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Analysis and Parameter Identification of Time-delay Systems using the Chebyshev Wavelets

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Abstract. A delay matrix T_d is derived and used along with the Chebyshev matrix of integration in a new algorithm for analysis and parameter identification of time-delay systems. The method reduces the problem to a set of algebraic equations. In addition, the Chebyshev wavelets are more successful in analyzing and identifying time-delay systems when compared with the other algorithms (polynomial series). The examples support this claim.

1. Introduction

Systems with time delay occur frequently in fields such as mechanical and electrical systems, industrial processes, population growth, epidemic growth, neural networks, etc. In general, they are difficult to be analyzed and identified, therefore much effort has been devoted to the analysis and estimation of delay systems. Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. Examples are the use of Walsh function (Chen and Shih, 1978; Chen and Hisao, 1975; Chen, 1982) [1, 2, 3], block-pulse function (Hwang and Shih, 1985; Chen and Chung, 1987; Hsu and Cheng, 1981; Hwang and Shih 1985) [4, 5, 6, 7], Shifted Jacobi polynomials (Horng and Chou) [8], Legendre Polynomials (Hwang and Chen, 1985; Wang and Chang, 1985; Razzaghi and Shafiee, 1997; Marzban and Razzaghi, 2004) [9, 10, 11, 12], Laguerre polynomials (Kung and Lee, 1983; Clement, 1982; Hwang and Shih, 1983) [13, 14, 15], Chebyshev polynomials (Horng and Chou, 1985; Paraskevopoulos) [16, 17], Taylor series (Mouroutsos and Sparis, 1985; Razzaghi, 1988; Yang and Chen, 1987) [18, 19, 20], Fourier series (Paraskevopoulos, Sparis and Mouroutsos, 1985; Razzaghi, 1988; Ardekani, Samavat and Rahmani, 1991; Samavat and Vali, 1992; Samavat and Rashidi, 1995; Ebrahimi, Samavat, Vali and Gharavisi, 2007) [21, 22, 23, 24, 25, 26].

Key words and phrases. Time-delay systems; Chebyshev wavelets; System Analysis; System identification; Delay matrix.

Wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering science; particularly, wavelets are successfully used in signal analysis for waveform representation and segmentations, identification, optimal control and many other applications. Wavelets permit the accurate representation of a variety of functions and operators. The examples are (Tavassoli Kajani, Ghasemi and Babolian, 2007; Razzaghi, Yousefi, 2002; Karimi, Lohmann, Moshiri and Maralani, 2006; Karami, Karimi, Moshiri and Maralani, 2008; Sharif, Vali, Samavat and Gharavisi, 2011) [27, 28, 29, 30, 31]. The main characteristic of the mentioned algorithms is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem.

In this paper, the Chebyshev wavelets are used as orthogonal basis to approximate the delayed systems. One of the advantages of the present method is that we introduced only one algorithm which covers the entire simulation time for analysis and identification of time delay systems. Also the derived T_d has a very simple form and has so many zero entries and therefore it makes it easy for computer programming. Moreover, in this paper, for the first time, Chebyshev wavelets are extended for arbitrary times from t_1 to t_2. The given examples show the effectiveness of the proposed method. As it is shown in the tables, the proposed method has more accurate results when compared with some of the existing mentioned references.

This article is organized as follows: section 2 is about the preliminaries and problem statement, section 3 focuses on the main results including some benchmark examples and finally section 4 is the conclusion.

2. **Preliminaries and problem statement**

2.1. *The definition and properties of kronecker product*

Definition. Let A and B be two $n \times n$ and $m \times m$ square matrices respectively. The kronecker product of A and B denoted by $A * B$ is defined as follows:

$$A * B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}.$$

Which is an $(nm) \times (nm)$ matrix.

Let A, B and C be an $n\times n$, $n\times m$, $m\times m$ matrices respectively. We can identify the matrix ABC with the vector $(C^T * A)\widehat{B}$, where $\widehat{B}$ is an $(nm)\times 1$ vector as $\widehat{B} = \begin{bmatrix} B_1^T \\ \vdots \\ B_n^T \end{bmatrix}$, and B_i^T is the transpose of the ith row B_i of B. So $(C^T * A)\widehat{B}$ is a $(nm) \times 1$ vector.

