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Analysis of An Age Dependent Epidemic Model

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Abstract. In this paper, we consider an age dependent epidemic model of type $S \to I \to \cdot$ and study the solutions of the model. In view of the above, we study a nonlinear model which is a generalisation of the age dependent epidemic model and obtain the existence, uniqueness, the semigroup property and the continuous dependence of the solutions on the initial data for the general model. Then, we show that the age dependent SI model is a particular case of the general model and hence conclude the wellposedness of the age dependent epidemic model.

1. Introduction

Mathematical and computational approaches provide powerful tools in the study of problems in population biology and ecosystems science. In particular, a great deal of research has been done in models in epidemiology [10, 4]. Study of epidemic models, both with and without age dependence have a long history with a vast variety of models and explanations for the spread and cause of epidemic outbreaks [5, 1]. In the current work, we are going to study and analyse an age dependent SI model [4, 10, 5]. We consider a population that is divided into two classes: Susceptibles S and Infectives I. Susceptibles are individuals of the population, who can catch the disease. Infectives are people who already have the disease and can infect others. $u(t)$ denotes the number of susceptibles and $v(a,t)$ denotes the number of infectives. The assumption is that the number of susceptibles depend only on time t whereas the number of infectives depend on a as well as t. Here, a is the time that has lapsed after susceptibles have entered the infective class. (i.e) it is the age from exposure to the disease. In this epidemic model, we start with an initial number of susceptibles and an initial number of infectives. The assumption is that, once infected, individuals from susceptible class move on to the infective class. A is a positive constant and is the inflow of individuals per time into the population due to births or immigration. μ_o is a

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positive constant and denotes the death rate due to natural causes and not due to infection.

Then, the rate of change of susceptibles is given by

$$\frac{du}{dt} = A - \mu_o u - u \int_0^\infty k(a)v(a,t)da$$

where $k(a)$ denotes the measure of infectiousness of the infectives. The last term on the right hand side in the above equation gives the removal rate of the susceptibles due to exposure from the infection.

By law of conservation, the change in the number of infectives is equal to the number of infectives removed. (i.e) mortality due to the infection. Therefore, if $\mu(a)$ denotes the death rate due to infection, then

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} = -\mu(a)v(a,t).$$

If $v(0,t)$ is the number of new individuals who enter the infective class, then

$$v(0,t) = u \int_0^\infty k(a)v(a,t)da$$

where the right hand side denotes the number of persons removed from the susceptible class due to infection.

$$u(0) = u_0 \quad \text{and} \quad v(a,0) = v_0(a)$$

denote the initial number of susceptibles and the initial number of infectives respectively.

The functions $k(a)$, $\mu(a)$ are nonnegative, bounded functions and depend on the age of infection a.

Thus, the model is given by

$$\frac{du}{dt} = A - \mu_o u - u \int_0^\infty k(a)v(a,t)da\,,$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} = -\mu(a)v(a,t),$$

$$v(0,t) = u \int_0^\infty k(a)v(a,t)da\,,$$

$$u(0) = u_0\,,$$

$$v(a,0) = v_0(a).$$

This paper is organised as follows: In section 2, we formulate a non-linear model which is a generalisation of the age dependent epidemic model. We study the model in an abstract space and choose $L^1 \times \mathfrak{R}$ as the tractable mathematical setting for our general model. Since, for population dynamics problems, $L^1 \times \mathfrak{R}$ is the natural choice for a mathematical setting in that, the physical interpretation of the density function requires that, it should be integrable and the mathematical treatment of the problem requires that, the density function belong to a complete normed linear space. Our choice of $L^1 \times \mathfrak{R}$ is also influenced by our intention to view age dependent population dynamics from the vantage point of view of theory of semigroup of operators in a Banach space [3]. We refer to [8, 6, 7] where the semigroup theory has been used to study general age dependent models with diffusion. We further state some propositions, to prove the existence and uniqueness, continuous dependence of the solutions on the initial data and the semigroup property of the solutions of the general model. In section 3, we show how the SI epidemic model is a particular case of the general non-linear model and conclude the wellposedness of the age dependent epidemic model.

2. **The General Nonlinear Model**

The general age dependent non linear model [11] is given by

$$Dl(a,t) = G(l(a,t), Q(t)), \tag{1}$$

$$l(0,t) = F(l(\cdot,t), Q(t)), \tag{2}$$

$$D(Q(t)) = g(l(a,t), Q(t)), \tag{3}$$

$$l(a,0) = \phi(a), \tag{4}$$

$$Q(0) = Q_0. \tag{5}$$

In this section, we prove the existence, uniqueness, continuous dependence of the solutions on the initial data and the semigroup property of the solutions of the general model. To prove the above, we convert the model into an equivalent integral system of equations, thereby study the properties of the solutions for this integral system of equations and conclude that these hold for the proposed age dependent model also. Our general model (1)-(5) can be written as

$$\lim_{h\to 0^+} \int_0^\infty |h^{-1}[l(a+h,t+h) - l(a,t)] - G(l(\cdot,t),Q(t))(a)|da = 0, \quad 0 \le t \le T, \tag{6}$$

$$\lim_{h\to 0^+} h^{-1} \int_0^h |l(a,t+h) - F(l(\cdot,t),Q(t))|da = 0, \quad 0 \le t \le T, \tag{7}$$

$$Q'(t) = g(l(a,t), Q(t)), \tag{8}$$

$$l(a,0) = \phi(a), \tag{9}$$

$$Q(0) = Q_0. \tag{10}$$

The equivalent integral system of equations [11] for our general model (6)-(10) is given by

$$l(a,t) = F(l(\cdot,t-a), Q(t-a)) + \int_{t-a}^{t} G(l(\cdot,s),Q(s))(s+a-t)ds \ \text{ a.e } \ a \in (0,t), \tag{11}$$

$$l(a,t) = \phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds \quad \text{a.e} \quad a \in (t,\infty), \quad (12)$$

$$Q(t) = Q_0 + \int_0^t g(l(a,s),Q(s))ds\,. \quad (13)$$

The following are the assumptions that we have on the functions F, G and g.

