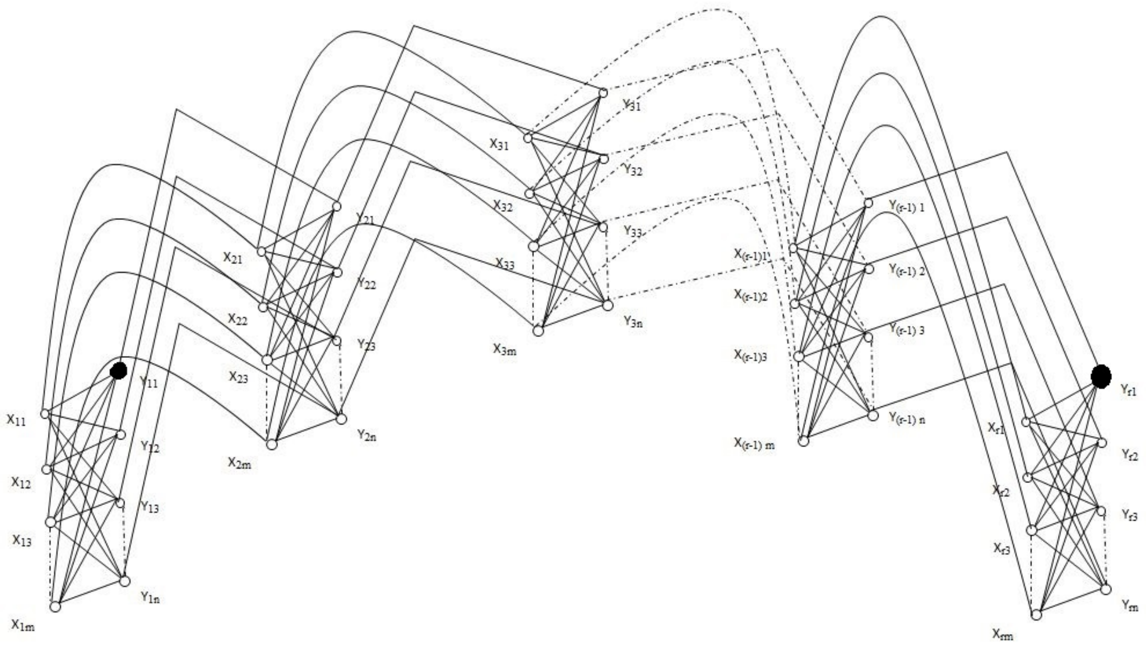

Figure 5. The vertex x is either at left most or right most position in the graph $G = K_{m,n} \times P_r$.

Lemma 2. *If* $x \in X_1(Y_1)$ *or* $x \in X_r(Y_r)$ *then*

$$|I_1(x)+I_2(x)| \le \begin{cases} (m+1)^2 & \text{if } m>n \\ (n+1)^2 & \text{if } n>m \\ (m+1)^2 & \text{if } m=n. \end{cases} \tag{3.3}$$

Proof. As $m>n$ already consider.

$m^2 \ge (m+n+1)$, $\forall\ m,n \in Z^+$ and $m \ge 2$, $n \ge 1$, $m>n$.

or $m^2+2m \ge 3m+n+1$

or $3m+n+2 \le (m+1)^2$

$|I_1(x)+I_2(x)| \le (m+1)^2$.

Similarly, we can prove the cases for $n>m$ and $m=n$. □

Claim 2: If $x \in X_2(Y_2)$ or $x \in X_{r-1}(Y_{r-1})$ then

$$|I_1(x)+I_2(x)| \le \begin{cases} 4m+n+4 & \text{if } m>n \\ 4n+m+4 & \text{if } n>m \\ 5m+4 & \text{if } m=n. \end{cases} \tag{3.4}$$

Proof. Let $x \in X_2$. Then the number of 1-nbd vertices of x is $n+2$. So, $|I_1(x)| \le n+2$.

Now, x is at distance two apart from $(m-1)$ number of vertices of X_2, n number of vertices of Y_1, n number of vertices of Y_3 and 1 vertices of Y_4.

Thus $|I_2(x)| \le (n+2)+(m-1)+2n+1 = 3n+m+2$.

So, $|I_1(x)+I_2(x)| \le (n+2)+(3n+m+2) = 4n+m+4$.

