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Research Article

Binomial Coefficients and Powers of One Type of Large Pentadiagonal Matrices

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Abstract. In this paper, we derive a general expression for the entries of the rth power ($r \in \mathbb{N}$) of one type of the $n \times n$ complex pentadiagonal matrix for all $n \geq 4(r-1)$, in terms of binomial coefficients.

Keywords. Pentadiagonal matrix; Powers of matrices; Binomial coefficients; Eigenvalues

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1. Introduction

A certain type of transformation of a set of numbers can be represented as the multiplication of a vector by a square matrix. Repetition of this operation is equivalent to multiplying the original vector by a power of the matrix. In solving some difference equations, differential and delay differential equations, and boundary value problems, we need to compute the arbitrary integer powers of a square matrix. Properties of powers of the matrices are thus of considerable importance ([3, 9, 10]).

An $n \times n$ pentadiagonal matrix is one having the form

$$S_n = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ d_1 & a_2 & b_2 & c_2 & \ddots & & \vdots \\ e_1 & d_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & e_2 & \ddots & \ddots & \ddots & c_{n-3} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & b_{n-2} & c_{n-2} \\ \vdots & & \ddots & e_{n-3} & d_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & 0 & e_{n-2} & d_{n-1} & a_n \end{bmatrix}.$$

In other words, $S_n = [s_{i,j}]_{1 \le i,j \le n}$ is pentadiagonal if $s_{i,j} = 0$ for $|i-j| > 2$. These types of matrices often occur in several areas, such as in numerical solutions of ordinary and partial differential equations (ODE and PDE), interpolation problems, high order harmonic spectral filtering theory, boundary value problems (BVP), etc. In many of these areas, the integer powers of pentadiagonal matrices are encountered as a problem ([1, 4, 6, 10]).

An $n \times n$ persymmetric matrix A is a square matrix which is symmetric in the northeast to southwest diagonal, i.e., $[A]_{i,j} = [A]_{n-j+1,n-i+1}$ with $1 \le i, j \le n$. For example, 5×5 persymmetric matrices are the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{23} & a_{13} \\ a_{41} & a_{42} & a_{32} & a_{22} & a_{12} \\ a_{51} & a_{41} & a_{31} & a_{21} & a_{11} \end{bmatrix}.$$

There is a vast literature concerned with powers of tridiagonal and pentadiagonal matrices. Rimas derived a general expression for powers of one type of $(0-1)$-symmetric pentadiagonal matrices depending on the Chebyshev polynomials (see [12] for the odd case and [11] for the even case). In [2], the authors derived a general expression for powers and inverse of one type of symmetric pentadiagonal matrices depending on the Chebyshev polynomials. In these papers, the powers and the inverses are given by using Chebyshev polynomials.

Recently, computing integer powers of square matrices using binomial coefficients instead of the Chebyshev polynomials have been a very popular problem since the expressions with binomial coefficients is simpler than Chebyshev polynomials, [5, 7] are the two of these papers.

In [8], Gutiérrez-Gutiérrez studied powers of an $n \times n$ complex tridiagonal matrices with constant diagonals given by

$$B_n = \operatorname{tridiag}_n(b, a, c) = \begin{bmatrix} a & c & 0 & \cdots & \cdots & 0 \\ b & a & c & \ddots & & \vdots \\ 0 & b & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c & 0 \\ \vdots & & \ddots & b & a & c \\ 0 & \cdots & \cdots & 0 & b & a \end{bmatrix}, \tag{1.1}$$

where $bc \neq 0$. He gave the powers of such matrices depending on the Chebyshev polynomials. After that, in [7], Gutiérrez interestingly obtained the following result for the powers of the matrix given by (1.1), in terms of binomial coefficients.

Theorem 1 ([7], Theorem 3). *Let $a \in \mathbb{C}$ and $b, c \in \mathbb{C}\backslash\{0\}$. Suppose $r, n, i, j \in \mathbb{N}$, $n \geq 2(r-1)$ and $1 \leq i,\ j \leq n$. If $A_n = tridiag_n(b,a,c)$ then*

$$[B_n^r]_{i,j} = \begin{cases} \sum\limits_{h\in H} \binom{r}{h} a^{r-h} c^{\frac{h+j-i}{2}} b^{\frac{h+i-j}{2}} \left[\binom{h}{\frac{h+i-j}{2}} - \binom{h}{\frac{h-i-j+2n+2}{2}}\right] & \text{if } i,j > n-r+1, \\ \sum\limits_{h\in H} \binom{r}{h} a^{r-h} c^{\frac{h+j-i}{2}} b^{\frac{h+i-j}{2}} \left[\binom{h}{\frac{h+i-j}{2}} - \binom{h}{\frac{h+i+j}{2}}\right] & \text{otherwise,} \end{cases}$$

with $H = \{h : 0 \leq h \leq r : h \equiv i+j \pmod 2\}$.

Here, note that the binomial coefficient $\binom{a}{b}$ is defined by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{cases} \frac{a!}{b!(a-b)!} & 0 \leq b \leq a \ \text{ and } \ b \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

when $a \in \mathbb{Z}$, $a \geq 0$ and $b \in \mathbb{R}$ (see [7]).