(1) F, G, g are defined as follows

$$F : L^1 \times \mathfrak{R} \to \mathfrak{R}$$

$$G : L^1 \times \mathfrak{R} \to L^1$$

$$g : L^1 \times \mathfrak{R} \to \mathfrak{R}$$

(2) F, G, g are Lipschitz continuous functions such that the following hold:

(i) $\exists$ an increasing function $c_1 : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}|F(\Phi_1,x_1) - F(\Phi_2,x_2)| &\le c_1(r)\|(\Phi_1,x_1)-(\Phi_2,x_2)\|_{L^1\times\mathfrak{R}} \quad (14)\\ &= c_1(r)[\|\Phi_1-\Phi_2\|_{L^1} + |x_1-x_2|]\end{aligned}$$

(ii) Similarly $\exists$ an increasing function $c_2 : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}\|G(\Phi_1,x_1) - G(\Phi_2,x_2)\|_{L^1} &\le c_2(r)\|(\Phi_1,x_1)-(\Phi_2,x_2)\|_{L^1\times\mathfrak{R}} \quad (15)\\ &= c_2(r)[\|\Phi_1-\Phi_2\|_{L^1} + |x_1-x_2|]\end{aligned}$$

(iii) Similarly $\exists$ an increasing function $c_3 : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}|g(\Phi_1,x_1) - g(\Phi_2,x_2)| &\le c_3(r)\|(\Phi_1,x_1)-(\Phi_2,x_2)\|_{L^1\times\mathfrak{R}} \quad (16)\\ &= c_3(r)[\|\Phi_1-\Phi_2\|_{L^1} + |x_1-x_2|]\end{aligned}$$

The following proposition proves that a solution of the integral equation (11)-(13) is also a solution of the general model (6)-(10).

Proposition 2.1. *Let equations* (14)-(16) *hold, let* $T > 0$*, let* $\phi \in L_1$*,* $Q_0 \in \mathfrak{R}$ *and let* $(l,Q) \in L_T$*. If* (l,Q) *is a solution of the integral equation* (11)-(13) *on* $[0,T]$*, then* (l,Q) *is a solution of the general model* (6)-(10) *on* $[0,T]$.

Proof. Given that $(l,Q) \in L_T$ is a solution of the integral equation (11)-(13) on $[0,T]$ where

$$L_T = \mathscr{C}([0,T]; L^1 \times \mathfrak{R})$$

and the norm on L_T is given by

$$\|(l,Q)\|_{L_T} = \sup_{0\le t\le T} [\|(l(\cdot,t)\|_{L^1} + |Q(t)|]\,.$$

We have to prove that (l,Q) is a solution of the general model (6)-(10) on $[0,T]$.

To prove: (l,Q) is a solution of equation (6) of the general model.
Let $0 \le t < T$ and let $0 < h < T - t$.

In the following equation,

$$\int_0^\infty |h^{-1}[l(a+h,t+h)-l(a,t)]-G(l(\cdot,t),Q(t))(a)|da\,.$$

Substitute for $l(a+h,t+h)$ and $l(a,t)$ using equations (11) and (12). Therefore, the above becomes

$$=\int_0^\infty |h^{-1}\int_t^{t+h}(G(l(\cdot,s),Q(s))(s+a-t)-G(l(\cdot,t),Q(t))(a))ds|da$$

$$\leq h^{-1}\int_t^{t+h}\int_0^\infty |G(l(\cdot,s),Q(s))(s+a-t)-G(l(\cdot,t),Q(t))(a)|dads\,.$$

Adding and subtracting $G(l(\cdot,t),Q(t))(s+a-t)$ and using triangle inequality, the above becomes

$$\leq h^{-1}\int_t^{t+h}\left[\int_0^\infty |G(l(\cdot,s),Q(s))(s+a-t)-G(l(\cdot,t),Q(t))(s+a-t)|da\right.$$

$$\left.+\int_0^\infty |G(l(\cdot,t),Q(t))(s+a-t)-G(l(\cdot,t),Q(t))(a)|da\right]ds$$

$$=h^{-1}\int_t^{t+h}\|G(l(\cdot,s),Q(s))-G(l(\cdot,t),Q(t))\|_{L^1\times\mathfrak{R}}ds$$

$$+h^{-1}\int_t^{t+h}\int_0^\infty |G(l(\cdot,t),Q(t))(s+a-t)-G(l(\cdot,t),Q(t))(a)|dads$$

$$\leq \sup_{t\leq s\leq t+h} c_2(r)(\|l(\cdot,s)-l(\cdot,t)\|_{L^1}+|Q(s)-Q(t)|)$$

$$+\sup_{t\leq s\leq t+h}\int_0^\infty |G(l(\cdot,t),Q(t))(s+a-t)-G(l(\cdot,t),Q(t))(a)|da\,.$$

As $h\to 0$, the above expression approaches 0 by the continuity of the function $G: L^1\times\mathfrak{R}\to L^1$. Therefore

$$\lim_{h\to 0^+}\int_0^\infty |h^{-1}[l(a+h,t+h)-l(a,t)]-G(l(\cdot,t),Q(t))(a)|da=0\,.$$

Hence, proved equation (6) of the general model.

To prove: (l,Q) is a solution of equation (7) of the general model.
Let $0\leq t<T$ and let $0<h<T-t$.
In the following equation,

$$h^{-1}\int_0^h |l(a,t+h)-F(l(\cdot,t),Q(t))|da\,.$$

Substituting for $l(a, t+h)$ using equation (11) of the integral equation and using triangle inequality, the above becomes

$$\begin{aligned}&\leq h^{-1}\int_0^h |F(l(\cdot, t+h-a), Q(t+h-a)) - F(l(\cdot, t), Q(t))|da \\ &\quad + h^{-1}\int_0^h \int_{t+h-a}^{t+h} |G(l(\cdot, s), Q(s))(s+a-t-h)|dsda \\ &\stackrel{\text{def}}{=} K_1 + K_2\,.\end{aligned}$$

Now, as $h \to 0$, $K_1 \to 0$ by the continuity of the function $F : L^1 \times \mathfrak{R} \to \mathfrak{R}$.
Changing the order of integration in case of K_2, we get

$$\begin{aligned}K_2 &= h^{-1}\int_t^{t+h}\left[\int_{t+h-s}^{h} |G(l(\cdot, s), Q(s))(s+a-t-h)|da\right]ds \\ &= h^{-1}\int_t^{t+h}\left[\int_0^{s-t} |G(l(\cdot, s), Q(s))(a)|da\right]ds\,.\end{aligned}$$

Now,

$$\begin{aligned}&\lim_{s\to t^+}\int_0^{s-t} |G(l(\cdot, s), Q(s))(a)|da \\ &\quad\leq \lim_{s\to t^+}\int_0^{s-t} |G(l(\cdot, s), Q(s))(a) - G(l(\cdot, t), Q(t))(a)|da \\ &\qquad + \lim_{s\to t^+}\int_0^{s-t} |G(l(\cdot, t), Q(t))(a)|da \\ &\quad\leq \lim_{s\to t^+}\int_0^{\infty} |G(l(\cdot, s), Q(s))(a) - G(l(\cdot, t), Q(t))(a)|da \\ &\qquad + \lim_{s\to t^+}\int_0^{s-t} |G(l(\cdot, t), Q(t))(a)|da \\ &\quad= \lim_{s\to t^+} \|G(l(\cdot, s), Q(s)) - G(l(\cdot, t), Q(t))\|_{L^1\times\mathfrak{R}} \\ &\qquad + \lim_{s\to t^+}\int_0^{s-t} |G(l(\cdot, t), Q(t))(a)|da \\ &\quad= 0\,.\end{aligned}$$

Therefore , $K_2 \to 0$, as $h \to 0$.
Hence,

$$\lim_{h\to 0^+} h^{-1}\int_0^h |l(a, t+h) - F(l(\cdot, t), Q(t))|da = 0\,.$$

Hence, proved equation (7) of the general model.

