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Auto-B cklund Transformation, Lax Pairs and Painlev Property of $u_t + p(t)uu_x + q(t)u_{xxx} + r(t)u = 0$

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Abstract. Using the *Painlev property* (PP) of partial differential equations, the *auto-B cklund transformation* (ABT) and Lax pairs for *Korteweg-de Vries* (KdV) equation with time-dependent coefficients are obtained. The Lax pair criterion also makes it possible for some new models of the variable coefficient KdV equation to be found that can represent nonsoliton dynamical systems. This can explain the wave breaking phenomenon in variable depth shallow water.

1. Introduction

Exciting and important discoveries have been made in nonlinear dynamics of dissipative and conservative systems. Numerical, analytical, and experimental works in the last two decades show that most of the nonlinear systems exhibit a transformation from regular to chaotic behaviour [1]. Recently, [2] the connection between movable singularities and algebraic integrability of dynamical systems is widely studied in different contexts. For an algebraically completely integrable system the independent, single-valued integrals of motions are part of a compact, complex tori on which tha motion is linear.

Ward [3] has extended the study of the Painlev property (PP), well known in the context of *ordinary differential equations* (ODE's) to *partial differential equations* (PDE's). A system of PDE's in n independent variables is considered in the complex domain, the coefficients being analytic on C^n. If S is an analytic noncharacteristic complex hypersurface in C^n, then the PDE that is analytic on S is meromorphic on C^n. A weaker form of the PP was suggested by Weiss *et al.* [4] while studying the Lorentz series expansion of the solutions in the neighborhood of a movable singularity.

It is a well-known conjecture that if a field equation has the PP then it is completely integrable [5]. The limitations of the conjecture, known as

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the *Ablowitz-Ramani-Segur* (ARS) conjecture, have been pointed out by Bountis [2]. The complete integrability is also defined in terms of the existence of the *inverse scattering transform* (IST) or the auto-B cklund transformation (ABT) [6]. The existence of an IST solution is assured by that of Lax pairs.

A well-known [6] model for an IST solvable and completely integrable dynamical system is the celebrated constant coefficient *Korteweg-de Vries* (KdV) equation:

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{1}$$

the coefficients α and β being constants and the suffix indicating a partial derivative with the respective variables. This equation yields a highly collisionally stable particlelike solution, called a soliton.

Here we report the results of the PP analysis of a KdV equation with variable coefficients. The PP is used to identify the values of the different parameters for which the system loses its integrability. We have found these parameter value using a property of Lax pairs obtained from the PP. The possible ABT is also developed, when the system is integrable.

Such an equation is particularly significant the study of the development of a steady solitary wave as it enters a region where the bottom is no longer level [7, 8, 9, 10, 11, 12]. It has been found both theoretically and experimentally that when the depth decreases to form a shelf, the solitary wave breaks into a number of 'solitons' while if the depth is increasing solitary wave degenerates into a cnoidal wave.

In the present paper, we give a detailed account of the Painlev analysis of the variable coefficient KdV equation

$$u_t + p(t) u u_x + q(t) u_{xxx} + r(t) u = 0, \tag{2}$$

where $p(t), q(t)$ and $r(t)$ are the functions of t by the work of Nirmala et al. [13].

The rest of the paper is organised as follows:

Section 2 deals with the Painlev property of (2). The conclusion of the present study is set forth in section 3.

2. **Painlev Property**

Equation (2) has the PP when its solutions $u(x,t)$ are single valued about the movable, singularity manifolds, determined from the singularity analysis of the Lorentz series expansion

$$u(x,t) = \phi^{\alpha}(x,t) \sum_{j=0}^{n} u_j(x,t) \phi^j(x,t), \tag{3}$$

where $\phi(x,t)$ and $u_j(x,t)$ are analytic functions in a neighborhood of the manifold

$$\phi(x,t) = 0 \tag{4}$$

and α is an integer to be determined. Substituting (3) into equation (2), a leading-order terms analysis uniquely determines the possible values of α. The PP requires that α be a negative integer. The resultant series expansion of (2) gives the required *Auto-B cklund transformation* (ABT) and Lax pair for the IST.