2.2. *The definition and properties of Chebyshev Wavelets*

Wavelets have been used by many researchers in many scientific and engineering fields. They constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b} = |a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R},\ a \neq 0\,.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$ where $a_0 > 1$, $b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n} = |a|^{\frac{k}{2}}\psi(a_0^k - t - nb_0)\,.$$

In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}$ forms an orthonormal basis. Chebyshev wavelets $\psi_{n,m}(t) = \psi(n,k,m,t)$ have four arguments:

$m = 0,1,\ldots,M-1$, $n = 1,2,\ldots,2^{k-1}$, $k = 0,1,2,\ldots$ the values of m are given in Eq. (2.3.1) and t is the normalized time. They are defined on the interval $[0,1)$ [32]:

$$\psi(t)_{n,m} = \begin{cases} 2^{\frac{k}{2}}\widetilde{T}_m(2^k t - 2n + 1), & \dfrac{n-1}{2^{k-1}} \leq t < \dfrac{n}{2^{k-1}}\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.2.1}$$

Where

$$\widetilde{T}_m(t) = \begin{cases} \dfrac{1}{\sqrt{\pi}}\,, & m = 0\,, \\ \sqrt{\dfrac{2}{\pi}}\,T_m(t)\,, & m > 0\,. \end{cases} \tag{2.2.2}$$

In Eq. (2.2.1) the coefficients are used for orthonormality. Here $T_m(t)$ are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$, on $[-1,1]$, and satisfy the following recursive formula:

$$T_0(t) = 1,\ T_1(t) = t,\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3,\ldots\,. \tag{2.2.3}$$

We should note that in dealing with Chebyshev wavelets the weight function $w(x)$ has to be dilated and translated as:

$$w_n(t) = w(2^k t - 2n + 1)\,.$$

Remark. The time interval $[0,1)$ in Chebyshev wavelets can be extended to an arbitrary interval $[t_1,t_2)$ as follows:

$$\psi(t)_{n,m}=\begin{cases}\dfrac{2^{\frac{k}{2}}}{\sqrt[2]{\Delta t}}\widetilde{T}_m\left(\dfrac{2^k}{\Delta t}t-2n+1\right), & t_1+\Delta t\dfrac{n-1}{2^{k-1}}\le t<t_1+\Delta t\dfrac{n}{2^{k-1}}\,,\\ 0, & \text{otherwise}\,.\end{cases} \tag{2.2.4}$$

Where $\Delta t=t_2-t_1$.

2.3. *Function approximation*

The function $f(t)$ can be approximated as:

$$f(t)=\sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}}C_{n,m}\psi(t)_{n,m}\,. \tag{2.3.1}$$

Where $C_{n,m}=\langle f(t),\psi(t)_{n,m}\rangle$, in which $\langle\cdot,\cdot\rangle$ denotes the inner product as:

$$C_{n,m}=\langle f(t),\psi(t)_{n,m}\rangle=\int_{-\infty}^{+\infty}f(t)\psi(t)_{n,m}w(t)dt\,. \tag{2.3.2}$$

Equation (2.3.1) can be written in a matrix form as:

$$f(t)=C^T\psi(t)\,. \tag{2.3.3}$$

Where C and $\psi(t)$ are $(2^{k-1}M)\times 1$ matrices which are given by:

$$C^T=[c_{0,0}\;\; c_{0,1}\;\;\cdots\;\; c_{0,M-1}\;\; c_{1,0}\;\;\cdots\;\; c_{1,M-1}\;\; c_{2^{k-1},0}\;\;\cdots\;\; c_{2^{k-1},M-1}] \tag{2.3.4}$$

$$\psi(t)=[\psi(t)_{0,0}\;\;\psi(t)_{0,1}\;\cdots\;\psi(t)_{0,M-1}\;\cdots\;\psi(t)_{1,0}\;\cdots\;\psi(t)_{1,M-1}\;\cdots\;\psi(t)_{2^{k-1},0}\;\cdots\;\psi(t)_{2^{k-1},M-1}]. \tag{2.3.5}$$

2.4. *Operational matrix of integration*

Integration of the vector $\psi(t)$ defined in Eq. (2.3.1) can be written as:

$$\int_0^t\psi(s)ds=P_{(2^{k-1}M)\times(2^{k-1}M)}\psi(t)_{(2^{k-1}M)\times 1}\,. \tag{2.4.1}$$

Where P is a $(2^{k-1}M)\times(2^{k-1}M)$ matrix called the operational matrix of integration.

