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Research Article

Secure Support Strong Domination in Graphs

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Abstract. Let $G = (V,E)$ be a simple finite undirected graph. Let D be a subset of $V(G)$. D is called a secure support strong dominating set of G (also called very excellent support strong dominating set), if D is a support strong dominating set of G and for any u in $V - D$, there exists a, $v \in D$ such that $uv \in E(G)$ and $supp(u) \geq supp(v)$ and $(D - \{v\}) \cup \{u\}$ is support dominating set. The minimum cardinality of a secure support strong dominating set of G is called the secure support strong domination number of G and is denoted by $\gamma_{sec}^{ss}(G)$. In this paper, properties of the new parameters are derived and its relationships with other parameters are studied.

Keywords. Support; Strong Dominating set; Secure support strong dominating set; Dominator coloring; Color class domination

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1. Introduction

Let $G = (V,E)$ be a simple graph. Secure domination and support strong domination are combined to defined secure support strong dominating sets. Hedetniemi *et al.* [7] introduced the concept of dominator coloring in graphs and Gera [2] developed this concept. A partition of $V(G)$ in to sets such that each vertex of G dominates at least one member of the partition is called a dominator partition of G. If we stipulate that each member of the partition is an independent subset, then we get dominator color partition of G. E. Sampathkumar [1] defined color class domination in graphs as follows: Let ϕ be a proper color partition of $V(G)$. This partition is said to be color class domination partition if every member of ϕ is dominated by a vertex of G. Let $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$. Then $\phi = \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$ is a proper color class domination partition of G, since each element of the partition is a singleton which is dominated by itself. The minimum cardinality of a color class domination partition of G is denoted by $\chi_{cd}(G)$. In this paper secure support strong color class domination is defined and interesting results are derived. Further secure support strong color class domination complete graph is defined and studied.

2. Secure Support Strong Dominating Set

Definition 2.1. Let $G = (V,E)$ be a simple graph. The support of a vertex in a graph is defined as the sum of the degrees of its neighbours, i.e., $\operatorname{supp}(u) = \sum_{v \in N(u)} \deg(v)$.

Definition 2.2. A subset D of V is a dominating set of G if every vertex $v \in V - D$ is dominated by some $u \in D$. Further, a subset D of V is said to be a strong dominating set of G if every vertex $v \in V - D$, there exists a vertex $u \in D$ such that u and v are adjacent and $\deg(u) \geq \deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of G.

Definition 2.3. Let $u, v \in V(G)$. u is said to support strong dominate v if $uv \in E(G)$ and $\operatorname{supp}(u) \geq \operatorname{supp}(v)$. A subset D of $V(G)$ is called a support strong dominating set if for every $v \in V - D$ there exists $u \in D$ such that u support strong dominates v.

Definition 2.4 ([8])**.** Let $G = (V,E)$ be a simple graph. A secure dominating set X of a graph G is a dominating set with the property that each vertex $u \in V - X$ is adjacent to a vertex $v \in X$ such that $(X - \{v\} \cup \{u\})$ is dominating. The minimum cardinality of such a set is called the secure domination number of G and is denoted by $\gamma_s(G)$.

Definition 2.5. Let $G = (V,E)$ be a simple finite undirected graph. Let D be a subset of $V(G)$. D is called a secure support strong dominating set of G (also called very excellent support strong dominating set), if D is a support strong dominating set of G and for any u in $V - D$, there exists a, $v \in D$ such that $uv \in E(G)$ and $\operatorname{supp}(v) \leq \operatorname{supp}(u)$ and $(D - \{v\}) \cup \{u\}$ is support strong dominating set.

Remark 2.6. (i) $V(G)$ is a secure support strong domination of G.

(ii) secure support strong dominating set is super hereditary.

Definition 2.7. The minimum(maximum) cardinality of a secure support strong dominating set of G is called the secure support strong domination number of G (upper secure support strong domination number of G) and is denoted by $\gamma^{ss}_{sec}(G)(\Gamma^{ss}_{serc}(G))$.

$\gamma^{ss}_{sec}(G)$ for Standard Graphs

(1) $\gamma^{ss}_{sec}(K_n) = 1$,

(2) $\gamma^{ss}_{sec}(\overline{K}_n) = n$,

(3) $\gamma^{ss}_{sec}(K_{1,n}) = n$ (the set of all pendent vertices of $K_{1,n}$ is a minimum secure support strong domination set of $K_{1,n}$),

(4) $\gamma^{ss}_{sec}(P_n) = \left\lceil \frac{n}{2} \right\rceil$,

(5) $\gamma^{ss}_{sec}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$,

(6) $\gamma^{ss}_{sec}(W_n) = \begin{cases} 2, & \text{if } n \geq 5 \\ 1, & \text{when } n = 4, \end{cases}$

(7) $\gamma^{ss}_{sec}(D_{r,s}) = n$,

(8) $\gamma^{ss}_{sec}(P) = 5$, where P is the Petersen graph.