The leading-order terms analysis gives the value

$$\alpha = -2. \tag{5}$$

The recursion relations for $u_j(x,t)$ are found to be

$$\begin{aligned} &u_{j-3,t} + (j-4)u_{j-2}\phi_t + p(t)\sum_{m=0}^{j}[u_{j-m}(u_{m-1,x} + (m-2)\phi_x u_m)] \\ &\quad + q(t)[u_{j-3,xxx} + (j-4)(3u_{j-2,xx}\phi_x + 3u_{j-2,x}\phi_{xx} + u_{j-2}\phi_{xxx}) \\ &\qquad + (j-3)(j-4)(3u_{j-1,x}\phi_x^2 + 3u_{j-1}\phi_x\phi_{xx}) \\ &\qquad + (j-2)(j-3)(j-4)\phi_x^3 u_j] + r(t)u_{j-3} = 0, \end{aligned} \tag{6}$$

where

$$\phi_x = \frac{\partial \phi}{\partial x}, \quad u_{j,x} = \frac{\partial u_j(x,t)}{\partial x}, \quad \text{etc.} \tag{7}$$

Collecting terms involving u_j, it is readily found that

$$q(t)\phi_x^3(j+1)(j-4)(j-6)u_j = F(u_{j-1},\cdots,u_0,\phi_t,\phi_x,\phi_{xx},\cdots) \quad \text{for } j = 0,1,2,\cdots \tag{8}$$

We note that the recursion relations (8) are not determined when $j = -1, 4$ and 6. These values of j are called the “resonances” of the recursion relation and, corresponding to these values of j, we can insert arbitrary functions of (x,t) instead of $u_j(x,t)$ into the series expansion (3). But for $j = -1$, the series expansion (3) not is defined and so the admissible values of resonances are $j = 4$ and 6 only.

Putting $j = 0, 1, 2, \cdots$ in (6), we get

$$j = 0, \ u_0 = -\frac{12q(t)\phi_x^2}{p(t)}, \tag{9}$$

$$j = 1, \ u_1 = \frac{12q(t)\phi_{xx}}{p(t)}, \tag{10}$$

$$j = 2, \ \frac{1}{p(t)}\phi_x\phi_t + u_2\phi_x^2 - \frac{3q(t)}{p(t)}\phi_{xx}^2 + \frac{4q(t)}{p(t)}\phi_x\phi_{xxx} = 0, \tag{11}$$

$$j = 3, \ \frac{q_t}{pq}\phi_x - \frac{p_t}{p^2}\phi_x + \frac{1}{p(t)}\phi_{xt} + u_2\phi_{xx} - u_3\phi_x^2 + \frac{q}{p}\phi_{xxxx} + \frac{r}{p}\phi_x = 0, \tag{12}$$

$$j = 4, \ \frac{\partial}{\partial x}\left(\frac{q_t}{pq}\phi_x - \frac{p_t}{p^2}\phi_x + \frac{1}{p(t)}\phi_{xt} + u_2\phi_{xx} - u_3\phi_x^2 + \frac{q}{p}\phi_{xxxx} + \frac{r}{p}\phi_x\right) = 0, \tag{13}$$

which is a compatibility condition. The compatibility condition at $j = 6$ involves extensive calculations.

When we assign $u_4 = u_6 = 0$ and for $u_3 = 0$, we can find

$$u_j = 0, \quad \text{for all } \; j \geq 3, \tag{14}$$

provided u_2 is a solution of (3), which implies that

$$u_{2,t} + p(t)u_2u_{2,x} + q(t)u_{2,xxx} + r(t)u_2 = 0. \tag{15}$$

From equation (3) and (9)-(15), we get

$$u_0 = -\frac{12q(t)}{p(t)}\phi_x^2, \tag{16}$$

$$u_1 = \frac{12q(t)}{p(t)}\phi_{xx}, \tag{17}$$

$$\frac{1}{p(t)}\phi_x\phi_t + u_2\phi_x^2 - \frac{3q(t)}{p(t)}\phi_{xx}^2 + \frac{4q(t)}{p(t)}\phi_x\phi_{xxx} = 0, \tag{18}$$

$$\frac{q'}{pq}\phi_x - \frac{p'}{p^2}\phi_x + \frac{1}{p(t)}\phi_{xt} + u_2\phi_{xx} + \frac{q}{p}\phi_{xxxx} + \frac{r}{p}\phi_x = 0, \tag{19}$$