By the use of Eq. (2.4.1), the P matrix is obtained as:

$$P=\frac{1}{2^k}\begin{bmatrix}F & S & \cdots & S\\ 0 & F & \cdots & S\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & F\end{bmatrix}_{2^{k-1}\times 2^{k-1}}\,,$$

$$S = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{-2\sqrt{2}}{3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{m+1}}{m+1} - \dfrac{1-(-1)^{m-1}}{m-1}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \dfrac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & \frac{1}{\sqrt[2]{2}} & 0 & \cdots & \cdots & 0 \\ -\frac{\sqrt[2]{2}}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 \\ -\frac{\sqrt[2]{3}}{4} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \ddots & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{m-1}}{m-1} - \frac{(-1)^{m+1}}{m+1}\right) & \vdots & 0 & \frac{-1}{2(m-1)} & \ddots & \frac{1}{2(m+1)} \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \frac{\sqrt{2}}{2}\left(\frac{(-1)^{M-1}}{M-1} - \frac{(-1)^{M+1}}{M+1}\right) & \vdots & 0 & \cdots & \ddots & \frac{-1}{2(M-2)} \end{bmatrix}_{(M)\times(M)}.$$

2.5. *Delay operational matrix*

The delay operational matrix T_d can be defined as follows:

$$\psi(t-d)_{(2^{k-1}M)\times 1} = T_{d_{(2^{k-1}M)}\times(2^{k-1}M)}\psi(t)_{(2^{k-1}M)\times 1}. \tag{2.5.1}$$

Where d is a known time and T_d is a constant matrix given by:

$$T_D = \begin{bmatrix} 0 & \cdots & I & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & I \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

This is true if $d = N \times \frac{1}{2^k}$, where N is an integer.

2.6. *Problem statement*

Consider the following time-delay system:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau_x) + Cu(t) + Eu(t-\tau_u), \tag{2.6.1}$$
$$x(t) = g(t) \quad \text{for} \quad t < 0.$$

Where $x(t)$ is the n dimensional state vector, $u(t)$ is the m dimensional input vector. τ_x, τ_u are fixed delays and A, B, C, E are $n\times n$, $n\times n$, $n\times m$ and $n\times m$ constant matrices respectively.

In section 3.1 by using Eq. (2.6.1), we assume that an input u is given and then by using the approximation and changing the system to a set of algebraic equation, we solve for $x(t)$. In section 3.2, we assume u and x are given, and then using the same method as above, we solve for the unknown coefficients.

3. **Main results**

3.1. *Analysis of time-delay systems*

Integrating Eq. (2.6.1) from zero to t yields:

$$\int_0^t \dot{x}(s)ds = A\int_0^t x(s)ds + B\int_0^t x(s-\tau_x)ds + C\int_0^t u(s)ds + E\int_0^t u(s-\tau_x)ds. \tag{3.1.1}$$

Using equations (3.1.1), (2.3.1) implies that:

$$\int_0^t x(s)ds = \int_0^t X^T\psi(s)ds = X^T\int_0^t \psi(s)ds = X^T P\psi(t). \tag{3.1.2}$$

Where X^T is a $n\times r$, coefficient matrix that its ith row X_i^T is the coefficient vector of $x_i(s)$.