Theorem 2.8. *Let G be a simple graph. $\gamma^{ss}_{sec}(G) = 1$ if and only if G is complete.*

Proof. If G is complete, then $\gamma^{ss}_{sec}(G) = 1$. Suppose $\gamma^{ss}_{sec}(G) = 1$. Let $D = \{u\}$ be a minimum secure support strong dominating set of G. Let v be any vertex of G. Then $\{v\}$ is a secure support strong dominating set of G. $\{v\}$ is adjacent with every vertex of G. Thus, every two vertices of G are adjacent. Therefore, G is complete. □

Corollary 2.9. *If $\gamma^{ss}_{sec}(G) = 1$ then G has a full degree vertex.*

Remark 2.10. The converse of the above Corollary need not true. For example, in W_n $(n \geq 5)$ has a full degree vertex with support $3n$. But $\gamma^{ss}_{sec}(G) \neq 1$.

Remark 2.11. A secure support strong dominating set in a graph is minimal if and only if it is 1-minimal.

Definition 2.12. Let D be a subset of V. Let $u \in D$. The support strong private neighbour set of u denoted by $pn_{ss}[u;D]$ is defined as $pn_{ss}[u;D] = N_{ss}[u] - N_{ss}[D-u]$, where $N_{ss}[x] = \{y \in V :$ y is adjacent with x, $\text{supp}(x) \geq \text{supp}(y)\} \cup x$

Theorem 2.13. *A secure support strong dominating set S is minimal if and only if for any $u \in S$, the following holds:*

(i) $pn_{ss}[u;S] \neq \phi$.

(ii) *There exists no vertex in $S-\{u\}$ which is support strong adjacent with u (a vertex v in $S-\{u\}$ is support strong adjacent with u, if u and v are adjacent and* $\operatorname{supp}(v) \geq \operatorname{supp}(u)$*).*

(iii) *There exists vertex $v \in V-S$ such that u is the only vertex in S which is adjacent with v and $(S-\{u\}) \cap \{v\}$ is a support strong dominating sets.*

(iv) *Either u is isolate of S or if for any $w \in S, w \neq u$ is adjacent with u, then $S-\{w\}$ is not a support strong dominating set.*

Proof. Suppose S is a secure support strong dominating set of G, which is minimal. Let $u \in S$. Then $S-\{u\}$ is not a secure support strong dominating set, that is $S-\{u\}$ is either not a support strong dominating set or $S-\{u\}$ is support strong dominating set but not a secure support strong dominating set. If $S-\{u\}$ is not a support strong dominating set then $pn_{ss}[u;S]=\phi$, a contradiction.

Suppose $S-\{u\}$ is support strong dominating set but $S-\{u\}$ is not a secure support strong dominating set then there exists a vertex $v \in (V-S) \cap \{u\}$ with $((S-\{u\})-\{w\}) \cap \{v\})$ is not a support strong dominating set for any $w \in S-\{u\}$, w being adjacent with v.

Suppose $v \in V-S, v \neq u$. Since S is a secure support strong dominating set, there exists a vertex $w \in S$ such that $(S-\{w\}) \cap \{v\}$ is a support strong dominating set with w being adjacent with v.

Since $S-\{u\}$ is not secure, $w=u$. That is there exists a vertex $v \in V-S$ such that u is only vertex in S which is adjacent with v and with $(S-\{u\}) \cap \{v\}$ is a support strong dominating set.

Either u is an isolate os S ($pn_{ss}[u;S]$ contains u) or if for any $w \in S, w \neq u$ is adjacent with u, then $(s-\{u\}-\{w\}) \cap \{w\}$ is not a support strong dominating set. That is, $S-\{w\}$ is not a support strong dominating set. Therefore, u satisfies one of the four conditions.

