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Research Article

Analysis of Fractional Schrödinger Equation Occurring in Quantum Mechanics

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Abstract. In this paper, we present Adomian decomposition method to solve linear fuzzy fractional integro-differential equation with fuzzy initial conditions. Results are compared with the results obtained using Fuzzy Laplace transform method.

Keywords. Fractioanl Schrödinger equaton; Modified Adomian decomposition method; Fractional partial differential equations

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1. Introduction

The space-time fractional one-dimensional Schrödinger equation

$$i\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}+\frac{1}{2}\frac{\partial^\gamma \psi(x,t)}{\partial x^\gamma}-V_d(x)\psi(x,t)-\beta_d|\psi|^2\psi(x,t)=0,\quad 0<\alpha\le 1, 1<\gamma\le 2,$$

with initial condition $\psi(x,0) = \psi_0(x)$ is considered in this paper. The fractional derivative considered here is in Caputo sense. Here V_d is the trapping potential and β_d is a real constant. Modified Adomian decomposition method is used to derive the solutions.

In 1926, the Austrian physicist Erwin Schrödinger formulated the Schrödinger equation [25] which is a partial differential equation in quantum mechanics that describes how the quantum state of some physical system changes with time. It is a model of the evolution of a one dimensional packet of surface waves on sufficiently deep water. It is widely used in basic models of nonlinear waves in many areas of physics. The space fractional Schrödinger equation was constructed by Laskin [12, 13]. Time fractional Schrödinger equation can be constructed if non-Marcovian evolution is considered. In this paper, both space and time fractional nonlinear Schrödinger equation is considered with Mittag Leffler function as initial condition and solutions are obtained in terms of Mittag Leffler functions. The equation is considered in more general form so that time fractional, space fractional and nonlinear Schrödinger equation will be the special case. Caputo fractional derivative is considered throughout the paper since it will be more realistic than Riemann Liouville derivative while modeling the physical problems.

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science [15] are well described by differential equations of fractional order. So far there have been several fundamental works on the fractional derivative and fractional differential equations, by Oldham and Spanier [17], Miller and Ross [16], Podlubny [19], Kilbas, Srivastava and Trujillo [11] and others [6, 8, 23]. Machado *et al.* [14] published a review article on recent history of fractional calculus. Hernández *et al.* [10] discussed the recent developments in the theory of abstract differential equations with fractional derivatives. In this work we use modified Adomian decomposition method to solve the fractional nonlineaer Schrödinger equation.

The Adomian decomposition method (ADM), introduced by Adomian [2] in 1980, provides an effective procedure for finding explicit and numerical solutions of a wider and general class of differential systems representing real physical problems. Linearization or perturbation is not required in this method. In the last two decades, extensive work has been done using ADM [1,3,5] as it provides analytical approximate solutions for nonlinear equations and considerable interest in solving fractional differential equations using ADM [7] has been developed. Recently [18] ADM is used to solve the system of fractional partial differential equations. In this paper we are going to use the modified ADM [9] which provides a easy mechanized algorithm to calculate Adomian polynomials that will be more effective when considering the system of nonlinear equations.

Saxena *et al.* [24] studied the linear space-time fractional Schrödinger equation using joint Laplace and Fourier transform technique. Several other authors studied the time fractional Schrödinger equations using different techniques such as finite difference method [26], variational iteration method [27], homotopy-perturbation method [22], differential transform method [20], Adomian decomposition method [21]. In this paper nonlinear space-time fractional Schrödinger equation is considered which is new in the literature.

The paper is organised as follows: Section 2 provides the basic preliminaries in fractional calculus. Basic idea of modified Adomian decomposition method is given in Section 3. In Section 4 fractional Schrödinger equation is solved with the help of modified ADM. Several examples with different trapping potential and initial conditions are discussed in Section 5. Concluding remarks are provided in the last section.

2. Preliminaries

In this section, we provide some basic definitions of fractional integral and differential operators

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$, if there exists a real number p $(> \alpha)$ such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0,\infty)$. Clearly, $C_\alpha \subset C_\beta$ if $\beta \le \alpha$.