For $i=1,\dots,n$, and $r=2^{k-1}M$. Also

$$\int_0^t \dot{x}(s)ds = x(t)-x(0) = X^T\psi(t) - X_0^T\psi(t) \tag{3.1.3}$$

Where X_0^T is a $n\times r$ coefficient matrix defined by:

$$x_0^T = \frac{\sqrt{2}}{2^{\frac{k}{2}}}\begin{bmatrix} x_1(0) & 0 & 0 & \cdots & 0 & x_1(0) & 0 & 0 & \cdots & 0 & \cdots & x_1(0) & 0 & 0 & \cdots & 0 \\ x_2(0) & 0 & 0 & \cdots & 0 & x_2(0) & 0 & 0 & \cdots & 0 & \cdots & x_2(0) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n(0) & 0 & 0 & \cdots & 0 & x_n(0) & 0 & 0 & \cdots & 0 & \cdots & x_n(0) & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times r},$$

$$\int_0^t u(s-\tau_x)ds = \int_0^t U^T\psi(s-\tau_x)ds = U^T P\psi(t-\tau_x) = U^T P T_{\tau_x}\psi(t). \tag{3.1.4}$$

Where U^T is an $m\times r$ coefficient matrix that its ith row u_i^T is the coefficient vector of $u_i^T(t)$.

For $i=1,\dots,m$.

$$\begin{aligned}\int_0^t u(s)ds &= \int_0^t U^T\psi(s)ds \\ &= U^T\int_0^t \psi(s)ds \\ &= U^T P\psi(t),\end{aligned} \tag{3.1.5}$$

$$\begin{aligned}\int_0^t x(s-\tau_x)ds &= \int_0^{\tau_x} x(s-\tau_x)ds + \int_{\tau_x}^t x(s-\tau_x)ds \\ &= \int_0^{\tau_x} g(s-\tau_x)ds + \int_0^{t-\tau_x} x(s)ds \\ &= G^T P\psi(t) + X^T P\psi(t-\tau_x) \\ &= G^T P\psi(t) + X^T P T_{\tau_x}\psi(t). \qquad (3.1.6)\end{aligned}$$

Where G^T is an $n \times r$ coefficient matrix which its ith row U_i^T is the coefficient vector of $u_i(t)$.

For $i = 1, \ldots, n$.
Substituting equations (3.1.2)-(3.1.6) into (3.1.1) gives:

$$\begin{aligned}\Rightarrow \quad X^T\psi(t) - X_0^T\psi(t) = AX^T P\psi(t) + BX^T P T_{\tau_x}\psi(t) + BG^T(t) \\ + CU^T P\psi(t) + EU^T P T_{\tau_x}\psi(t). \qquad (3.1.7)\end{aligned}$$

Or equivalently

$$X^T - X_0^T = AX^T P + BX^T P T_{\tau_x} + BF^T P + CU^T P + EU^T P T_{\tau_x}. \qquad (3.1.8)$$

Applying the operation of kronecker product to Eq.(3.1.8) gives:

$$\begin{aligned}&[I_{r\times r} * I_{n\times n}]\widehat{X} - [P_{r\times n}^T * A_{n\times n}]\widehat{X} - [(P_{r\times r} T_{\tau_{x_{r\times r}}})^T * B_{n\times n}]\widehat{X} \\ &\quad = [I_{r\times r} * I_{n\times n}]\widehat{X}_0 + [(P_{r\times r})^T * B_{n\times n}]\widehat{G} + [P_{r\times r}^T * C_{n\times m}]\widehat{U} \\ &\qquad + [(P_{r\times r} T_{\tau_{x_{r\times r}}})^T * E_{n\times n}]\widehat{U}. \qquad (3.1.9)\end{aligned}$$

Where $\widehat{X}$, $\widehat{U}$, $\widehat{G}$, $\widehat{X}_0$ are the same as $\widehat{B}$ in section 2.1.
Eq. (3.1.9) can be written in a compact form:

$$R_{(nr)\times(nr)} \times \widehat{X}_{(nr)\times 1} = M_{(nr)\times 1}. \qquad (3.1.10)$$

Where

$$M = [(I_{r\times r} * I_n)\widehat{X}_0 + (P^T * B)\widehat{G} + (P^T * C)\widehat{U} + ((PT_{\tau_x})^T * E)\widehat{U}]$$

and R can be calculated as:

$$R = [(I_{r\times r} * I_{n\times n}) - (P^T * A) - ((PT_{\tau_X})^T * B)].$$

Then

$$\widehat{X}_{(nr)\times 1} = R^{-1}_{(nr)\times(nr)} M_{(nr)\times 1}.$$

Two examples are given next which show that the results are very accurate.