Substituting $t = 0$ in equation (13) of the integral equation, we find that (l,Q) satisfies equation (9) of the general model. Differentiating equation (13) with respect to t, we find that (l,Q) satisfies equation (8) of the general model.
Similarly, substituting $t=0$ in equation (12) of the integral equation, we find that (l,Q) is a solution of equation (10) of the general model.
Thus we have proved that a solution of the integral equation (11)-(13) is also a solution of the general model (6)-(10).
In the following proposition, we prove that a unique solution to the integral equation (11)-(13) exists. □

Proposition 2.2. *Let equations* (14)-(16) *hold, let $r > 0$. There exists $T > 0$ such that if $\phi \in L^1$, $Q_0 \in \Re$ and $\|\phi\|_{L_1} \le r$, then there is a unique function $(l,Q) \in L_T$ such that (l,Q) is a solution of the integral equation* (11)-(13) *on* $[0,T]$.

Proof. Choose $T > 0$ such that

$$\frac{\begin{pmatrix} T[c_1(2r)+c_2(2r)+2c_3(2r)+|F(0,0)|+\|G(0,0)\|_{L^1} \\ +2|g(0,0)|]+(c_1(2r)+c_2(2r)+2c_3(2r))R+2|Q_0| \end{pmatrix}}{2r} + 1/2 \le 1\,.$$

Let $\phi \in L^1$, such that $\|\phi\|_{L^1} \le r$. Define

$$M \stackrel{\text{def}}{=} \{(l,Q) \in L_T : l(\cdot,0) = \phi, Q(0) = Q_0 \ \text{ and } \ \|(l,Q)\|_{L_T} \le 2r\}\,.$$

Define a mapping K on M as follows:
For $(l,Q) \in M$, $t \in [0,T]$,

$$K(l(a,t),Q(t))$$

$$\stackrel{\text{def}}{=} \begin{cases} \Big(F(l(\cdot,t-a),Q(t-a)) + \int_{t-a}^{t} G(l(\cdot,s),Q(s))(s+a-t)ds, \\ Q_0 + \int_0^t g(l(a,s),Q(s))ds \Big) \quad \text{a.e } \ a \in (0,t), \\ \Big(\phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds, Q_0 + \int_0^t g(l(a,s),Q(s))ds \Big) \\ \qquad \text{a.e } \ a \in (t,\infty). \end{cases}$$

To prove this proposition, we need to prove the following:

(i) M is a closed subset of L_T.
(ii) K maps M into M.
(iii) K is a strict contraction in M.

To prove: (i) M is a closed subset of L_T.
Now, M is closed iff for a sequence $(l_n,Q_n) \in M, (l_n,Q_n) \to (l,Q) \Rightarrow (l,Q) \in M$.

Now,

$$(l_n, Q_n) \in M$$

$$\Rightarrow \quad (l_n, Q_n) \in L_T \text{ and } l_n(\cdot, 0) = \phi \text{ and } \|(l_n, Q_n)\|_{L_T} \leq 2r$$

Now given that

$$(l_n, Q_n) \to (l, Q)$$

$$\Rightarrow \quad l_n \to l \text{ and } Q_n \to Q$$

$$\Rightarrow \quad l_n(\cdot, 0) \to l(\cdot, 0)$$

But

$$l_n(\cdot, 0) = \phi$$

$$\Rightarrow \quad l(\cdot, 0) = \phi$$

Also as,

$$(l_n, Q_n) \to (l, Q)$$

$$\Rightarrow \quad \|(l_n, Q_n)\|_{L_T} \to \|(l, Q)\|_{L_T}$$

But

$$\|(l_n, Q_n)\|_{L_T} \leq 2r$$

$$\Rightarrow \quad \|(l, Q)\|_{L_T} \leq 2r$$

Therefore, we have

$$(l, Q) \in L_T, l(\cdot, 0) = \phi, \|(l, Q)\|_{L_T} \leq 2r$$

$$\Rightarrow \quad (l, Q) \in M.$$

Therefore, M is a closed subset of L_T.
To prove (ii) K maps M into M. For that, we first prove

$$\|K(l(a,t), Q(t))\|_{L_T} \leq 2r$$

i.e.

$$\sup_{0 \leq t \leq T} [\|K' + K''\|] \leq 2r$$

i.e.

$$\sup_{0 \leq t \leq T} [\|K'\|_{L^1} + |K''|] \leq 2r$$

where

$$K(l(a,t), Q(t)) = (K', K'').$$

Let $(l,Q) \in M,\ t \in [0,T]$. By using equations (14)-(16)and by interchanging the order of integration,we get

$$
\begin{aligned}
&\int_0^\infty |K(l(a,t),Q(t))|da \\
&= \int_0^\infty |K'|da + |K''| \\
&= \int_0^t \left[\left|F(l(\cdot,t-a),Q(t-a)) + \int_{t-a}^t G(l(\cdot,s),Q(s))(s+a-t)ds\right|\right]da \\
&\quad + \left|Q_0 + \int_0^t g(l(a,s),Q(s))ds\right| \\
&\quad + \int_t^\infty \left[\left|\phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds\right|\right]da \\
&\quad + \left|Q_0 + \int_0^t g(l(a,s),Q(s))ds\right| \\
&\le \int_0^t \left[|F(l(\cdot,t-a),Q(t-a))| + \int_{t-a}^t |G(l(\cdot,s),Q(s))(s+a-t)|ds\right]da \\
&\quad + \int_t^\infty \left[|\phi(a-t)| + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)|ds\right]da + 2|Q_0| \\
&\quad + 2\int_0^t |g(l(a,s),Q(s))|ds \\
&\le \int_0^t |F(l(\cdot,s),Q(s))|ds + \int_0^t \left[\int_{t-s}^t |G(l(\cdot,s),Q(s))(s+a-t)|da\right]ds \\
&\quad + \int_0^\infty |\phi(a)|da + \int_0^t \left[\int_t^\infty |G(l(\cdot,s),Q(s))(s+a-t)|da\right]ds + 2|Q_0| \\
&\quad + 2\int_0^t |g(l(a,s),Q(s))|ds \\
&\le \int_0^t |F(l(\cdot,s),Q(s)) - F(0,0)|ds + \int_0^t |F(0,0)|ds + \|\phi\|_{L^1} \\
&\quad + \int_0^t \left[\int_0^\infty |G(l(\cdot,s),Q(s))(s+a-t)|da\right]ds + 2|Q_0| \\
&\quad + 2\int_0^t |g(l(a,s),Q(s))|ds \quad \text{(adding and subtracting } F(0,0))
\end{aligned}
$$