Similarly, $x \in Y_2$ then $|I_1(x)| \le m+2$

$$|I_2(x)| \le (m+2)+(n-1)+2m+1 = 3m+n+2.$$

So, $\quad |I_1(x)+I_2(x)| \le (m+2)+(3m+n+2) = 4m+n+4.$

Clearly, if $m > n$ then $4m+n+4 > 4n+m+4$,

if $\quad m < n$ then $4m+n+4 < 4n+m+4$.

and if $\quad m = n$ then $4m+n+4 = 4n+m+4$.

Hence, we conclude the following result

$$|I_1(x)+I_2(x)| \le \begin{cases} 4m+n+4 & \text{if } m > n \\ 4n+m+4 & \text{if } n > m \\ 5m+4 & \text{if } m = n. \end{cases}$$

□

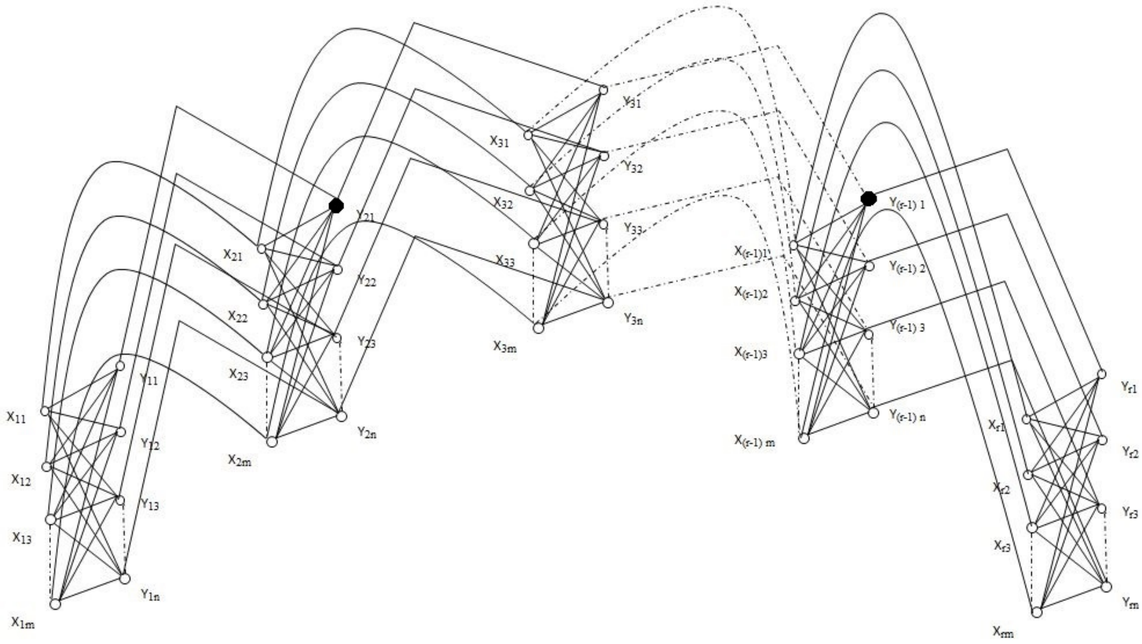

Figure 6. The vertex x is at second copy of the graph $K_{m,n}$ from both the end of $K_{m,n} \times P_r$.

Lemma 3. *If $x \in X_2(Y_2)$ or $x \in X_{r-1}(Y_{r-1})$ then*

$$|I_1(x)+I_2(x)| \le \begin{cases} (m+2)^2 & \textit{if } m > n \\ (n+2)^2 & \textit{if } n > m \\ (m+2)^2 & \textit{if } m = n. \end{cases} \tag{3.5}$$

Proof. As $m > n$, $\forall\ m,n \in Z^+$ and $m \ge 2, n \ge 1, m > n$.

$m^2 > m+n \quad$ or $\quad m^2+4m+4 > 4m+n+4 \quad$ or $\quad (m+2)^2 > 4m+n+4$

i.e. $\quad |I_1(x)+I_2(x)| = 4m+n+4 < (m+2)^2$.

Similarly, we can prove the cases for $n > m$ and $m = n$. □

Claim 3: If $x \in X_l(Y_l)$ such that $X_{l-2}(Y_{l-2})$ and $X_{l+2}(Y_{l+2})$ exist for some $l < r$, then

$$|I_1(x)+I_2(x)| \le \begin{cases} 4m+n+5 & \text{if } m>n \\ 4n+m+5 & \text{if } n>m \\ 5n+5 & \text{if } m=n. \end{cases} \tag{3.6}$$

Proof. Let $x \in X_l$. Then the number of 1-nbd vertices of x is $n+2$. So, $|I_1(x)| \le n+2$.