Example 1. Consider the time delay system modeled by:

$$\begin{aligned}&\dot{x} = x(t-1) + u(t), \\ &x = 1 \quad \text{for} \ \ t \le 0.\end{aligned}$$

Where $u(t)$ is given by:

$$u(t)=\begin{cases}-2.1+1.05t, & 0<t<1,\\ -1.05, & 1\le t<2\end{cases}$$

and the exact solution is:

$$x(t)=\begin{cases}1-1.1t+0.525t^2, & 0<t<1,\\ -0.25+1.575t-1.075t^2+0.175t^3, & 1\le t<2.\end{cases}$$

In Table 3.1.1, a comparison is made between the approximated and exact values of $x(t)$ and some of other algorithms.

Table 3.1.1. Results for approximated values of $x(t)$, using the Chebyshev wavelets for $K=M=4$, $K=M=6$ and [8], [9], [16] and the exact values

Time	Exact solution of $x(t)$	Approximation of $x(t)$ for $K=4, M=4$	Approximation of $x(t)$ for $K=6, M=6$	Approximation of $x(t)$ in ref [8]	Approximation of $x(t)$ in ref. [9]	Approximation of $x(t)$ in ref. [16]
0.2	0.80100	0.80102	0.80100	0.80204	0.80078	0.80084
0.4	0.64400	0.64400	0.64399	0.64315	0.64440	0.64444
0.6	0.52900	0.52900	0.52899	0.52700	0.52865	0.52856
0.8	0.45600	0.45538	0.45600	0.45942	0.45597	0.45605
1	0.42500	0.42507	0.42499	0.42473	0.42495	0.42489
1.2	0.39440	0.39443	0.39440	0.29134	0.39449	0.39436
1.4	0.32820	0.32821	0.32819	0.33050	0.32846	0.32848
1.6	0.23480	0.23481	0.23479	0.23510	0.23450	0.23432
1.8	0.12260	0.12263	0.12260	0.12120	0.12263	0.12265

In Table 3.1.2, a comparison is made between the approximated of $x(t)$, using the norm of error, for the proposed method and the methods of [8], [9], [16].

Example 2. Let us consider the following time-delay system:

$$\dot{x}=-x(t)-2x(t-0.25)+2u(t-0.25),$$
$$x(t)=0 \quad \text{for} \quad t\le 0.$$

Where $u(t)$ is:

$$u(t)=\begin{cases}0, & t\le 0,\\ 1, & 0<t<1.\end{cases}$$

Table 3.1.3 shows the exact and approximated values of $x(t)$.

Table 3.1.2. A comparison is made between Two Norm of Error approximations by using the Chebyshev wavelets for $K = M = 4$, $K = M = 6$ and the methods given in [8], [9], [16]

Time	Two norm error approximations between all samples	Number of samples
Approximation of $x(t)$ for $K = 6$, $M = 6$	2.23×10^{-5}	9
Approximation of $x(t)$ for $K = 4$, $M = 4$	6.25×10^{-4}	9
Approximation of $x(t)$ in ref. [8]	0.1032	9
Approximation of $x(t)$ in ref. [9]	7.07×10^{-4}	9
Approximation of $x(t)$ in ref. [16]	8.60×10^{-4}	9

Table 3.1.3. Exact and approximated values of $x(t)$, using the Chebyshev wavelets for $K = 3$, $M = 4$ and $K = 4$, $M = 4$ and $K = 6$, $M = 6$

Time	Exact solution of$x(t)$	Approximation of $x(t)$ for $K = 3$, $M = 4$	Approximation of $x(t)$ for $K = 4$, $M = 4$	Approximation of $x(t)$ for $K = 6$, $M = 6$
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0.0975411	0.0975393	0.0975412	0.0975411
0.4	0.2785840	0.2785855	0.2785839	0.2785840
0.5	0.4423984	0.4424131	0.4423994	0.4423984
0.6	0.5719084	0.5719191	0.5719076	0.5719084
0.7	0.6546513	0.65463872	0.6546518	0.6546512
0.8	0.6985156	0.69848867	0.6985173	0.6985156
0.9	0.7137236	0.71374648	0.7137220	0.7137235