$$
\begin{aligned}
&\le c_1(2r)\int_0^t [\|l(\cdot,s)\|_{L^1} + |Q(s)|]ds + \int_0^t |F(0,0)|ds + r \\
&\quad + \int_0^t \|G(l(\cdot,s),Q(s))\|_{L^1} ds + 2|Q_0| + 2\int_0^t |g(l(a,s),Q(s))|ds \\
&\le c_1(2r)\int_0^t \|l(\cdot,s)\|_{L^1} ds + c_1(2r)\int_0^t |Q(s)|ds + \int_0^t |F(0,0)|ds + r \\
&\quad + c_2(2r)\int_0^t [\|l(\cdot,s)\|_{L^1} ds + |Q(s)|]ds + \int_0^t \|G(0,0)\|_{L^1} ds + 2|Q_0| \\
&\quad + 2c_3(2r)\int_0^t [\|l(\cdot,s)\|_{L^1} + |Q(s)|]ds \\
&\quad + 2\int_0^t |g(0,0)|ds \quad \text{(adding and subtracting } G(0,0), g(0,0)) \\
&= (c_1(2r)+c_2(2r)+2c_3(2r))\int_0^t \|l(\cdot,s)\|_{L^1} ds + \int_0^t |F(0,0)|ds \\
&\quad + \int_0^t \|G(0,0)\|_{L^1} ds + 2\int_0^t |g(0,0)|ds + r + 2|Q_0| \\
&\quad + (c_1(2r)+c_2(2r)+2c_3(2r))\int_0^t |Q(s)|ds \\
&\le (c_1(2r)+c_2(2r)+2c_3(2r))\int_0^t \|l(\cdot,s)\|_{L^1} ds + \int_0^t |F(0,0)|ds \\
&\quad + \int_0^t \|G(0,0)\|_{L^1} ds + 2\int_0^t |g(0,0)|ds + r + 2|Q_0| \\
&\quad + (c_1(2r)+c_2(2r)+2c_3(2r))R
\end{aligned}
$$

(as t ranges from 0 to T, $\int_0^t |Q(s)|ds$ will be bounded by a constant R)

$$
\begin{aligned}
&\le \left[\frac{\begin{pmatrix} t(c_1(2r)+c_2(2r)+2c_3(2r)+|F(0,0)|+\|G(0,0)\|_{L^1} \\ +2|g(0,0)|)+(c_1(2r)+c_2(2r)+2c_3(2r))R+2|Q_0| \end{pmatrix}}{2r} + 1/2\right]2r \\
&\le 2r
\end{aligned}
$$

where we have chosen $T > 0$ such that

$$
\frac{\begin{pmatrix} T[c_1(2r)+c_2(2r)+2c_3(2r)+|F(0,0)|+\|G(0,0)\|_{L^1} \\ +2|g(0,0)|]+(c_1(2r)+c_2(2r)+2c_3(2r))R+2|Q_0| \end{pmatrix}}{2r} + 1/2 \le 1\,.
$$

Thus, we have proved that

$$\|K(l(a,t),Q(t))\|_{L^1\times\mathfrak{R}} \le 2r$$

$$\Rightarrow \quad \sup_{0\le t\le T} \|K(l(a,t),Q(t))\|_{L^1\times\mathfrak{R}} = \|K(l(a,t),Q(t))\|_{L_T} \le 2r\,.$$

Next, we prove the continuity of the function $K(l(\cdot,t),Q(t))$.

To prove: For $(l,Q)\in M$, the function $t\to K(l(\cdot,t),Q(t))$ is continuous from [0,T] to $L^1\times\mathfrak{R}$.

Proof. Let $(l,Q)\in M$ and let $0\le t<\hat t\le T$. Then,

$$\begin{aligned}
&\|K(l(\cdot,t),Q(t))-K(l(\cdot,\hat t),Q(\hat t))\|_{L^1\times\mathfrak{R}}\\
&\le \int_0^t \Big[|F(l(\cdot,t-a),Q(t-a)) + \int_{t-a}^t G(l(\cdot,s),Q(s))(s+a-t)ds\\
&\quad - F(l(\cdot,\hat t-a),Q(\hat t-a)) - \int_{\hat t-a}^{\hat t} G(l(\cdot,s),Q(s))(s+a-\hat t)ds|\Big]da\\
&\quad + \int_t^{\hat t} \Big[|\phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds\\
&\quad - F(l(\cdot,\hat t-a),Q(\hat t-a)) - \int_{\hat t-a}^{\hat t} G(l(\cdot,s),Q(s))(s+a-\hat t)ds|\Big]da\\
&\quad + \int_{\hat t}^{\infty} \Big[|\phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds - \phi(a-\hat t)\\
&\quad - \int_0^{\hat t} G(l(\cdot,s),Q(s))(s+a-\hat t)ds|\Big]da\\
&\quad + 2\Big[\Big|\int_0^{t-\hat t} g(l(a,s),Q(s))ds\Big|\Big]\\
&\overset{\text{def}}{=} J_1+J_2+J_3+J_4\,.
\end{aligned}$$

