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Research Article

Characterization of Delta Operator for Poisson-Charlier Polynomials

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Abstract. The aim of the paper is to study the characterization of delta operator associated with some Sheffer polynomials. In this paper, we consider Poisson-Charlier polynomials and investigate the characterization of delta operator via sequential representation of delta operator. From our investigation, we are able to prove an interesting propositions for the above mentioned.

Keywords. Delta operator; Sheffer polynomials; Poisson-Charlier polynomials; Operational methods

MSC. 05A10; 05A40; 11B83; 47A56

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1. Introduction

The Poisson-Charlier polynomials $C_n(x;a)$ (also called Charlier polynomials) are introduced by Carl Charlier in 1906. They form a Sheffer sequences [1] with

$$g(t) = e^{a(e^t-1)} \quad \text{and} \quad f(t) = a\,(e^t - 1).$$

The Sheffer identity is

$$C_n(x+y;a) = \sum_{k=0}^{n} \binom{n}{k} a^{k-n}\, C_k(y;a)\,(x)_{n-k}\,,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is a falling factorial.

As is well known, they can be expressed in terms of the generalized hypergeometric function by

$$C_n(x;a) = {}_2F_0\left(-n, -x, -\frac{1}{a}\right).$$

An alternative expression for $C_n(x;a)$ is given by

$$C_n(x;a) = (-1)^n\ n!\, L_n{}^{(-1-x)}\left(-\frac{1}{a}\right)$$

where L are Laguerre polynomials.

Special polynomials play an important role in Mathematical Physics. Many of the models in Applied Mathematics particularly Physics are expressed in terms of classical special polynomials. Recently, Kim *et al*. [1] studied the linear differential equations for the generating function of the Poisson-Charlier polynomials and its applications. Kim *et al*. [2] derived various identities involving Poisson-Charlier polynomials from Umbral Calculus. Some recurrence relations for Poisson-Charlier polynomials are derived in [3]. The properties of several Sheffer polynomials are discussed by Rainville [4], and Boas and Buck [5].

Operational Methods are used to reduce the differential problems into algebraic problems. In 1975, Rota [6] introduced Finite Operator Calculus which is a systematic approach to delta operators on the algebra of polynomials. It contains a detailed study of delta operators associated with basic polynomial sequences and Sheffer sequences. The main objective of this paper is to investigate the characterization of delta operator for the Poisson-Charlier polynomials.

This paper is organized as follows. We begin with the fundamentals of Finite Operator Calculus in the Section 2. Section 3 is focusing on sequential representation of delta operator. The characterization of delta operator for the Poisson-Charlier polynomials is investigated in the Section 4. Section 5 is devoted to the construction of Delta triangle for Poisson-Charlier and their related polynomials.

2. Finite Operator Calculus

In this section, we list the main definitions and results of Finite Operator Calculus which we shall use in next section. These results were derived by Rota [6]. The proofs of known results are skipped, but they are easily read from the reference Rota [6].

Let F be a Field of characteristic zero, preferably the real number field. By a *polynomial sequence* we shall denote a sequence of polynomials $p_n(x)$, $n = 0, 1, 2, \ldots$, where $p_n(x)$ is exactly of degree n for all $n \in \mathbb{Z}^+ \cup \{0\}$.

The objective of Rota [6] was a unified theory of special polynomials associated with some operators. We start with such operators and their properties.

An operator E^a is said to be a shift operator if

$$E^a : p(x) \to p(x+a),$$

for all polynomials $p(x)$ in one variable and for all real a in the field F.

A linear operator T which commutes with all shift operators E^a is called a shift invariant operator. In symbol,

$$TE^a = E^a T, \quad \text{for all } a \in F.$$

A delta operator Q is a shift invariant operator such that

$$Qx = \text{const} \neq 0.$$

For example, the forward difference operator

$$(\Delta f)(x) = f(x+1) - f(x)$$

is a delta operator.