Where exact solution is:

$$x(t) = \begin{cases} 0, & 0 \le t < 0.25, \\ 2(1 - e^{-(t-0.25)}), & 0.25 \le t < 0.5, \\ 2(1 - e^{-(t-0.25)}) - 4(1 - (t + 0.5)e^{-(t-0.25)}), & 0.5 \le t < 0.75, \\ (1 - e^{-(t-0.25)}) - 4(1 - (t + 0.5)e^{-(t-0.25)}) & \\ \quad + 8\left(1 - \left(\frac{1}{2}t^2 + \frac{1}{4}t + \frac{17}{32}\right)e^{-(t-0.75)}, & 0.75 \le t < 1. \end{cases}$$

In Table 3.1.4, a comparison is made between the approximated and exact values of $x(t)$, using the norm of error.

Table 3.1.4. A comparison is made between two norm of error approximations by using the Chebyshev wavelets for $K=3$, $M=4$ and $K=4$, $L=4$ and $K=6$, $L=6$

Time	Two norm of error approximations between all samples	Number of samples
Approximation of $x(t)$ for $K=3$, $M=4$	8.26×10^{-4}	4000
Approximation of $x(t)$ for $K=4$, $M=4$	7.38×10^{-5}	8000
Approximation of $x(t)$ for $K=6$, $M=6$	3.64×10^{-6}	32000

3.2. *Identification of time-delay systems*

Again, consider the system (2.6.1). Given the state vector $x(t)$ and the input vector $u(t)$, the identification problem is to estimate the unknown matrices A, B, C and E. Therefore a total of $2(n^2+nm)$ elements are to be estimated.

Let us integrate (2.6.1) to obtain:

$$\int_0^t \dot{x}(s)ds = A\int_0^t (s)ds + B\int_0^t x(s-\tau_x)ds + C\int_0^t u(s)ds + E\int_0^t u(s-\tau_u)ds\,. \tag{3.2.1}$$

Using Eq. (3.1.2)-(3.1.6) and substituting in equation (3.2.1) yields:

$$\begin{aligned} X^T\psi(t) - X_0^T\psi(t) &= AX^TP\psi(t) + BX^TPT_{\tau_x}\psi(t) + BG^TP\psi(t) \\ &\quad + CU^TP\psi(t) + EU^TPT_{\tau_x}\psi(t) \end{aligned} \tag{3.2.2}$$

$$\Rightarrow \qquad X^T - X_0^T = AX^TP + BX^TPT_{\tau_x} + BG^TP + CU^TP + P + EU^TPT_{\tau_x}. \tag{3.2.3}$$

Using the operation of kronecker product and solving for the coefficients A, B, C, E gives:

$$\begin{aligned} &\widehat{x}_{(nr)\times 1} - \widehat{X}_{0_{(nr)\times 1}} + \widehat{BG^TP} \\ &\quad = ((X^TP)^T_{r\times n} * I_n)\widehat{A}_{(n^2)\times 1} + ((X^TPT_{\tau_X})^T_{r\times n} * I_n)\widehat{B}_{(n^2)\times 1} \\ &\qquad + ((U^TP)^T_{r\times m} * I_n)\widehat{C}_{(nm)\times 1} + ((U^TPT_{\tau_X})^T_{r\times m} * I_n)\widehat{E}_{(nm)\times 1} \end{aligned} \tag{3.2.4}$$

Where $\widehat{X}$, $\widehat{X}_0$, $\widehat{BG^TP}$, $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{E}$ are same as $\widehat{B}$ in section (2.1).