Definition 2. A function $f(t)$, $t > 0$, is said to be in the space C_α^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 3 (Riemann-Liouville Fractional Integral Operator)**.** The (left sided) Riemann-Liouville fractional integral of order $\mu \ge 0$ of a function $f \in C_\alpha$, $\alpha \ge -1$, is defined as

$$D_t^{-\mu} f(t,x) = \begin{cases} \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t \dfrac{f(s,x)}{(t-s)^{1-\mu}}, & \mu > 0, t > 0, \\ f(t) & \mu = 0. \end{cases}$$

Definition 4 (Caputo Fractional Derivative Operator)**.** The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in N \cup \{0\}$, is defined as

$$^C D_t^\mu = \begin{cases} D_t^{-(m-\mu)} \dfrac{\partial^m}{\partial t^m} f(t,x), & m-1 < \mu < m, \ m \in N, \\ \dfrac{\partial^m}{\partial t^m} f(t,x), & \mu = m. \end{cases}$$

Note that

$$D_t^{-\mu}\, {}^C D_t^\mu f(t,x) = f(t,x) - \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(0,x) \frac{t^k}{k!}, \quad m-1 < \mu \le m, \ m \in N.$$

In this paper, the derivative is considered in Caputo sense, since it provides us the advantage of requiring initial conditions given in terms of integer-order derivatives. Also it allows us to specify inhomogeneous initial conditions for fractional differential equations with the Caputo derivative if it is desired.

Definition 5 (Mittag-Leffler function)**.** A one-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (\alpha > 0).$$

Definition 6 (Generalised Cosine and Sine function [4]). We define the generalised Cosine and Sine function as

$$\cos_\alpha(x) = \mathrm{Re}(E_\alpha(ix^\alpha)), \quad \sin_\alpha(x) = \mathrm{Im}\, g(E_\alpha(ix^\alpha)).$$

It is easy to show that these generalised trigonometric functions have a behavior analogous in many respects to that of the corresponding classical functions. They also have the following properties. For $0 < \alpha \le 1$

$$^C D_{0+}^\alpha \sin_\alpha(\lambda x) = \lambda \cos_\alpha(\lambda x), \quad ^C D_{0+}^\alpha \cos_\alpha(\lambda x) = -\lambda \sin_\alpha(\lambda x).$$

3. Modified Adomian Decomposition Method

In this section we briefly describe the Adomian decomposition method [1]. Consider the differential equation

$$Lu + Ru + Nu = g, \tag{3.1}$$

where
L is the highest order derivative operator and invertible,
R is the linear differential operator of order less that L,
Nu is the Nonlinear terms and g is the source term.

Solving Lu from (3.1), we have

$$Lu = g - Ru - Nu.$$

Because L is invertible, the equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{3.2}$$

If L is a second-order operator, for example, L^{-1} is a two fold integration operator and $L^{-1}Lu = u - u(0) - tu'(0)$, then (3.2) yields,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu.$$

Now the solution u can be presented as a series

$$u = \sum_{n=0}^{\infty} u_n$$

with u_0 identified as $u(0) + tu'(0) + L^{-1}g$ and u_n, $n > 0$, is to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n,$$

where A_n is calculated using the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(v(\lambda))\right]_{\lambda=0}, \quad n = 0,1,\dots,$$

where

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n.$$

Now the series solution $u = \sum_{n=0}^{\infty} u_n$ to the differential equation is calculated iteratively as follows:

$$u_0 = u(0) + tu'(0) + L^{-1}g,$$
$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 0.$$

The above described method can be easily extended to a system of differential equations and the resulting equations will be of the form

$$u_{i,0} = \Phi_i + L^{-1}g_i,$$
$$u_{i,k+1} = -L^{-1}(Ru_{i,k}) - L^{-1}(A_{i,k}), \quad k \geq 0,$$

where $1 \leq i \leq n$ and Φ_i represent the terms arising from the given initial and boundary conditions for a system containing n equations.

The above described method is the original Adomian decomposition method. Modified decomposition method differ from the original in calculating the Adomian polynomials. In modified ADM [9], the Adomian polynomials for nonlinear terms are calculated using the following method.