Consider J_1 first.
As $|t-\hat t|\to 0$, $\int_0^t [|F(l(\cdot,t-a),Q(t-a)) - F(l(\cdot,\hat t-a),Q(\hat t-a))|]da \to 0$ by the continuity of F from $L^1\times\mathfrak{R}\to\mathfrak{R}$.
Now, let $0<\hat t-t<t$ and $0\le t<\hat t<T$.
Then,

$$\int_0^t \Big[\Big|\int_{t-a}^t G(l(\cdot,s),Q(s))(s+a-t)ds - \int_{\hat t-a}^{\hat t} G(l(\cdot,s),Q(s))(s+a-\hat t)ds\Big|\Big]da$$

$$\leq \int_0^{\hat{t}-t} \left[\int_{t-a}^{t} |G(l(\cdot,s),Q(s))(s+a-t)|ds + \int_{\hat{t}-a}^{\hat{t}} |G(l(\cdot,s),Q(s))(s+a-\hat{t})|ds \right] da$$

$$+ \int_{\hat{t}-t}^{t} \left[\int_{t-a}^{\hat{t}-a} |G(l(\cdot,s),Q(s))(s+a-t)|ds + \int_{\hat{t}-a}^{t} |G(l(\cdot,s),Q(s))(s+a-t) \right.$$

$$\left. - G(l(\cdot,s),Q(s))(s+a-\hat{t})|ds \; + \int_{t}^{\hat{t}} |G(l(\cdot,s),Q(s))(s+a-\hat{t})|ds \right] da$$

$$\leq \int_{\hat{t}-t}^{t} \left[\int_0^{\hat{t}-t} |G(l(\cdot,s),Q(s))(b)|db \right] ds + \int_{\hat{t}-t}^{t} \left[\int_{\hat{t}-t}^{s} |G(l(\cdot,s),Q(s))(s+a-t) \right.$$

$$\left. - G(l(\cdot,s),Q(s))(s+a-\hat{t})|da \right] ds + \int_0^{\hat{t}-t} \|G(l(\cdot,s),Q(s))\|_{L^1\times\mathfrak{R}} ds$$

$$+ \int_t^{\hat{t}} \|G(l(\cdot,s),Q(s))\|_{L^1\times\mathfrak{R}} ds \quad \text{(by changing the order of integration)}$$

$$\stackrel{\text{def}}{=} I_1 + I_2 + I_3 + I_4\,.$$

Now, $I_3, I_4 \to 0$ as $|t-\hat{t}| \to 0$ by the continuity of the function G from $L^1 \times \mathfrak{R} \to L^1$.
$I_2 \to 0$ as $|t-\hat{t}| \to 0$ by uniform continuity of translation on the compact set.
$I_1 \to 0$ as $|t-\hat{t}| \to 0$ since, $\int_0^{t-\hat{t}} |G(l(\cdot,s),Q(s))(b)|db \to 0$ when $|t-\hat{t}| \to 0$.
Thus $J_1 \to 0$ as $|t-\hat{t}| \to 0$ for $0 < \hat{t}-t < t$ and when $t = 0$.
Consider J_2.

$$J_2 = \int_t^{\hat{t}} \left[\left| \phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds \right.\right.$$

$$\left.\left. - F(l(\cdot,\hat{t}-a),Q(\hat{t}-a)) - \int_{\hat{t}-a}^{\hat{t}} G(l(\cdot,s),Q(s))(s+a-\hat{t})ds \right| \right] da$$

$$\leq \int_0^{\hat{t}-t} |\phi(a)|da + \int_0^{\hat{t}-t} \left[\int_0^t |G(l(\cdot,s),Q(s))(s+c)|ds \right] dc$$

$$+ \int_0^{\hat{t}-t} |F(l(\cdot,s),Q(s))|ds + \int_{t-\hat{t}}^{0} \left[\int_{-c}^{\hat{t}} |G(l(\cdot,s),Q(s))(s+c)|ds \right] dc\,.$$

Since, $\int_0^t |G(l(\cdot,s),Q(s))(s+c)|ds$, $\int_0^{\hat{t}} |G(l(\cdot,s),Q(s))(s+c)|ds$, $\int_0^{\hat{t}-t} |F(l(\cdot,s),Q(s))|ds$ are integrable, $\Rightarrow J_2 \to 0 as |t-\hat{t}| \to 0$.
Consider J_3.

$$J_3 = \int_{\hat{t}}^{\infty} \left[\left| \phi(a-t) + \int_0^t G(l(\cdot,s),Q(s))(s+a-t)ds - \phi(a-\hat{t}) \right.\right.$$

$$\left.\left. - \int_0^{\hat{t}} G(l(\cdot,s),Q(s))(s+a-\hat{t})ds \right| \right] da$$

$$\leq \int_{\hat{t}}^{\infty} |\phi(a-t)-\phi(a-\hat{t})|da + \int_{\hat{t}}^{\infty} \left[\int_0^t |G(l(\cdot,s),Q(s))(s+a-t) \right.$$
$$\left. - G(l(\cdot,s),Q(s))(s+a-\hat{t})|ds \right] da + \int_{\hat{t}}^{\infty} \left[\int_t^{\hat{t}} |G(l(\cdot,s),Q(s))(s+a-\hat{t})|ds \right] da$$
$$\leq \int_0^{\infty} |\phi(a+\hat{t}-t)-\phi(a)|da + \int_0^t \left[\int_{\hat{t}}^{\infty} |G(l(\cdot,s),Q(s))(s+a-t) \right.$$
$$\left. - G(l(\cdot,s),Q(s))(s+a-\hat{t})|da \right] ds + \int_t^{\hat{t}} \left[\int_{\hat{t}}^{\infty} |G(l(\cdot,s),Q(s))(s+a-\hat{t})|da \right] ds$$

Therefore, $J_3 \to 0$ as $|t-\hat{t}| \to 0$ by the continuity of G from $L^1 \times \mathfrak{R} \to L^1$.
Consider J_4.

$$J_4 = 2\left[\left| \int_0^{t-\hat{t}} g(l(a,s),Q(s))ds \right| \right].$$

Now $J_4 \to 0$, as $|t-\hat{t}| \to 0$.
Thus, $||K(l(\cdot,t),Q(t)) - K(l(\cdot,\hat{t}),Q(\hat{t}))||_{L^1\times\mathfrak{R}} \to 0$.
Therefore, for $(l,Q) \in M$, the function $t \to K(l(\cdot,t),Q(t))$ is continuous from $[0,T]$ to $L^1 \times \mathfrak{R}$.
Therefore, K maps M into M.
To prove: (iii) K is a strict contraction in M.