Eq. (3.2.4) can be written in a compact form:

$$R_{(nr)\times(2n^2+2mn)} \times N_{(2n^2+2mn)\times 1} = M_{(nr)\times 1}\,, \tag{3.2.5}$$

$$N_{(2n^2+2mn)\times 1} = \begin{bmatrix} \widehat{A} \\ \widehat{B} \\ \widehat{C} \\ \widehat{E} \end{bmatrix},$$

$$M = \widehat{X}_{(nr)\times 1} - \widehat{X}_{0_{(nr)\times 1}}\,.$$

and the matrix R can be calculated as:

$$R=[((X^TP)^T_{r\times n}*I)\ \ ((X^TPT_{\tau_x})^T_{r\times n}*I_n)\ \ ((U^TP)^T_{r\times m}*I_n)\ \ ((U^TPT_{\tau_x})^T_{r\times m}*I_n)]_{nr\times(2n^2+2mn)}$$

N can be obtained using the Eq. (3.2.5) as:

$$N_{(2n^2+2mn)\times 1}=(R^TR)^{-1}R^TM_{nr\times 1}\,. \tag{3.2.6}$$

In order to show that the mentioned algorithm is very accurate, two examples are given.

Example 3. Consider the following time-delay system:

$$\dot{x}=x(t-1)+u(t),$$
$$x=1 \quad \text{for}\ \ t\leq 0\,.$$

Where the following data are given:

$$u(t)=\begin{cases}-2.1+1.05t, & 0<t<1,\\ -1.05, & 1\leq t<2,\end{cases}$$

$$x(t)=\begin{cases}1-1.1t+0.525t^2, & 0<t<1,\\ -0.25+1.575t-1.075t^2+0.175t^3, & 1\leq t<2.\end{cases}$$

Using the present algorithm, the approximated values of the parameters are given in Table 3.2.1.

Table 3.2.1. Results for approximated values, using the Chebyshev wavelets for $K=M=6$ and the methods of [8], [23] and the exact values

	A	B	C	D
Exact values	0.00000	1.00000	1.00000	0.00000
$K=6$, $M=6$	0.00000	0.99999	0.99999	0.00000
Ref. [8]	0.00000	1.03015	0.99963	0.00000
Ref. [23]	0.01000	0.98000	0.99000	0.01000

Example 4. Consider the linear time-delay system:

$$\dot{x}=-x(t)-2x(t-0.25)+2u(t-0.25),$$
$$x(t)=0 \quad \text{for}\ \ t\leq 0.$$

Where $u(t)$ is:

$$u(t)=\begin{cases}0, & t\leq 0,\\ 1, & 0<t<1\end{cases}$$

and $x(t)$ is given by:

$$x(t)=\begin{cases} 0, & 0\le t<0.25,\\ 2(1-e^{-(t-0.25)}), & 0.25\le t<0.5,\\ 2(1-e^{-(t-0.25)})-4(1-(t+0.5)e^{-(t-0.25)}), & 0.5\le t<0.75,\\ (1-e^{-(t-0.25)})-4(1-(t+0.5)e^{-(t-0.25)}) & \\ \qquad +8\left(1-\left(\frac{1}{2}t^2+\frac{1}{4}t+\frac{17}{32}\right)e^{-(t-0.75)}\right), & 0.75\le t<1. \end{cases}$$

By using the method given in section 3.2, the results are shown in Table 3.2.2.

Table 3.2.2. Exact and approximated values, using the Chebyshev wavelets for different values of K, L and the method of [9]

K, M	A	B	C	D
6, 4	−0.999999867	−1.999999694	0.000000004	1.999999699
6, 6	−0.999999867	−1.999999700	0.000000004	1.999999699
Ref. [9]	−1.056700000	−1.942500000	−0.012000000	2.026900000

4. Conclusion

In general, it has been very difficult and tedious to obtain the solution of time delay systems. In this paper, we have used integral operational matrix and delay operational matrix to transform the system into a set of algebraic equations thus greatly simplifying the problem. Using these algebraic equations, the analysis and identification of linear time delay systems are obtained. As it is shown by the given examples, the convergences of the Chebyshev wavelets (when compared with the existing polynomial series approach) are excellent, and the number of terms required for the approximation is not too large. Therefore, we may conclude that the Chebyshev wavelets provide an efficient and simple tool for the analysis and parameter identification of time-delay systems. Also it is possible to extend this algorithm for analysis and identification of time varying and nonlinear delay systems.

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