Now, x is at distance two away from $(m-1)$ number of vertices of X_l, n number of vertices of Y_{l-1}, n number of vertices of Y_{l+1}, 1 vertices of Y_{l-2} and 1 vertices of Y_{l+2}.

Thus $I_2(x) \le (n+2)+(m-1)+2n+2 = 3n+m+3$.

So, $|I_1(x)+I_2(x)| \le (n+2)+(3n+m+3) = 4n+m+5$.

Similarly, $x \in Y_2$ then $|I_1(x)| \le m+2$

$|I_2(x)| \le (m+2)+(n-1)+2m+2 = 3m+n+3$.

So, $|I_1(x)+I_2(x)| \le (m+2)+(3m+n+3) = 4m+n+5$.

Clearly, if $m>n$ then $4m+n+5 > 4n+m+5$,

if $m<n$ then $4m+n+5 < 4n+m+5$.

and if $m=n$ then $4m+n+5 = 4n+m+5$.

Hence, we conclude the following result

$$|I_1(x)+I_2(x)| \le \begin{cases} 4m+n+5 & \text{if } m>n \\ 4n+m+5 & \text{if } n>m \\ 5n+5 & \text{if } m=n. \end{cases}$$

□

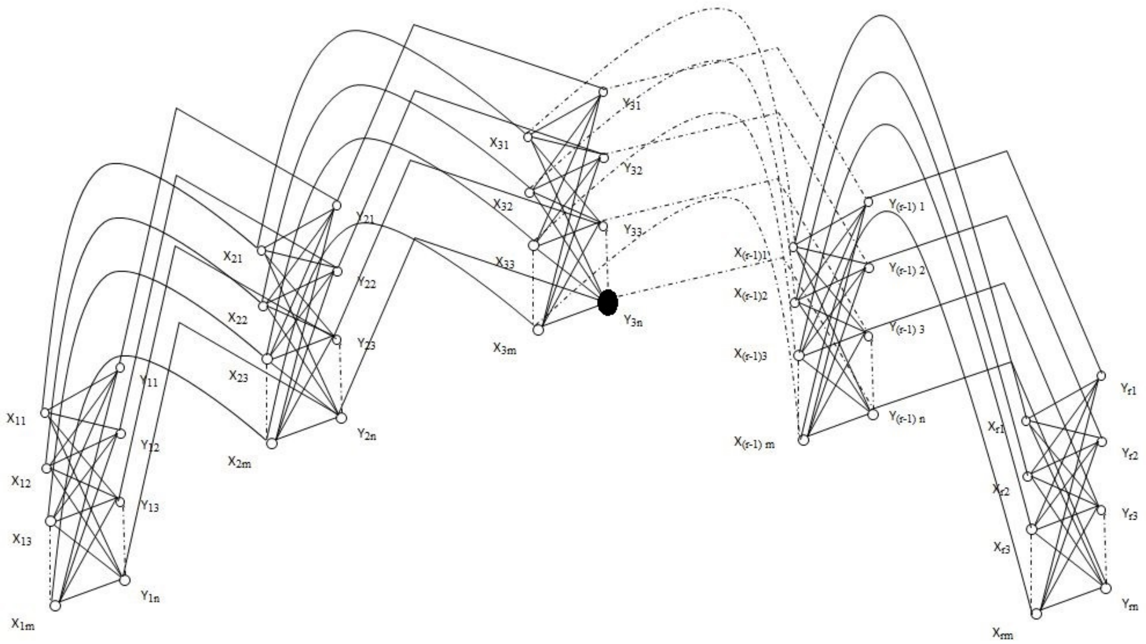

Figure 7. The vertex x is at third or more copy of the graph $K_{m,n}$ from both the end of $K_{m,n} \times P_r$

Lemma 4. *If $x \in X_l(Y_l)$ such that $X_{l-2}(Y_{l-2})$ and $X_{l+2}(Y_{l+2})$ exist for some $l < r$, then*

$$|I_1(x)+I_2(x)| \le \begin{cases} (m+2)^2 & \text{if } m>n \\ (n+2)^2 & \text{if } n>m \\ (n+2)^2 & \text{if } m=n. \end{cases} \tag{3.7}$$

Proof. As $m > n$, $\forall\ m,n \in Z^+$ and $m \ge 2$, $n \ge 1$.