Let $\{u_i\}_{i=0}^{\infty}$ be a sequence such that $u = \sum_{j=0}^{\infty} u_j$ be the solution and G be any nonlinear operator. Consider the operator T with the following properties.

(i) $T(v_i) = (i+1)v_{i+1}$

(ii) $T(G^k(u_0)) = u_1 G^{k+1}(u_0)$

(iii) $T(v_{i_1},\cdots,v_{i_j}) = T(v_{i_1})(v_{i_2,\cdots v_{i_j}}) + v_{i_i}T(v_{i_2,\cdots v_{i_j}})$

(iv) $T(v_{i_1},v_{i_2},\cdots,v_{i_l}G^k(u_0)) = T(v_{i_1},\cdots,v_{i_l})G^k(u_0) + (v_{i_1},\cdots,v_{i_l})T(G^k(u_0))$.

Theorem 1. *Let $u = \sum_{j=0}^{\infty} u_j$ be the solution and A_m's be Adomian polynomials. We have*

$$A_{m+1} = \frac{1}{m+1}T(A_m).$$

The proof of the theorem and effectiveness of using this kind of Adomian polynomials is found in [9].

4. Fractional Schrödinger Equation

The Schrödinger equation is the fundamental equation of physics for describing nonrelativistic quantum mechanical behavior. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wave function of a physical system evolves over time. The fractional one-dimensional Schrödinger equation is given by

$$i\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}+\frac{1}{2}\frac{\partial^\gamma \psi(x,t)}{\partial x^\gamma}-V_d(x)\psi(x,t)-\beta_d|\psi|^2\psi(x,t)=0, \quad 0<\alpha\le 1, 1<\gamma\le 2, \tag{4.1}$$

with initial condition $\psi(x,0)=\psi_0(x)$. Here the fractional derivative is considered in Caputo sense.

By setting $\psi(x,t)=u(x,t)+iv(x,t)$ the equation (4.1) can be transformed into a system of fractional partial differential equation as

$$\begin{aligned} &{}^C D_t^\alpha u+\frac{1}{2}\,{}^C D_x^\gamma v-V_d v-\beta_d(vu^2+v^3)=0,\\ &{}^C D_t^\alpha v-\frac{1}{2}\,{}^C D_x^\gamma u-V_d u+\beta_d(uv^2+u^3)=0 \end{aligned} \tag{4.2}$$

with the initial conditions

$$\begin{aligned} &u(x,0)=f(x),\\ &v(x,0)=g(x), \end{aligned}$$

where $f(x)+ig(x)=\psi_0(x)$.

In this section we are going to solve the system of fractional differential equation (4.2) using modified Adomian decomposition method. By using the property of Caputo fractional derivative the (4.2) can be rewritten as

$$\begin{aligned} Du(x,t)&={}^C D_t^{1-\alpha}\left(-\frac{1}{2}\,{}^C D_x^\gamma v+V_d v+\beta_d(vu^2+v^3)\right),\\ Dv(x,t)&={}^C D_t^{1-\alpha}\left(\frac{1}{2}\,{}^C D_x^\gamma u+V_d u-\beta_d(uv^2+u^3)\right). \end{aligned}$$

Now integrating on both sides we get

$$\begin{aligned} u(x,t)&=u(x,0)+\int_0^t {}^C D_s^{1-\alpha}\left(-\frac{1}{2}\,{}^C D_x^\gamma v(x,s)+V_d v(x,s)+\beta_d(vu^2+v^3)(x,s)\right)ds,\\ v(x,t)&=v(x,0)+\int_0^t {}^C D_s^{1-\alpha}\left(\frac{1}{2}\,{}^C D_x^\gamma u(x,s)+V_d u(x,s)-\beta_d(uv^2+u^3)(x,s)\right)ds. \end{aligned}$$