Proof.

$$||K(l_1,Q_1) - K(l_2,Q_2)||_{L^1\times\mathfrak{R}}$$
$$\leq \int_0^t |F(l_1(\cdot,t-a),Q_1(t-a)) - F(l_2(\cdot,t-a),Q_2(t-a))|da$$
$$+ \int_0^t \left[\int_{t-a}^t |G(l_1(\cdot,s),Q_1(s))(s+a-t) - G(l_2(\cdot,s),Q_2(s))(s+a-t)|ds \right] da$$
$$+ \int_t^{\infty} \left[\int_0^t |G(l_1(\cdot,s),Q_1(s))(s+a-t) - G(l_2(\cdot,s),Q_2(s))(s+a-t)|ds \right] da$$
$$+ 2\left[\int_0^t |g(l_1(a,s),Q_1(s)) - g(l_2(a,s),Q_2(s))|ds \right]$$
$$\leq \int_0^t |F(l_1(\cdot,s),Q_1(s)) - F(l_2(\cdot,s),Q_2(s))|ds + \int_0^t [||G(l_1(\cdot,s),Q_1(s))$$
$$- G(l_2(\cdot,s),Q_2(s))||_{L^1}]ds + 2\left[\int_0^t |g(l_1(a,s),Q_1(s)) - g(l_2(a,s),Q_2(s))|ds \right]$$

by changing the order of integration.

Using, Lipschitz continuity of F, G, g, we have the above equation to be

$$\begin{aligned}
&\leq c_1(2r)\int_0^t \|l_1(\cdot,s),Q_1(s)) - (l_2(\cdot,s),Q_2(s))\|_{L^1\times\mathfrak{R}}ds \\
&\quad - c_2(2r)\int_0^t \|l_1(\cdot,s),Q_1(s)) - (l_2(\cdot,s),Q_2(s))\|_{L^1\times\mathfrak{R}}ds \\
&\quad + 2c_3(2r)\int_0^t \|l_1(\cdot,s),Q_1(s)) - (l_2(\cdot,s),Q_2(s))\|_{L^1\times\mathfrak{R}}ds \\
&= (c_1(2r)+c_2(2r)+2c_3(2r))\int_0^t \|l_1(\cdot,s) - (l_2(\cdot,s)\|_{L^1}ds \\
&\quad + (c_1(2r)+c_2(2r)+2c_3(2r)\int_0^t |Q_1(s)-Q_2(s)|ds \\
&= t(c_1(2r)+c_2(2r)+2c_3(2r))[\|l_1(\cdot,s)-(l_2(\cdot,s)\|_{L^1}+|Q_1(s)-Q_2(s)|] \\
&\leq \tfrac{1}{2}[\|l_1(\cdot,s)-(l_2(\cdot,s)\|_{L^1}+|Q_1(s)-Q_2(s)|]
\end{aligned}$$

where $0 \leq t \leq T$ and $T > 0$ such that

$$\frac{\begin{pmatrix} T[c_1(2r)+c_2(2r)+2c_3(2r)+|F(0,0)|+\|G(0,0)\|_{L^1} \\ +2|g(0,0)|]+(c_1(2r)+c_2(2r)+2c_3(2r))K+2|Q_0| \end{pmatrix}}{2r} \leq 1/2.$$

Therefore,

$$\begin{aligned}
&\|K(l_1(\cdot,t),Q_1(t)) - K(l_2(\cdot,t),Q_2(t))\|_{L_T} \\
&\quad = \sup_{0\leq t\leq T}\tfrac{1}{2}[\|l_1(\cdot,s)-(l_2(\cdot,s)\|_{L^1}+|Q_1(s)-Q_2(s)|] \\
&\quad \leq \tfrac{1}{2}\|(l_1,Q_1)-(l_2Q_2)\|_{L_T}.
\end{aligned}$$

Hence, K is a strict contraction in M and by the contraction mapping theorem [2], there is a unique fixed point $(l,Q) \in M$such that $K(l,Q) = (l,Q)$. This unique fixed point (l,Q) of K in M is the unique solution to the integral equation (11)-(13). Hence, proved that a unique solution to the integral equation (11)-(13) exists. □

The following proposition shows that the solutions of the general model depend continuously on the initial age distributions.

Proposition 2.3. *Let equations* (14)-(16) *hold,let* $\phi, \hat{\phi} \in L^1, Q_0, \hat{Q}_0 \in \mathfrak{R}$, *let* $T > 0$ *and let* $(l,Q),(\hat{l},\hat{Q}) \in L_T$ *such that* $(l,Q),(\hat{l},\hat{Q})$ *is the solution of the general model on* $[0,T]$ *for* (ϕ,Q_0), $(\hat{\phi},\hat{Q}_0)$ *respectively.*
Let $r > 0$ *such that* $\|(l,Q)\|_{L_T}, \|(\hat{l},\hat{Q})\|_{L_T} \leq r$. *Then,*

$$\begin{aligned}
&\|(l(\cdot,t),Q(t)) - (\hat{l}(\cdot,t),\hat{Q}(t))\|_{L^1\times\mathfrak{R}} \\
&\quad \leq \exp[(c_1(r)+c_2(r))t][\|\phi-\hat{\phi}\|_{L^1}+|Q_0-\hat{Q}_0|] \quad \textit{for } 0\leq t\leq T.
\end{aligned} \tag{17}$$

Proof. Let $0 \le t \le T$ and define

$$V(t) = \int_0^\infty |l(a,t) - \hat{l}(a,t)|da + |Q(t) - \hat{Q}(t)|$$
$$= \int_{-t}^\infty |l(t+c,t) - \hat{l}(t+c,t)|dc + |Q(t) - \hat{Q}(t)|\,.$$

For that, it is enough if we prove that

$$\limsup_{h\to 0^+} h^{-1}[V(t+h) - V(t)] \le (c_1(r) + c_2(r))V(t) \ \text{ for } \ 0 \le t \le T, \quad (18)$$

since equation (17) follows from (18) [9].
For $0 < h < T - t$, we have

$$h^{-1}[V(t+h) - V(t)]$$
$$= h^{-1} \int_{-t-h}^{-t} |l(t+h+c, t+h) - \hat{l}(t+h+c, t+h)|dc$$
$$+ h^{-1} \int_{-t}^{\infty} [|l(t+h+c, t+h) - \hat{l}(t+h+c, t+h)|$$
$$- |l(t+c,t) - \hat{l}(t+c,t)|]dc + h^{-1}|Q(t+h) - \hat{Q}(t+h)| - h^{-1}|Q(t) - \hat{Q}(t)|$$
$$\le h^{-1} \int_0^h [|l(a, t+h) - F(l(\cdot,t), Q(t))| + |F(l(\cdot,t), Q(t)) - F(\hat{l}(\cdot,t), \hat{Q}(t))|$$
$$+ |F(\hat{l}(\cdot,t), \hat{Q}(t)) - \hat{l}(a, t+h)|]da + \int_0^\infty [|h^{-1}[l(a+h, t+h) - l(a,t)]$$
$$- G(l(\cdot,t), Q(t))(a)| + |G(l(\cdot,t), Q(t))(a) - G(\hat{l}(\cdot,t), \hat{Q}(t))(a)|$$
$$+ |h^{-1}[\hat{l}(a+h, t+h) - \hat{l}(a,t)] - G(\hat{l}(\cdot,t), \hat{Q}(t))(a)|]da$$
$$+ h^{-1}|Q(t+h) - \hat{Q}(t+h)| - h^{-1}|Q(t) - \hat{Q}(t)|\,.$$