$$u_{2,t} + p(t)u_2u_{2,x} + q(t)u_{2,xxx} + r(t)u_2 = 0, \tag{20}$$

and

$$u_j = 0, \quad \text{for } \; j \geq 3. \tag{21}$$

Substituting equations (16)-(21) in equation (3), we have

$$u(x,t) = -\frac{12q(t)\phi_x^2}{p(t)\phi^2} + \frac{12q(t)\phi_{xx}}{p(t)\phi} + u_2, \tag{22}$$

or

$$u(x,t) = \frac{12q(t)}{p(t)}\frac{\partial^2}{\partial x^2}(\log\phi) + u_2, \tag{23}$$

where $u(x,t)$ and u_2 are exact solutions of (3) and (15), respectively.

Equations (16)-(23) define the auto-B cklund transformation for the equation (3) provided (18) and (19) are consistent. If any one of the solutions $u_2(x,t)$ is known then another solution $u(x,t)$ of (3) can be determined using the auto-B cklund transformation. The consistency of equations (18) and (19) can be verified by using a property of the Lax pairs.

The Lax pairs are obtained from the equations (18) and (19) by using a transformation

$$\phi_x = V^2. \tag{24}$$

Substituting (24) into equation (19) yields

$$\frac{q'}{2pq}V - \frac{p'}{p^2}\frac{V}{2} + \frac{1}{p}V_t + u_2V_x + \frac{q}{p}V_{xxx} + \frac{3q}{p}V_x\frac{V_{xx}}{V} = 0. \tag{25}$$

Equation (18) also transforms to

$$\frac{1}{p}V_t + u_2V_x + \frac{1}{2}Vu_{2,x} + \frac{4q}{p}V_{xxx} = 0. \tag{26}$$

Eliminating V_t from equations (25) and (26), we get

$$\left(\frac{q'}{2pq} - \frac{p'}{2p^2}\right) - \frac{1}{2}u_{2,x} - \frac{3q}{p}\left(\frac{V_{xx}}{V}\right)_x = 0. \tag{27}$$

Integrating equation (27) with respect to x gives

$$\frac{q}{p}\frac{V_{xx}}{V} + \frac{1}{6}u_2 - \frac{1}{3}\left(\frac{q'}{2pq} - \frac{p'}{2p^2}\right)x \quad = \quad \lambda(t). \tag{28}$$

or

$$f(t)\left[\frac{q}{p}D^2 + \frac{1}{6}u_2 - \frac{1}{3}\left(\frac{q'}{2pq} - \frac{p'}{2p^2}\right)x\right]V \quad = \quad f(t)\lambda(t)V. \tag{29}$$

Thus we get the linear eigen value problem

$$LV = \mu V, \tag{30}$$

where $\mu = \lambda(t)f(t)$ and L is a linear operator defined by

$$L = f(t)\left[\frac{q}{p}D^2 + \frac{1}{6}u_2 - \frac{1}{3}\left(\frac{q'}{2pq} - \frac{p'}{2p^2}\right)x\right], \tag{31}$$

From equation (26) we get

$$V_t = -p\left[\frac{4q}{p}D^3 + u_2 D + \frac{1}{2}u_{2,x}\right]V, \tag{32}$$

or

$$V_t = -BV, \tag{33}$$

where the operator B is defined by

$$B = p\left[\frac{4q}{p}D^3 + u_2 D + \frac{1}{2}u_{2,x}\right]. \tag{34}$$

Equations (30) and (34) define the Lax pairs L and B. However, equation (33) implies that the eigenfunction V is in time evolution so that

$$L_t = LB - BL. \tag{35}$$

The L_t in equation (35) denotes the derivative with respect to both the explicit time dependence of L and the implicit dependence through $u_2(x,t)$.

3. **Results**

The equation variable coefficient KdV equation (2) that have introduced is a new member in the families of integral as well as nonintegrable PDE's depending on the acidents. The PP analysis leads to the auto-B cklund transformation and Lax pairs it is integrable. The operator identity (35) of Lax reveals that the system (3) can be integrable. The soliton solutions are the products of IST solvable class of nonlinear PDE. The above study shows that the equation (3) does not always have a soliton. Hence in general a solitary wave solution of (3) need not be a soliton and so it need not be collisionly stable always.

The equation (2) that introduced is a model for explaining the observations of soliton-type solution's instability reported earlier in different contexts [8, 9, 10, 11, 12, 13].

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