The following result establishes the fundamental properties of the delta operator Q.

Theorem 2.1. (i) *If Q is a delta operator, then $Qa = 0$ for every constant 'a'.*

(ii) *If $p(x)$ is a polynomial of degree n, then $Qp(x)$ is a polynomial of degree $n-1$.*

The delta operators possess many of the properties of the usual derivative D. The above theorem is a good example.

Definition 2.2. A polynomial sequence $p_n(x)_{n\geq 0}$; $\deg p_n = n$; such that

(i) $p_0(x) = 1$,

(ii) $p_n(0) = 0$, whenever $n > 0$,

(iii) $Qp_n(x) = np_{n-1}(x)$

is called the basic polynomial sequence of the delta operator Q.

A trivial example for basic polynomials sequence is $\{x^n\}$.

Theorem 2.3. *Every delta operator has a unique sequence of basic polynomials.*

Definition 2.4. A polynomial sequence $p_n(x)$ $(n \geq 0)$, where $p_n(x)$ is exactly of degree n for all n, is said to be binomial type if it satisfies the infinite sequence of following identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y), \quad n = 0, 1, 2, \dots.$$

The simplest sequence of binomial type is $\{x^n\}$.

Definition 2.5. A polynomial sequence $s_n(x)$ is called a Sheffer set or a set of Sheffer polynomials for the delta operator Q if

(i) $s_0(x) = c \neq 0$, and

(ii) $Qs_n(x) = ns_{n-1}(x)$.

Sheffer polynomials are a large class of polynomial sequences that include Monomials, Upper Factorials, Lower Factorials, Striling polynomials, Poisson-Charlier polynomials, Bell polynomials, First and second kind of Abel polynomials, Laguerre polynomials, Boole polynomials and many others.

In the next section, we study more about the sequential structure of delta operator.

3. Sequential Representation of Delta Operator

As is well known, the Monomials $\{x^n\}$ is a trivial example for basic set as well as Sheffer set. Using the expressions for $Q(x^2), Q(x^3), \ldots, Q(x^n)$, we formulated $Q(x^n)$ as a sequential representation of delta operator by considering $\{x^n\}$ is a basic set in [7]. In the same context, we formulate $Q(x^n)$ as a sequential structure by considering the monomials $\{x^n\}$ as a Sheffer set in the following theorem.

Theorem 3.1. *For the monomial $\{x^n : n \in \mathbb{Z}^+ \cup \{0\}\}$, and for each α_r an arbitrary real constant,*

$$Q(x^n) = \sum_{r=1}^{n} \binom{n}{r} \alpha_r x^{n-r}. \tag{1}$$

Proof. If $n = 1$, then from the definition of delta operator, $Q(x)$ is a non zero constant.

Let it be α_1. Therefore, $Q(x) = \alpha_1 \neq 0$ and hence the result is true for $n = 1$

Let $n = 2$. By Theorem 2.1(ii), construct $Q(x^2) = c_0 x + c_1$.

Since Q is shift invariant, $E^a Q(x^2) = Q E^a (x^2)$.

$$E^a Q(x^2) = E^a (c_0 x + c_1) = c_0 E^a(x) + c_1 = c_0(x + a) + c_1 = c_0 x + c_0 a + c_1.$$

Since $Q(a) = 0$, $Q(x) = \alpha_1$ and by Table 1, we have

$$Q E^a (x^2) = Q(x + a)^2 = Q(x^2 + a^2 + 2ax) = Q(x^2) + 2aQ(x) = c_0 x + c_1 + 2a\alpha_1.$$

Equating the corresponding terms in $E^a Q(x^2)$ and $Q E^a (x^2)$, we get $c_0 = 2\alpha_1$.

c_1 is a new independent constant which may be taken as α_2.

Hence $Q(x^2) = 2\alpha_1 x + \alpha_2$.

Therefore, the result is true for $n = 2$.

Let us assume that the result is true for all $n = k$.