Applying $\limsup_{h\to 0^+}$ on both sides of the above inequality and using equations (6) and (7) (i.e.) the balance law and the birth law of the general model, we get

$$\limsup_{h\to 0^+} h^{-1}[V(t+h) - V(t)]$$
$$\le \limsup_{h\to 0^+} h^{-1} \left[\int_0^h |F(l(\cdot,t), Q(t)) - F(\hat{l}(\cdot,t), \hat{Q}(t))|da \right.$$
$$\left. + \int_0^\infty |G(l(\cdot,t), Q(t))(a) - G(\hat{l}(\cdot,t), \hat{Q}(t))(a)|da \right]$$
$$= |F(l(\cdot,t), Q(t)) - F(\hat{l}(\cdot,t), \hat{Q}(t))| + \|G(l(\cdot,t), Q(t)) - G(\hat{l}(\cdot,t), \hat{Q}(t))\|_{L^1}\,.$$

Using Lipschitz continuity of F and G, the above becomes

$$\begin{aligned}
&\leq c_1(r)\|(l(\cdot,t),Q(t))-(\hat{l}(\cdot,t),\hat{Q}(t))\|_{L^1\times\Re} \\
&\quad + c_2(r)\|(l(\cdot,t),Q(t))-(\hat{l}(\cdot,t),\hat{Q}(t))\|_{L^1\times\Re} \\
&=(c_1(r)+c_2(r))\|(l(\cdot,t),Q(t))-(\hat{l}(\cdot,t),\hat{Q}(t))\|_{L^1\times\Re} \\
&=(c_1(r)+c_2(r))V(t).
\end{aligned}$$

Hence, we have proved equation (18) which implies that the solutions of the general model depend continuously on the initial age distributions.
The following proposition proves that the solutions of the integral equation (11)-(13) possess the semigroup property. □

Proposition 2.4. *Let F, G and g be Lipschitz continuous functions so that equations (14)-(16) hold. Let $\phi \in L^1$, $Q_0 \in \Re$, let $T>0$ and let $(l,Q)\in L_T$ such that (l,Q) is a solution of the integral equation (11)-(13) on $[0,T]$. Let $\hat{T}>0$,and let $(l,Q)\in L_{\hat{T}}$ such that for $t\in[0,\hat{T}]$*

$$\hat{l}(a,t)=\begin{cases} F(\hat{l}(\cdot,t-a),\hat{Q}(t-a))+\displaystyle\int_0^a G(\hat{l}(\cdot,s+t-a),\hat{Q}(s+t-a))(s)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a.e\ \ a\in(0,t) \\ l(a-t,T)+\displaystyle\int_{a-t}^a G(\hat{l}(\cdot,s+t-a),\hat{Q}(s+t-a))(s)ds \ \ a.e\ \ a\in(t,\infty) \end{cases}$$

$$\hat{Q}(t)=Q_0+\int_0^{t+T} g(\hat{l}(a,s),\hat{Q}(s))ds.$$

Define $l(\cdot,t)=\hat{l}(\cdot,t-T)$ and $Q(t)=\hat{Q}(t-T)$ for $t\in(T,T+\hat{T}]$.
Then, $(l,Q)\in L_{T+\hat{T}}$ and (l,Q) is a solution of the integral equation (11)-(13) *on* $[0,T+\hat{T}]$.

Proof. We are given that $(l,Q)\in L_T$ and (l,Q) is a solution of the integral equation (11)-(13) on $[0,T]$. Now, we have to prove $(l,Q)\in L_{T+\hat{T}}$ and (l,Q) is a solution of the integral equation (11)-(13) on $[0,T+\hat{T}]$. For that ,it is sufficient if we prove (l,Q) is a solution of the integral equation (11)-(13) on $(T,T+\hat{T}]$.
Let $t\in(T,T+\hat{T}]$. For almost all $a\in(0,t-T)$

$$\begin{aligned}
l(a,t)&=\hat{l}(a,t-T) \\
&=F(\hat{l}(\cdot,t-T-a),\hat{Q}(t-T-a)) \\
&\quad+\int_0^a G(\hat{l}(\cdot,s+t-T-a),\hat{Q}(s+t-T-a))(s)ds \\
&=F(l(\cdot,t-a),Q(t-a))+\int_0^a G(l(\cdot,s+t-a),Q(s+t-a))(s)ds
\end{aligned} \tag{19}$$

by replacing $(\hat{l},\hat{Q})$ by (l,Q).
For almost all $a \in (t-T,t)$

$$\begin{aligned} l(a,t) &= \hat{l}(a,t-T) \\ &= l(a-t+T,T) + \int_{a-t+T}^{a} G(\hat{l}(\cdot,s+t-T-a),\hat{Q}(s+t-T-a))(s)ds \\ &= F(l(\cdot,t-a),Q(t-a)) + \int_{0}^{a-t+T} G(l(\cdot,s+t-a),Q(s+t-a))(s)ds \\ &\quad + \int_{a-t+T}^{a} G(l(\cdot,s+t-a),Q(s+t-a))(s)ds \\ &= F(l(\cdot,t-a),Q(t-a)) + \int_{0}^{a} G(l(\cdot,s+t-a),Q(s+t-a))(s)ds \end{aligned} \tag{20}$$

by substituting for $l(a-t+T,T)$ using equation (11) of the integral equation.
For almost all $a \in (t,\infty)$

$$\begin{aligned} l(a,t) &= \hat{l}(a,t-T) \\ &= l(a-t+T,T) + \int_{a-t+T}^{a} G(\hat{l}(\cdot,s+t-T-a),\hat{Q}(s+t-T-a))(s)ds \\ &= \phi(a-t) + \int_{a-t}^{a-t+T} G(l(\cdot,s+t-a),Q(s+t-a))(s)ds \\ &\quad + \int_{a-t+T}^{a} G(\hat{l}(\cdot,s+t-T-a),\hat{Q}(s+t-T-a))(s)ds \\ &= \phi(a-t) + \int_{a-t}^{a} G(l(\cdot,s+t-a),Q(s+t-a))(s)ds \end{aligned} \tag{21}$$

by substituting for $l(a-t+T,T)$ using integral equation (12).
Hence using equations (19)-(21), we conclude that $(l,Q) \in L_{T+\hat{T}}$ and (l,Q) is a solution of the integral equation (11) and (12) on $[0,T+\hat{T}]$.
Next, we prove that (l,Q) is a solution of the integral equation (13) on $[0,T+\hat{T}]$.
Replace t by $t-T$ in the equation,

$$\hat{Q}(t) = Q_0 + \int_{0}^{t+T} g(\hat{l}(a,s),\hat{Q}(s))ds$$

and then replace $\hat{l}$ by l and $\hat{Q}$ by Q.
Then,

$$Q(t) = \hat{Q}(t-T) = Q_0 + \int_{0}^{t} g(l(a,s),Q(s))ds\,.$$

Thus, (l,Q) is a solution of the integral equation (13) on $[0,T+\hat{T}]$.