Here we take $u(x,0)$ and $v(x,0)$ as u_0 and v_0 and the iterative scheme for this problem is given by

$$u_n(x,t)=\int_0^t {}^C D_s^{1-\alpha}\left(-\frac{1}{2}\,{}^C D_x^\gamma v_{n-1}(x,s)+V_d v_{n-1}(x,s)+\beta_d A_{n-1}(x,s)\right)ds,$$

$$v_n(x,t) = \int_0^t {}^C D_s^{1-\alpha}\left(\frac{1}{2}\,{}^C D_x^{\gamma} u_{n-1}(x,s) + V_d u_{n-1}(x,s) - \beta_d B_{n-1}(x,s)\right) ds\,,$$

where A_n and B_n are called as Adomian polynomials and is given by [9]

$$\begin{aligned}
A_0 &= v_0 u_0^2 + v_0^3 \\
A_1 &= 2u_0 u_1 v_0 + (u_0^2 + 3v_0^2)v_1 \\
A_2 &= (u_1^2 + 2u_0 u_2 + 3v_1^2)v_0 + 2u_0 u_1 v_1 + (u_0^2 + 3v_0^2)v_2 \\
&\;\vdots
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= u_0 v_0^2 + u_0^3 \\
B_1 &= 2v_0 v_1 u_0 + (v_0^2 + 3u_0^2)u_1 \\
B_2 &= (v_1^2 + 2v_0 v_2 + 3u_1^2)u_0 + 2v_0 v_1 u_1 + (v_0^2 + 3u_0^2)u_2 \\
&\;\vdots
\end{aligned}$$

and so on.

Hence by Adomian decomposition method the solution of (4.2) is given by

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t)\,, \\
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t)
\end{aligned}$$

and therefore the solution of (4.1) is given by

$$\psi(x,t) = u(x,t) + iv(x,t).$$

5. Examples

In this section we provide some examples with different V_d and β_d to show the effectiveness of this method.

5.1 Linear Fractional Schrödinger Equation

Consider the linear fractional Schrödinger equation with $V_d = \beta_d = 0$ in (4.1) which takes the form

$$\begin{aligned}
&i\frac{\partial^\alpha \psi}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta \psi}{\partial x^\beta}\right) = 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1, \qquad (5.1) \\
&\psi(x,0) = \psi_0(x).
\end{aligned}$$

Now by applying the above described Adomian decomposition method we get the iteration scheme as

$$u_n(x,t) = \int_0^t {}^C D_s^{1-\alpha}\left(-\frac{1}{2}\,{}^C D_x^{\beta}\left({}^C D_x^{\beta} v_{n-1}(x,s)\right)\right) ds,$$
$$v_n(x,t) = \int_0^t {}^C D_s^{1-\alpha}\left(\frac{1}{2}\,{}^C D_x^{\beta}\left({}^C D_x^{\beta} u_{n-1}(x,s)\right)\right) ds$$

with $u_0 = f(x)$ and $v_0 = g(x)$, where $f(x) + ig(x) = \psi_0(x)$. Let us take the initial condition $\psi_0(x)$ as $E_\beta(ix^\beta)$, the Mittag Leffler function.

By using the generalised sine and cosine function [4], we can write the Mittag Leffler function as

$$E_\beta(ix^\beta) = \cos_\beta(x) + i\sin_\beta(x).$$

Therefore we can take $f(x) = \cos_\beta(x)$ and $g(x) = \sin_\beta(x)$. Hence

$$u_0 = \cos_\beta(x),$$
$$v_0 = \sin_\beta(x),$$
$$u_1 = \sin_\beta(x)\frac{1}{2}\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$v_1 = -\cos_\beta(x)\frac{1}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$u_2 = \cos_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$v_2 = \sin_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3 = \sin_\beta(x)\left(\frac{1}{2}\right)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$v_3 = -\cos_\beta(x)\left(\frac{1}{2}\right)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

and so on. The solution by Adomian decomposition method is given by $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$. Hence

$$\begin{aligned} u(x,t) &= \cos_\beta(x)\left[1 + \left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(\frac{1}{2}\right)^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right] \\ &\quad + \sin_\beta(x)\left[\left(\frac{1}{2}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{1}{2}\right)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right] \\ &= \cos_\beta(x)\cos_\alpha(-t/2) - \sin_\beta(x)\sin_\alpha(-t/2) \end{aligned}$$

$$v(x,t) = \sin_\beta(x)\left[1 + \left(\frac{1}{2}\right)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(\frac{1}{2}\right)^4 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right]$$
$$- \cos_\beta(x)\left[\left(\frac{1}{2}\right) \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{1}{2}\right)^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$
$$= \sin_\beta(x)\cos_\alpha(-t/2) + \cos_\beta(x)\sin_\alpha(-t/2).$$