Therefore,

$$Q(x^k) = \sum_{r=1}^{k} \binom{k}{r} \alpha_r x^{k-r} = \binom{k}{1} \alpha_1 x^{k-1} + \binom{k}{2} \alpha_2 x^{k-2} + \cdots + \binom{k}{r} \alpha_r x^{k-r} + \cdots + \alpha_k. \tag{2}$$

Since $\{x^n\}$ is a Sheffer sequence, it satisfies $Q p_n(x) = n p_{n-1}(x)$ and hence, we have

$$Q(x^k) = k\, x^{k-1}. \tag{3}$$

From (3), we see that the delta operator Q is a usual derivative D.

From (2) and (3),

$$\binom{k}{1} \alpha_1 x^{k-1} + \binom{k}{2} \alpha_2 x^{k-2} + \cdots + \binom{k}{r} \alpha_r x^{k-r} + \cdots + \alpha_k = k x^{k-1}. \tag{4}$$

By comparing the corresponding terms, we have $\alpha_1 = 1$ and $\alpha_j = 0$, $j = 2,3,\dots,k$.

Therefore, the result is true for $n = k$ means that

$$\alpha_1 = 1 \text{ and } \alpha_j = 0 \ (j = 2,3,\dots k). \tag{5}$$

Now, we have to show that this result is true for $n = k+1$

$$\begin{aligned} Q(x^{k+1}) &= Q(x^k x) \\ &= Q(x^k)x + Q(x)\,x^k \qquad \text{(from the Definition 2.5, } Q = D \text{ for } \{x^n\}) \\ &= \left\{\binom{k}{1}\alpha_1 x^{k-1} + \binom{k}{2}\alpha_2 x^{k-2} + \dots + \binom{k}{r}\alpha_r x^{k-r} + \dots + \alpha_k\right\} x + \alpha_1 x^k \\ &= \alpha_1(k\,x^k + x^k) + \alpha_2\binom{k}{2}x^{k-1} + \alpha_3\binom{k}{3}x^{k-2} + \dots + \alpha_k\,x \\ &= (k+1)\,x^k \qquad \text{(by (5))}. \end{aligned}$$

Thus we have

$$Q(x^{k+1}) = (k+1)\,x^k. \tag{6}$$

On other hand, using the property that $Qp_n(x) = n\,p_{n-1}(x)$, we have

$$Q(x^{k+1}) = (k+1)\,p_k(x) = (k+1)\,x^k. \tag{7}$$

From (6) and (7), we conclude that the result is true for all $n = k+1$.

Thus, we proved the Theorem 3.1. □

The following table contains the expressions for $Q(x), Q(x^2), Q(x^3), \dots$.

Table 1. First few polynomials $Q(x^n)$, $n = 1,2,3,\dots$.

$Q(x) = 1\alpha_1$
$Q(x^2) = 2\alpha_1 x + \alpha_2$
$Q(x^3) = 3\alpha_1 x^2 + 3\alpha_2 x + \alpha_3$
$Q(x^4) = 4\alpha_1 x^3 + 6\alpha_2 x^2 + 4\alpha_3 x + \alpha_4$
$Q(x^5) = 5\alpha_1 x^4 + 10\alpha_2 x^3 + 10\alpha_3 x^2 + 5\alpha_4 x + \alpha_5$
$Q(x^6) = 6\alpha_1 x^5 + 15\alpha_2 x^4 + 20\alpha_3 x^3 + 15\alpha_4 x^2 + 6\alpha_5 x + \alpha_6$
$Q(x^7) = 7\alpha_1 x^6 + 21\alpha_2 x^5 + 35\alpha_3 x^4 + 35\alpha_4 x^3 + 21\alpha_5 x^2 + 7\alpha_6 x^5 + \alpha_7$
$Q(x^8) = 8\alpha_1 x^7 + 28\alpha_2 x^6 + 56\alpha_3 x^5 + 70\alpha_4 x^4 + 56\alpha_5 x^3 + 28\alpha_6 x^2 + 8\alpha_7 x + \alpha_8$

Here, $Q(x_n)$ has n independent parameters α_i $(i = 1,2,3,\dots)$. These parameters are unique. Allowing n being large, we get an infinite sequence of real numbers. We note that the values of α_i's determines the characterization of delta operator Q. Theorem 3.1 play vital role to investigate the characterization of delta operator for basic set and Sheffer set.