Hence, we have proved that $(l, Q) \in L_{T+\hat{T}}$ and (l, Q) is a solution of the integral equation (11)-(13) on $[0, T+\hat{T}]$. □

Thus from Propositions 2.1, 2.2, 3.1, 3.2, we conclude that our general model (1)-(5) has a unique solution and this solution depends continuously on the initial age distributions and has the semigroup property.

3. Conclusion

In this section, we show that the age dependent epidemic model is a particular case of the general nonlinear model and conclude the existence, uniqueness, continuous dependence of the solution on the initial age distributions and the semigroup property of the solution of the age dependent epidemic model.
Comparing the age dependent epidemic model in section 2 with the general model (1)-(5), we get

$$F(\Phi, x) = x \int_0^\infty k(a)\Phi(a)da\,,$$

$$G(\Phi, x) = -\mu(a)\Phi\,,$$

$$g(\Phi, x) = A - \mu_o x - x \int_0^\infty k(a)\Phi(a)da$$

where F, G, g are defined from

$$F : L^1 \times \mathfrak{R} \to \mathfrak{R}$$

$$G : L^1 \times \mathfrak{R} \to L^1$$

$$g : L^1 \times \mathfrak{R} \to \mathfrak{R}$$

Our aim now is to prove the wellposedness and the semigroup property of this model using the results from sections 3. For that, it is enough if we prove the Lipschitz continuity of the functions F, G and g.
We assume that for $\Phi \in L^1$ and $x \in \mathfrak{R}, \|(\Phi, x)\|_{L^1 \times \mathfrak{R}} \le r$.

To prove: F is Lipschitz continuous, we need to prove equation (14).

Proof.

$$|F(\Phi_1, x_1) - F(\Phi_2, x_2)| = \left| x_1 \int_0^\infty k(a)\Phi_1(a)da - x_2 \int_0^\infty k(a)\Phi_2(a)da \right|.$$

Adding and subtracting $x_2 \int_0^\infty k(a)\Phi_1(a)da$ and using triangle inequality, the above becomes

$$\le |x_1 - x_2| \int_0^\infty |k(a)|\,|\Phi_1(a)|da + |x_2| \int_0^\infty |k(a)|\,|\Phi_1(a) - \Phi_2(a)|da\,.$$

Assuming that $k(a)$ is bounded by a positive constant M_1, the above becomes

$$\le M_1 |x_1 - x_2| \|\Phi_1\| + |x_2| \|\Phi_1 - \Phi_2\|_{L^1}\,.$$

Since, $\|(\Phi, x)\|_{L^1\times\Re} \le r$, we get

$$\le rM_1|x_1 - x_2| + r\|\Phi_1 - \Phi_2\|_{L^1}$$

$$\le c_1(r)[|x_1 - x_2| + \|\Phi_1 - \Phi_2\|_{L^1}]$$

where $c_1(r) = rM_1$.
Therefore, we have proved that F is Lipschitz continuous.

To prove: G is Lipschitz continuous, we need to prove equation (15).

Proof.

$$\|G(\Phi_1, x_1) - G(\Phi_2, x_2)\|_{L^1} = \int_0^\infty |G(\Phi_1, x_1) - G(\Phi_2, x_2)|da$$
$$= \int_0^\infty |\mu(a)||\Phi_1 - \Phi_2|da$$

Assuming that the mortality rate $\mu(a)$ is bounded by a positive constant M_2, the above becomes

$$\le M_2 \int_0^\infty |\Phi_1 - \Phi_2|da$$

$$\le M_2|x_1| \int_0^\infty |\Phi_1 - \Phi_2|da + M_2|x_1||x_1 - x_2|.$$

Since, $\|(\Phi, x)\|_{L^1\times\Re} \le r$, we have the above to be

$$\le M_2 r[\|\Phi_1 - \Phi_2\|_{L^1} + |x_1 - x_2|]$$

where $c_2(r) = rM_2$.
Hence, G is Lipschitz continuous.

To prove: g is Lipschitz continuous, we have to prove equation (16).

Proof.

$$|g(\Phi_1, x_1) - g(\Phi_2, x_2)|$$
$$= \left|A - \mu_o x_1 - x_1 \int_0^\infty k(a)\Phi_1(a)da - A + \mu_o x_2 + x_2 \int_0^\infty k(a)\Phi_2(a)da\right|$$
$$= \left| - \mu_o(x_1 - x_2) - x_1 \int_0^\infty k(a)\Phi_1(a)da + x_2 \int_0^\infty k(a)\Phi_2(a)da\right|.$$

Adding and subtracting $x_2 \int_0^\infty k(a)\Phi_1(a)da$ and using triangle inequality, the above becomes

$$\le \left[|\mu_o| + \int_0^\infty |k(a)||\Phi_1(a)|da\right]|x_1 - x_2| + |x_2| \int_0^\infty |k(a)||\Phi_1(a) - \Phi_2(a)|da.$$

Since $k(a)$ is bounded, the above becomes,

$$\le (|\mu_o| + M_1\|\Phi_1\|_{L^1})|x_1 - x_2| + |x_2|M_1\|\Phi_1 - \Phi_2\|_{L^1} + |\mu_o|\|\Phi_1 - \Phi_2\|_{L^1}.$$

Since, $\|(\Phi, x)\|_{L^1 \times \mathfrak{R}} \leq r$, we have the above to be

$$\leq (|\mu_o| + M_1 r)(\|\Phi_1 - \Phi_2\|_{L^1} + |x_1 - x_2|)$$

where $c_3(r) = |\mu_o| + M_1 r$.

Hence, proved that g is Lipschitz continuous.

Since, F, G, g are Lipschitz continuous, it follows from the propositions that the age dependent epidemic model has a unique solution and this solution depends continuously on the initial age distributions and has the semigroup property. □

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