$m^2 \ge m+n.$

or $(m+2)^2 \ge 4m+n+5.$

$|I_1(x)+I_2(x)| = 4m+n+5 \le (m+2)^2.$

The proves are similar in case of $n > m$ and $m = n$ □

Theorem 1. *For the graph $G = K_{m,n} \times P_r$, if $x \in X_1(x \in Y_1)$ or $x \in X_r(x \in Y_r)$, then*

$$\lambda_{2,1}(G) \le \begin{cases} 3m+n+2 & \text{if } m>n \\ 3n+m+2 & \text{if } n>m \\ 4n+2 & \text{if } m=n. \end{cases} \tag{3.8}$$

Proof. In a graph $G = K_{m,n} \times P_r$, if $x \in X_1(Y_1)$ or $x \in X_r(Y_r)$, then from the equation 3.1, $\lambda_{2,1}(G) \le |I_2(x)| + |I_1(x)|$ and from the equation 3.2, for $m > n$

$|I_2(x)| + |I_1(x)| \le 3m+n+2.$

$\lambda_{2,1}(G) \le 3m+n+2.$

Hence the theorem.

Similarly, we can prove the cases for $n > m$ and $m = n$. □

Theorem 2. *For the graph $G = K_{m,n} \times P_r$, if $x \in X_2(Y_2)$ or $x \in X_{r-1}(Y_{r-1})$, then*

$$\lambda_{2,1}(G) \le \begin{cases} 4m+n+4 & \text{if } m>n \\ 4n+m+4 & \text{if } n>m \\ 5n+4 & \text{if } m=n. \end{cases} \tag{3.9}$$

Proof. In the graph $G = K_{m,n} \times P_r$, if $x \in X_2(Y_2)$ or $x \in X_{r-1}(Y_{r-1})$, then from the equation 3.1, $\lambda_{2,1}(G) \le |I_2(x)| + |I_1(x)|$ and from the equation 3.4, for $m > n$.

$|I_2(x)| + |I_1(x)| \le 4m+n+4.$

$\lambda_{2,1}(G) \le 4m+n+4.$

Proof is similar for the cases $n > m$ and $m = n$. Hence the theorem. □

Theorem 3. *For the graph $G = K_{m,n} \times P_r$, if $x \in X_l(Y_l)$ such that $X_{l-2}(Y_{l-2})$ and $X_{l+2}(Y_{l+2})$ exist for some $l < r$, then*

$$\lambda_{2,1}(G) \le \begin{cases} 4m+n+5 & \text{if } m>n \\ 4n+m+5 & \text{if } n>m \\ 5n+5 & \text{if } m=n. \end{cases} \tag{3.10}$$

Proof. In a graph $G = K_{m,n} \times P_r$, if $x \in X_l(Y_l)$ such that $X_{l-2}(Y_{l-2})$ and $X_{l+2}(Y_{l+2})$ exist for some $l < r$, then from the equation 3.1, $\lambda_{2,1}(G) \le |I_2(x)| + |I_1(x)|$ and from 3.6, for $m > n$.

$|I_2(x)| + |I_1(x)| \le 4m + n + 5.$

$\lambda_{2,1}(G) \le 4m + n + 5.$

Similarly we can prove the cases for $n > m$ and $m = n$. Hence the theorem. □

Theorem 4. *Girggs and Yeh conjecture is true for the graph* $G = K_{m,n} \times P_r$.

Proof. In the graph $K_{m,n} \times P_r$, from the lemma 2, lemma 3 and lemma 4 we can write $|I_2(x) + I_1(x)| \le \Delta^2$, where Δ denote the maximum degree of the graph $K_{m,n} \times P_r$. Also, from the equation 3.1, we have $\lambda_{2,1}(G) \le |I_2(x)| + |I_1(x)|$.

Thus, $\lambda_{2,1}(G) \le \Delta^2$.

Hence the theorem. □

4. Conclusion

In this paper, we designed an algorithm to label the graph obtained by Cartesian product between complete bipartite graph and path. For each cases our propose algorithm satisfy Griggs and Yeh [9] conjecture. The upper bound of $\lambda_{2,1}$ for the graph $K_{m,n} \times P_r$ is linear with respect to m and n and it does not depends on the value of r. For a complex type network assigning of frequency is really a tough job, so we try to draw a simple way of its graphical representation and proved that its follows the above conjecture. Generally, the graph $L(h,k)$-labeling is the general labeling technique. Here we consider $L(2,1)$-labeling problem which is a special type of $L(h,k)$-labeling. It is a very interesting problem for the researcher to label a graph in general way and considering more and more complex structure.

Acknowledgment

The work is supported by the Department of Science and Technology, New Delhi, India, Ref. No.SB/S4/MS: 894/14.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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