4. Delta Operator for Poisson-Charlier Polynomials

In [8], the characterization of delta operator for some Sheffer polynomials such as the Euler, Bernoulli of second kind, and Mott polynomials are investigated. In the same context, we investigate the characterization of delta operator for the Poisson-Charlier polynomials is investigated in this section.

As is well known, the generating function for the Poisson-Charlier polynomials $C_n(x;a)$ is given by

$$e^{-t}\left(1+\frac{t}{a}\right)^x = \sum_{n=0}^{\infty} C_n(x;a)\frac{t^n}{n!}, \quad (a \neq 0). \quad \text{(see [1], [9], [10])}$$

The Poisson-Charlier polynomials $C_n(x;a)$ are defined by

$$C_n(x;a) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{x}{k} k! a^{-k},$$

where $a > 0$ and $x \in N_0$.

The Poisson-Charlier polynomials satisfy the following recurrence relation

$$C_{n+1}(x;a) = a^{-1}xC_n(x-1;a) - C_n(x;a).$$

The first few Poisson-Charlier polynomials $C_n(x;a)$ are given by [11]

$$C_0(x;a) = 1,$$

$$C_1(x;a) = -\frac{a-x}{a},$$

$$C_2(x;a) = \frac{a^2 - x - 2ax + x^2}{a^2},$$

$$C_3(x;a) = -\frac{a^3 - 2x - 3ax - 3a^2x + 3x^2 + 3ax^2 - x^3}{a^3}.$$

If we take $a = 1$, we have

$$C_0(x;1) = 1,$$

$$C_1(x;1) = x - 1,$$

$$C_2(x;1) = x^2 - 3x + 1,$$

$$C_3(x;1) = x^3 - 6x^2 + 8x - 1 \text{ and so on.}$$

For $n = 1$, $QC_n = nC_{n-1}$ becomes $QC_1 = 1C_0$.

From Table 1,

$$QC_1 = \alpha_1 \text{ and } 1C_0 = 1 \Rightarrow \alpha_1 = 1.$$

For $n = 2$, $QC_n = nC_{n-1}$ becomes $QC_2 = 2C_1$.

By Table 1,

$$QC_2 = 2\alpha_1 x + \alpha_2 - 3\alpha_1 \text{ and } 2C_1 = 2x - 2 \Rightarrow \alpha_1 = 1 \text{ and } \alpha_2 = 1.$$

For $n = 3$, $QC_n = nC_{n-1}$ becomes $QC_3 = 3C_2$.

From Table 1,

$$QC_3 = 3\alpha_1 x^2 + (3\alpha_2 - 12\alpha_1)x + \alpha_3 - 6\alpha_2 + 8\alpha_1 \quad \text{and} \quad 3C_2 = 3x^2 - 9x + 3.$$

Equating the corresponding terms, we get

$$\alpha_1 = 1, \quad \alpha_2 = 1 \quad \text{and} \quad \alpha_3 = 1.$$

Applying the same procedure for $n = 4$, $n = 5$ and $n = 6$ and so on, we get

$$\alpha_r = 1, \quad r \geq 1.$$

Thus we have the following proposition:

Proposition 4.1. *For the Poisson-Charlier polynomials $C_n(x;a)$, the characterization of delta operator Q being:*

$$\alpha_r = 1, \quad \textit{for all } r \geq 1.$$

Remark 1. *In this case, we have*

$$\begin{aligned} Q(x^n) &= \sum_{r=1}^{n} \binom{n}{r} x^{n-r} \\ &= \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \cdots + \binom{n}{r} x^{n-r} + \cdots + 1 \\ &= (x+1)^n - x^n. \end{aligned}$$

As is well known, the following relation

$$(-1)\, L_n^{(a-n)}(x) = \frac{x^n}{n!}\, c_n(a;x) \quad \text{(see [3])}$$

indicates a relationship between the Poisson-Charlier polynomials and the Laguerre polynomials.