Hence the solution to (5.1) is given by

$$\begin{aligned}\psi(x,t) &= u(x,t) + iv(x,t)\\ &= \cos_\beta(x)\cos_\alpha(-t/2) - \sin_\beta(x)\sin_\alpha(-t/2) + i\sin_\beta(x)\cos_\alpha(-t/2) + \cos_\beta(x)\sin_\alpha(-t/2)\\ &= \cos_\beta(x)(\cos_\alpha(-t/2) + i\sin_\alpha(-t/2)) + i\sin_\beta(x)(\cos_\alpha(-t/2) + i\sin_\alpha(-t/2))\\ &= E_\beta(ix^\beta)E_\alpha(i(-1/2)t^\alpha)\end{aligned}$$

which is the exact solution of the linear fractional Schrödinger equation. If we take $\alpha = \beta = 1$, the solution reduces to

$$\psi(x,t) = e^{i(x-t/2)},$$

the exact solution of linear Schroödinger equation as shown in [20–22, 27].

5.2 Nonlinear Fractional Schrödinger Equation

Consider the nonlinear fractional Schroödiner equation with zero trapping potential $V_d = 0$ and $\beta_d = -1$.

$$i\frac{\partial^\alpha \psi}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta \psi}{\partial x^\beta}\right) + |\psi|^2\psi = 0, \quad 0 < \alpha \le 1,\ 0 < \beta \le 1 \tag{5.2}$$

subject to the initial condition $\psi(x,0) = E_\beta(ix^\beta)$.

Proceeding like the above example we get the iteration scheme as

$$u_n(x,t) = \int_0^t {}^C D_s^{1-\alpha}\left(-\frac{1}{2}\,{}^C D_x^\beta\left({}^C D_x^\beta v_{n-1}(x,s)\right) - A_n(x,s)\right)ds,$$
$$v_n(x,t) = \int_0^t {}^C D_s^{1-\alpha}\left(\frac{1}{2}\,{}^C D_x^\beta\left({}^C D_x^\beta u_{n-1}(x,s)\right) + B_n(x,s)\right)ds,$$

where A_n and B_n are called Adomian polynomials correspond to the nonlinear terms $vu^2 + v^3$ and $uv^2 + u^3$ respectively.

By using the property $E_\beta(ix^\beta) = \cos_\beta(x) + i\sin_\beta(x)$, we can take the initial condition as

$$u_0 = \cos_\beta(x), \quad v_0 = \sin_\beta(x).$$

The few terms of the solution are calculated below.

$$u_1 = -\sin_\beta(x)\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$v_1 = \cos_\beta(x)\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = \cos_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$v_2 = \sin_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3 = -\sin_\beta(x)\left(\frac{1}{2}\right)^3\frac{t^{2\alpha}}{\Gamma(3\alpha+1)},$$

$$v_3 = \cos_\beta(x)\left(\frac{1}{2}\right)^3\frac{t^{2\alpha}}{\Gamma(3\alpha+1)}.$$

The few Adomian polynomials are calculated below

$$A_0 = v_0u_0^2 + v_0^3 = \sin_\beta(x),$$

$$A_1 = 2u_0u_1v_0 + (u_0^2+3v_0^2)v_1 = \cos_\beta(x)\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$A_2 = (u_1^2+2u_0u_2+3v_1^2)v_0 + 2u_0u_1v_1 + (u_0^2+3v_0^2)v_2 = \sin_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

and so on. Similarly

$$B_0 = u_0v_0^2 + u_0^3 = \cos_\beta(x),$$

$$B_1 = 2v_0v_1u_0 + (v_0^2+3u_0^2)u_1 = \sin_\beta(x)\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$B_2 = (v_1^2+2v_0v_2+3u_1^2)u_0 + 2v_0v_1u_1 + (v_0^2+3u_0^2)u_2 = \sin_\beta(x)\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

and so on.