Conversely, suppose u satisfies one of the four conditions. If u satisfies (i), then $S-\{u\}$ is not a dominating set. If u satisfies (ii), then $S-\{u\}$ is not support strong dominating set. If u satisfies (iii), then $S-\{u\}$ is not a secure support strong dominating set. If u satisfies (iv), then either $S-\{u\}$ is not a dominating set or $S-\{u\}$ is not a secure support strong dominating set. Hence in any case $S-\{u\}$ is not a secure support strong dominating set. Therefore, S is a minimal secure support strong dominating set. Hence the theorem. □

Definition 2.14. If S is a subset of $V(G)$ and if for any $u \in S$, u satisfies one of the above four conditions, then S is called a sss-irredundant set of G.

Remark 2.15. sss-irredundance is hereditary.

Definition 2.16. The minimum (maximum) cardinality of a maximal sss-irredundant set of G is called the minimum(maximum) sss-irredundance number of G and is denoted by $ir^{ss}_{sec}(G)$ ($IR^{ss}_{sec}(G)$).

Theorem 2.17. *A minimal secure support strong dominating set of G is a maximal sss-irredundant set of G.*

Remark 2.18. $ir^{ss}_{sec}(G) \le \gamma^{ss}_{sec}(G) \le i^{ss}_{sec}(G) \le \beta^{ss}_{sec}(G) \le \Gamma^{ss}_{sec}(G) \le IR^{ss}_{sec}(G)$.

3. Secure Support Strong Color Class Domination

Definition 3.1. Let $G = (V,E)$ be a simple graph. A partition of $V(G)$ into sets such that each vertex of G dominates at least one member of the partition is called a dominator partition of G.

Definition 3.2. Let $\pi = \{V_1, V_2, \dots, V_k\}$ be a partition of $V(G)$. π is called class domination if every V_i, $1 \le i \le k$, there exists a vertex $u \in V(G)$ such that V_i is dominated by u. If every V_i, $1 \le i \le k$ is independent, we get color class domination. For any graph G on n vertices with $V(G) = \{v_1, v_2, \dots, v_n\}$, the partition $\{v_1\}, \{v_2\}, \dots \{v_n\}$ is a class domination partition of G. The class domination partition number of G denoted by $\pi_{cd}(G)$ is the minimum value of k for which the graph G has a class domination partition of size k.

Definition 3.3. Let $G = (V,E)$ be simple graph. Let $\pi = \{V_1, V_2, \dots, V_k\}$ be a proper color partition of G. π is called a color class domination partition of G if for every i, $1 \le i \le k$, there exists a vertex $u \in V(G)$ such that v_i dominated by u. The minimum cardinality of a color class domination partition of G is denoted by $\chi_{cd}(G)$ and is called color class domination number of G.

Definition 3.4. A dominator coloring of a graph is a proper coloring in which each vertex of the graph dominates every vertex of some color class.

Definition 3.5. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph of G.

Remark 3.6. A dominator color partition of G need not be a proper color class domination partition of G.

Definition 3.7. A proper color class partition of $V(G)$ is called a support strong color class domination partition if every color class is support strong dominated by a vertex of the graph. The trivial color class partition consisting of n color classes is a support strong color class domination partition(ssccd partition). The minimum cardinality of support strong color class domination partition is called the support strong color class domination number and is denoted by $\chi_{ssccd}(G)$.

Remark 3.8. $\chi(G) \le \chi_{cd}(G) \le \chi_{ssccd}(G)$

Theorem 3.9. $\max\{\chi(G), \gamma_{ssd}(G)\} \le \chi_{ssccd}(G)$.

Proof. Clearly, $\chi(G) \le \chi_{cd}(G)$. Let $\pi = \{v_1, v_2, \dots v_{\chi_{cd}}\}$ be a minimum proper color class domination partition of G. Since there exists $\chi_{ssccd}(G)$ vertices to support strong dominate $V(G)$, $\gamma_{ssd}(G) \le \chi_{ssccd}(G)$. Hence $\max\{\chi(G), \gamma_{ssd}(G)\} \le \chi_{ssccd}(G)$. □

Remark 3.10. There is no relation between $\chi_d(G)$ and $\chi_{ssccd}(G)$.

Example 3.11. In $D_{r,s}$, $*\chi_{ssccd}(G) = 2$ and $\chi_d(G) = 3$.

Example 3.12. In C_{20}, $*\chi_{ssccd}(G) = 10$ and $\chi_d(G) = 8$.

Remark 3.13. If a graph G has a full degree vertex, then $\chi(G) = \chi_{cd}(G) = \chi_{ssccd}(G)$.