The Laguerre polynomials of degree n is

$$L_n(x) = a_0 \sum_{r=0}^{n} (-1)^r \frac{(n!)}{(n-r)!\,(r!)^2}\, x^r.$$

Some authors define the Laguerre polynomial $L_n(x)$ by taking $a_0 = n!$, i.e.,

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{(n!)^2}{(n-r)!\,(r!)^2}\, x^r.$$

The recurrence relations are

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - xL_{n-1}(x).$$

The first few Laguerre polynomials are:

$$L_0(x) = 1,$$

$$L_1(x) = 1 - x,$$

$$L_2(x) = 2 - 4x + x^2,$$

$$L_3(x) = 6 - 18x + 9x^2 - x^3,$$

$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4.$$

By applying the same procedure as above in the Proposition 4.1, we get

$$\alpha_1 = -1,\ \alpha_2 = -2,\ \alpha_3 = -6 \text{ and } \alpha_4 = -24 \text{ and so on.}$$

We conclude that the characterization of delta operator Q for $L_n(x)$ is

$$\alpha_n = (-1)(n!).$$

Thus, we have the following proposition:

Proposition 4.2. *For the Leguerre polynomials* $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{(n!)^2}{(n-r)!(r!)^2} x^r$, *the characterization of delta operator* Q *being:*

$$\alpha_r = (-1)(r!), \quad \textit{for all } r \geq 1.$$

Remark 2. Here, $Q(x^n) = \sum_{r=1}^{n} \binom{n}{r} (-1)(r!)x^{n-r}$.

It is also known that the Hermite polynomials $H_n(x)$ are obtained by setting $a \to (2a)^{\frac{n}{2}} x + a$ and letting $a \to \infty$ in Poisson-Charlier polynomials $C_n(x;a)$. That is

$$\lim_{x\to\infty} (2a)^{\frac{n}{2}}\ c_n\big((2a)^{\frac{1}{2}} x + a; a\big) = (-1)^n\ H_n(x).$$

The Hermite polynomials $H_n(x)$ can be defined by the relation

$$H_n(x) = \sum_{m=0}^{n/2} (-1)^m\ \frac{n!}{m!(n-2m)!}\ (2x)^{n-2m}.$$

The pure recurrence relation for $H_n(x)$ is:

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x).$$

Since $H_0(x) = 1 \neq 0$, it is a Sheffer set.

The first few Hermite polynomials are:

$$H_1(x) = 2x,$$

$$H_2(x) = 4x^2 - 2,$$

$$H_3(x) = 8x^3 - 12x,$$

$$H_4(x) = 16x^4 - 48x^2 + 12,$$

$$H_5(x) = 32x^5 - 160x^3 + 120x \text{ and so on.}$$

By applying the same procedure as above in the Proposition 4.1, we get

$$\alpha_1 = \frac{1}{2},\ \alpha_2 = 0 \text{ and } \alpha_3 = 0 \text{ and so on.}$$

Hence the characterization of the delta operator for Hermite polynomials being $\alpha_1 = \frac{1}{2}$, and $\alpha_r = 0$ for all $r \geq 2$.

Thus, we obtain the following proposition.

Proposition 4.3. *For the Hermit's polynomial* $H_n(x) = \sum_{m=0}^{n/2} (-1)^m \frac{n!}{m!(n-2m)!}(2x)^{n-2m}$, *the characterization of delta operator* Q *being:*

$$\alpha_1 = \frac{1}{2} \text{ and } \alpha_r = 0, \quad \text{for all } r \geq 2.$$

Remark 3. Here, $Q(x^n) = \frac{1}{2}\, n\, x^{n-1}$ and hence the delta operator Q is the constant multiple of the usual derivative D.

