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# On a Problem of Z. Moszner

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**Abstract.** In this note, we provide an answer to a problem of Z. Moszner regarding the stability of the additive Cauchy functional equation on semigroups.

## 1. Introduction

The aim of this note is to provide a solution to a problem of Z. Moszner in [4]. Let $S$ be a semigroup and $\mathbb{N}$ be the set of natural numbers. We consider the additive Cauchy functional equation (i.e. the equation of homomorphisms)

$$f(xy) - f(x) - f(y) = 0, \quad \forall x, y \in S, \tag{1.1}$$

where $f : S \to \mathbb{R}$ is a real-valued function defined in the semigroup $S$. The additive Cauchy functional equation (1.1) is said to be stable if for any function $g : S \to \mathbb{R}$ satisfying condition

$$|g(xy) - g(x) - g(y)| \leq c, \quad \forall x, y \in S \tag{1.2}$$

for some positive number $c$ there exist a solution $f$ of equation (1.1) and a positive number $\delta$ such that

$$|f(x) - g(x)| \leq \delta, \quad \forall x \in S. \tag{1.3}$$

The interested readers are referred to the books [3] and [6] for treatments on the stability of functional equations.

In a recent paper [4], Z. Moszner wrote *G. L. Forti says in* [2] (*Theorem 5, p. 151*) *that J. R tz has proved in* [5] *that the condition*

$$\forall\, x, y \in G \;\; \exists\, n \in \mathbb{N} \text{ and } n \geq 2 : \;\; (xy)^n = x^n y^n \tag{1.4}$$

*implies the stability of the equation of homomorphisms. This implication is not in the paper* [5]. Then he asked the following problem.

**Problem 1.1.** Is this implication true?

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A look at R tz's paper [5] reveals that (see [5, Remark 5 and Theorem 4]) if

$$(xy)^n = x^n y^n \quad \forall\, x, y \in S \text{ holds for a fixed } n \in \mathbb{N},\ n \geq 2, \tag{1.5}$$

then one can obtain the stability of the equation of homomorphisms. However, the conditions 1.4 and 1.5 are not the same. In this note, we provide an answer to this problem in affirmative. Our solution of Problem 1.1 is based on theory of quasicharacters and pseudocharacters.

## 2. **Some basic notions**

To give an answer to the above Problem 1.1 we need to recall some notions and facts regarding spaces of quasicharacters and pseudocharacters. A *quasicharacter* of a semigroup $S$ is a real-valued function $f$ defined on $S$ such that the set

$$\{f(xy) - f(x) - f(y) | x, y \in S\}$$

is bounded. A *pseudocharacter* $f : S \to \mathbb{R}$ is a quasicharacter satisfying $f(x^n) = nf(x)$ for any $x \in S$ and any $n \in \mathbb{N}$. We denote by $KX(S)$ (respectively, by $PX(S)$) the set of quasicharacters of $S$ (respectively, the set of pseudocharacters of $S$). Further, we denote by $X(S)$ (respectively, by $B(S)$) the set of real additive characters (respectively, the set of real bounded functions) of $S$. In [1], it was shown that the space of quasicharacters of the semigroup $S$ is a direct sum of $PX(S)$ and $B(S)$, that is

$$KX(S) = PX(S) \oplus B(S).$$

From this decomposition it follows that the additive Cauchy functional equation (1.1) is stable on $S$ if and only if

$$PX(S) = X(S). \tag{2.1}$$

## 3. **Answer to Moszner's problem**

**Theorem 3.1.** *Let $S$ be a semigroup such that for any $x, y \in S$ there exists $n \geq 2$ such that*

$$(xy)^n = x^n y^n, \tag{3.1}$$

*then equation* (1.1) *is stable on $S$.*

***Proof.*** To prove the equation of homomorphisms is stable on the semigroup $S$, we need to show $PX(S) = X(S)$.

First note that if elements $x, y$ satisfy condition (3.1), then by induction on $k$ one can show that there is a sequence of integer numbers $m_k$ such that $m_k \to \infty$ and the following relation

$$(xy)^{m_k} = x^{m_k} y^{m_k} \tag{3.2}$$

holds. Indeed, we can put $m_1 = n$, then for elements $x^{m_1}$ and $y^{m_1}$ there is $n_1 \geq 2$ such that $(x^{m_1} y^{m_1})^{n_1} = (x^{m_1})^{n_1} (y^{m_1})^{n_1} = x^{m_1 n_1} y^{m_1 n_1}$. So, we can put $m_2 = m_1 n_1$. Now for elements $x^{m_1 n_1}$ and $y^{m_1 n_1}$ there is $n_2 \geq 2$ such that $(x^{m_1 n_1} y^{m_1 n_1})^{n_2} =$

$x^{m_1n_1n_2}y^{m_1n_1n_2}$. Therefore, $(xy)^{m_1n_1n_2} = (x^{m_1n_1}y^{m_1n_1})^{n_2} = x^{m_1n_1n_2}y^{m_1n_1n_2}$ and we can put $m_3 = m_2n_2$. Continuing this way we see that there is a sequence of integer numbers $m_k \to \infty$ such that the relation (3.2) holds.

It is easy to see that $X(S) \subseteq PX(S)$. Hence it enough to show that $PX(S) \subseteq X(S)$. Let $f \in PX(S)$ and $c > 0$ such that

$$|f(uv) - f(u) - f(v)| \le c, \quad \forall\, u, v \in S. \tag{3.3}$$

Suppose that $x, y \in S$ and $\{m_k\}_{k=1}^{\infty}$ is a sequence of integer numbers such that $m_k \to \infty$ and the relation (3.2) holds. From (3.2) and (3.3) we obtain

$$|f((xy)^{m_k}) - f(x^{m_k}) - f(y^{m_k})| \le c, \quad \forall\, x, y \in S,\ \forall\, k \in \mathbb{N}. \tag{3.4}$$

Taking into account that $f$ is a pseudocharacter from (3.4) it follows that

$$m_k\, |f(xy) - f(x) - f(y)| \le c, \quad \forall\, x, y \in S,\ \forall\, k \in \mathbb{N}. \tag{3.5}$$

The last relation implies $f(xy) = f(x) + f(y)$, thus $f \in X(S)$. This completes the proof. □

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