Proof. Suppose G has a full degree vertex, say u. Then $\pi = \{\{u\}, \{V_1\}, \{V_2\}, \dots \{V_{\chi_{cd}}\}$ be a minimum proper color class domination partition of G and also a minimum support strong color class domination partition. Therefore $\chi(G) = \chi_{cd}(G) = \chi_{ssccd}(G)$. □

Example 3.14. In K_n, $\chi(G) = \chi_{cd}(G) = \chi_{ssccd}(G) = n$.

Remark 3.15. The converse is not true. For example, in $G = C_4$, $\chi(G) = \chi_{cd}(G) = \chi_{ssccd}(G)$, but C_4 has no full degree vertex.

Theorem 3.16. $\left\lceil \dfrac{n}{\Delta_{\text{supp}}} \right\rceil \le \chi_{ssccd}(G)$.

Proof. Suppose G has maximum support degree Δ_{supp}. Let $\pi = \{V_1, V_2, \dots V_{\chi_{ssccd}}\}$ be a minimum support strong color class domination partition of G. Then $|V_i| \le \Delta_{\text{supp}}(G)$, $1 \le i \le \chi_{ssccd}(G)$. Therefore, $n \le \chi_{ssccd}(G). \Delta_{\text{supp}}(G)$. Hence $\left\lceil \dfrac{n}{\Delta_{\text{supp}}} \right\rceil \le \chi_{ssccd}(G)$. □

Observation 3.17. $\chi_{ssccd}(G) = n$ *if and only if* $G = K_n$ *or* $G = tK_2$.

Observation 3.18. $\chi_{ssccd}(G) = 2$ *if and only if* $G = K_{m,n}$.

Observation 3.19. $\chi_{ssccd}(G)$ *need not be less than* $\gamma(G) + \chi(G)$, *e.g.* $G = C_{20}$.

Remark 3.20. Given any positive integer k, there exists a graph G such that $\chi_{ssccd}(G) - \chi(G) = k$.

Proof. When k is an even $\chi_{ssccd}(G) - \chi(G) = k$ where $G = C_{2k+4}$. When k is odd $\chi_{ssccd}(G) - \chi(G) = k$ where $G = C_{2k+5}$. □

Definition 3.21. A support strong color class domination partition $\pi(G)$ is called a secure support strong color class domination partition if each color class is secure. The trivial partition is a secure support strong color class domination partition. Hence, minimum cardinality of a secure support strong color class domination partition is considered. This is denoted by $\chi_{sssccd}(G)$.

$\chi_{sssccd}(G)$ for Standard Graphs

(1) $\chi_{sssccd}(K_n) = n$.

(2) $\chi_{sssccd}(K_{1,n}) = n$.

(3) $\chi_{sssccd}(C_n) = \chi_{sccd}(C_n)$. ($\chi_{sccd}(C_n)$ is the minimum cardinality of a secure color class domination partition).

Definition 3.22. A proper color class partition is said to be secure support strong color class domination complete if $\chi_{sssccd}(G) = n$

The following theorem characterizes secure support strong color class domination complete graphs.

Theorem 3.23. *A graph G is secure support strong color class domination complete if for any two vertices u,v either $d(u,v) \leq 2$ or any two vertices are not support strong dominated by a single vertex.*

Proof. Suppose $\chi_{sssccd}(G) = n$. Therefore for any two distinct vertices $u,v \in V(G)$, u and v cannot belong to the same secure support strong color class domination partition of G. Therefore, either u and v are addjacent or u and v have common neighbour or u and v are not support strong dominated by a single vertex. That is $d(u,v) \leq 2$ or u and v are not support strong dominated by a single vertex.

Conversely, suppose for any two vertices u, v either $d(u,v) \leq 2$ or any two vertices are not support strong dominated by a single vertex. If u and v are not support strong dominated by a single vertex, then u and v can not belong to the same class of a secure support strong color class domination partition of G. If $d(u,v) \leq 2$ then also u and v can not belong to the same class of a secure support strong color class domination partition of G. Therefore $\chi_{sssccd}(G) = n$. Therefore a graph G is secure support strong color class domination complete. □

4. Conclusion

In this paper, two new concepts namely secure support strong dominating sets and secure support strong color class domination are introduced and studied. In a social network, the strength of an individual can be calculated by the number of friends he has. Even if an individual has many friends,they may not have social or financial status. So a new concept called support of an individual is introduced. Suppose each individual is given a weight according to the status,the sum of the weights of the friends of an individual gives a better assessment of the strength of the individuals. This concept in the graph representation of a social network leads to a better assessment of the individual combined with domination and partition,we get a better idea of the social network. The work can be further continued and the idea of the resolving sets,equitability etc., may be combined with the concept of support.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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