The solution by Adomian decomposition method is given by $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $v(x,t) = \sum_{n=0}^{\infty} v_n(x,t)$. Hence

$$\begin{aligned} u(x,t) &= \cos_\beta(x)\left[1+\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\left(\frac{1}{2}\right)^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right] \\ &\quad -\sin_\beta(x)\left[\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}+\left(\frac{1}{2}\right)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right] \\ &= \cos_\beta(x)\cos_\alpha(t/2) - \sin_\beta(x)\sin_\alpha(t/2) \end{aligned}$$

$$\begin{aligned} v(x,t) &= \sin_\beta(x)\left[1+\left(\frac{1}{2}\right)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\left(\frac{1}{2}\right)^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right] \\ &\quad +\cos_\beta(x)\left[\left(\frac{1}{2}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}+\left(\frac{1}{2}\right)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right] \\ &= \sin_\beta(x)\cos_\alpha(t/2) + \cos_\beta(x)\sin_\alpha(t/2). \end{aligned}$$

Hence the solution to (5.1) is given by

$$\begin{aligned}\psi(x,t) &= u(x,t) + iv(x,t)\\ &= \cos_\beta(x)\cos_\alpha(t/2) - \sin_\beta(x)\sin_\alpha(t/2) + i\sin_\beta(x)\cos_\alpha(t/2) + \cos_\beta(x)\sin_\alpha(t/2)\\ &= \cos_\beta(x)(\cos_\alpha(t/2) + i\sin_\alpha(t/2)) + i\sin_\beta(x)(\cos_\alpha(t/2) + i\sin_\alpha(t/2))\\ &= E_\beta(ix^\beta)E_\alpha(i(1/2)t^\alpha)\end{aligned}$$

which is the exact solution of the nonlinear fractional Schrödinger equation. If we take $\alpha = \beta = 1$, the solution reduces to

$$\psi(x,t) = e^{i(x+t/2)},$$

the exact solution of nonlinear Schroödinger equation shown in [20–22, 27].

Consider the nonlinear fractional Schroödiner equation with trapping potential $V_d = \cos^2_\beta(x)$ and $\beta_d = -1$.

$$i\frac{\partial^\alpha\psi}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta\psi}{\partial x^\beta}\right) - \psi\cos^2_\beta(x) - |\psi|^2\psi = 0, \quad 0<\alpha\le 1,\ 0<\beta\le 1 \tag{5.3}$$

subject to the initial condition $\psi(x,0) = \sin_\beta(x)$.

Proceeding like the above examples, the solution to the above equation is given by

$$\psi(x,t) = \sin_\beta(x)E_\alpha(-i(3/2)t^\alpha).$$

If we take $\alpha = \beta = 1$, the solution reduces to $\sin(x)e^{\frac{-3it}{2}}$ which is the exact solution derived in [20, 21].

Remark. The above described method can be easily applied for different type of initial conditions other than mentioned in the examples. These examples are available in the literature and they are convergent to the classical solution when α and β are integers.

6. Conclusion

In this paper Adomian decomposition method is efficiently used to find the solution of generalised linear and nonlinear Schrödiner equation which is used in quantum mechanics. The equation is transformed into a system of coupled equation which avoids the use of complex functions in the calculation. The Adomian polynomials used here are calculated by using the new method proposed by Gu *et al.* [9] which easily provide computer algorithms so that the calculations are less when compared to original method of calculating the Adomian polynomials. All the solutions are derived in terms of Mittag Leffler function, so that the equations can be easily analysed numerically. Also its convergence to the exact solution, when the system is of integer order, proves that this method is very much effective to solve the quantum mechanics problems. Moreover the necessity of discretization or perturbation is not needed, the method can be easily implemented using any symbolic mathematical softwares.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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