5. Delta Triangle for the Poisson-Charlier Polynomials

The coefficients of $Q(x^n)$ in Remarks 1, 2 and 3 are arranged by a triangular array, say *Delta triangle*. In this section, the Delta triangles for Poisson-Charlier and their related polynomials are discussed.

For the Poisson-Charlier polynomials, $\alpha_r = 1$, $r \geq 1$.

From Table 1, we have

$$Q(x) = 1, \; Q(x^2) = 2x + 1, \; Q(x^3) = 3x^2 + 3x + 1, \; Q(x^4) = 4x^3 + 6x^2 + 4x + 1,$$
$$Q(x^5) = 5x^4 + 10x^3 + 10x^2 + 5x + 1, \; Q(x^6) = 6x^5 + 15x^2 + 20x^3 + 15x^2 + 6x + 1.$$

The Delta triangle for Poisson-Charlier polynomials is

1
2 1
3 3 1
4 6 4 1
5 10 10 5 1
6 15 20 15 6 1
...

Similar to Pascal triangle, it is also a triangular arrangements of rows. The tip of the triangle is number "1" which makes up the first row. In Pascal triangle, each row, begins and ends with "1". But in delta triangle, the ith rows begins with i, for $i = 1, 2, \cdots$ but ending with "1". The "Pascal Triangle sum" result holds good.

For the Leguerre polynomials, $\alpha_r = (-1)(r!)$, for all $r \geq 1$.

From Table 1, we have

$$Q(x) = -1, \; Q(x^2) = -2x - 2, \; Q(x^3) = -3x^2 - -6x - 6, \; Q(x^4) = -4x^3 - 12x^2 - 24x - 24,$$
$$Q(x^5) = -5x^4 - 20x^3 - 60x^2 - 120x - 120, \; Q(x^6) = -6x^5 - 30x^4 - 120x^3 - 360x^2 - 720x - 720.$$

The Delta triangle for Leguerre polynomials is

−1
−2 −2
−3 −6 −6
−4 −12 −24 −24
−5 −20 −60 −120 −120
−6 −30 −120 −360 −720 −720
...

Here, the tip of the Delta triangle is number "−1" which makes up the first row. All the entries in this triangle are negative numbers. First element in each rows is decreased by 1 compare with previous one. Last two entries in each rows are same, except first row.

For the Hermite polynomial $H_n(x)$, the characterization of delta operator Q being:

$$\alpha_1 = \frac{1}{2} \quad \text{and} \quad \alpha_r = 0 \quad \text{for all } r \geq 2.$$

From Table 1, we have

$$Q(x) = \frac{1}{2},\ \ Q(x^2) = x,\ \ Q(x^3) = \frac{3}{2}x^2,\ \ Q(x^4) = 2x^3,\ \ Q(x^5) = \frac{5}{2}x^4,\ \ Q(x^6) = 3x^5.$$

The Delta triangle for the Hermite polynomials $H_n(x)$ is

0.5
1 0
1.5 0 0
2 0 0 0
2.5 0 0 0 0
3 0 0 0 0 0
...

Here, the tip of the Delta triangle is number "0.5" which makes up the first row. First element in each row is increased by 0.5 compare with previous one. Except first element in each row, all entries are zero.

6. Concluding Remarks

The present attempt is made to introduce a new approach to the Poisson-Charlier polynomials via sequential representation of delta operator. From above discussion, we get a way opened to study the special polynomials by a new method of investigating definite delta operator numerically. Moreover, Kwasniewski [12] proposed Finite Operator q-Calculus by using q-delta operator and q-basic polynomial sequence. This is the good starting point for further investigation of the characterization of q-delta operator for q-Sheffer polynomials.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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