In this paper, we present a general expression for the entries of the rth power ($r \in \mathbb{N}$) of the $n \times n$ symmetric pentadiagonal matrix given by

$$A_n = \begin{bmatrix} a-c & b & c & 0 & \cdots & \cdots & 0 \\ b & a & b & c & \ddots & & \vdots \\ c & b & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & c & \ddots & \ddots & \ddots & c & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & b & c \\ \vdots & & \ddots & c & b & a & b \\ 0 & \cdots & \cdots & 0 & c & b & a-c \end{bmatrix}, \tag{1.2}$$

where $a, b, c \in \mathbb{C}$, n is large enough, that is, $n \geq 4(r-1)$, in terms of binomial coefficients differently from Chebyshev polynomials.

2. Main Result

We start with the following result in the proof of our main theorem.

Theorem 2 ([2, Theorem 2]). *Let $a, b, c \in \mathbb{C}$ and $n \in \mathbb{N}$. Then $P_n \Lambda_n P_n^{-1}$ is eigenvalue decomposition of symmetric pentadiagonal matrix A_n given by* (1.2)*, where the entries of eigenvector matrix P_n are*

$$[P_n]_{i,j} = \sin\frac{ij\pi}{n+1}, \quad 1 \leq i, j \leq n, \tag{2.1}$$

and $\Lambda_n = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with

$$\lambda_j = a + 2b\cos\frac{j\pi}{n+1} + 2c\cos\frac{2j\pi}{n+1}, \quad 1 \leq j \leq n.$$

Next, we continue with the following lemma which gives the inverse of the matrix P_n in terms of itself, and allows us to use the previous theorem in the proof of our main theorem.

Lemma 3 ([13, Lemma 2.2]). *Let P_n be the matrix given by* (2.1). *Then*

$$P_n^{-1} = \frac{2}{n+1} P_n .$$

We can finally give the last result that we use in our proof. This lemma is the one that allows us to use the binomial coefficients in our main result.

Lemma 4 ([7, Lemma 2]). *If $r \in \mathbb{N} \cup \{0\}$, then*

$$\frac{2}{\pi}\int_0^{\pi} (2\cos x)^r \sin(ix)\sin(jx)dx = \binom{r}{\frac{r+i-j}{2}} - \binom{r}{\frac{r+i+j}{2}}$$

for all $i,j \in \mathbb{N}$.

We can now give the main result of our paper. The following theorem will give a general expression for the powers of A_n in terms of the binomial coefficients.

Theorem 5. *Let A_n be the symmetric pentadiagonal matrix having form in* (1.2). *Suppose $r,n,i,j \in \mathbb{N}$, $n \geq 4(r-1)$ and $1 \leq i,\ j \leq n$. Then*

$$[A_n^r]_{i,j} = \begin{cases} \sum\limits_{h\in H}\sum\limits_{k\in K_h} \binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k} b^k c^h \left[\binom{k+2h}{\frac{k+2h+i-j}{2}} - \binom{k+2h}{\frac{k+2h-i-j+2n+2}{2}}\right] & \text{if } i,j > n-r+1, \\ \sum\limits_{h\in H}\sum\limits_{k\in K_h} \binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k} b^k c^h \left[\binom{k+2h}{\frac{k+2h+i-j}{2}}\binom{k+2h}{\frac{k+2h+i+j}{2}}\right] & \text{otherwise}, \end{cases}$$

with $H = \{h : 0 \leq h \leq r\}$ and $K_h = \{k : 0 \leq k \leq r-h;\ k \equiv i+j \pmod 2\}$.

Proof. First of all, assume that $i \leq n-r+1$ or $j \leq n-r+1$. Hence we have $[A_n^r]_{i,j} = [A_m^r]_{i,j}$ for all $m \geq n \geq 4(r-1)$ (see [4, Theorem 3], that is a result on the structure of the natural powers of large banded Toeplitz matrices). Thus we get $[A_n^r]_{i,j} = \lim\limits_{m\to\infty}[A_m^r]_{i,j}$. By Theorem 2, we have

$$\begin{aligned}
[A_n^r]_{i,j} &= \lim_{m\to\infty}[A_m^r]_{i,j} \\
&= \lim_{m\to\infty}[(V_m D_m V_m^{-1})^r]_{i,j} \\
&= \lim_{m\to\infty}[V_m D_m^r V_m^{-1}]_{i,j} \\
&= \lim_{m\to\infty}\sum_{h=1}^{m}[V_m]_{i,h}[D_m^r V_m^{-1}]_{h,j} \\
&= \lim_{m\to\infty}\sum_{h=1}^{m}[V_m]_{i,h}\lambda_h^r\,[V_m^{-1}]_{h,j} \\
&= \lim_{m\to\infty}\frac{2}{m+1}\sum_{h=1}^{m}\left(a+2b\cos\frac{h\pi}{m+1}+2c\cos\frac{2h\pi}{m+1}\right)^r \sin\frac{ih\pi}{m+1}\sin\frac{jh\pi}{m+1}.
\end{aligned}$$

Now, by using the definition of the definite integral, we get

$$[A_n^r]_{i,j} = \frac{2}{\pi}\int_0^{\pi}(a+2b\cos x+2c\cos 2x)^r \sin ix \sin jx dx$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\pi}(a-2c+2b\cos x+4c\cos^2 x)^r \sin ix \sin jx dx \\
&= \frac{2}{\pi}\int_0^{\pi}\sum_{h=0}^{r}\binom{r}{h}(a-2c+2b\cos x)^{r-h}(4c\cos^2 x)^h \sin ix \sin jx dx \\
&= \frac{2}{\pi}\int_0^{\pi}\sum_{h=0}^{r}\binom{r}{h}\left(\sum_{k=0}^{r-h}\binom{r-h}{k}(a-2c)^{r-h-k}(2b\cos x)^k\right)(4c\cos^2 x)^h \sin ix \sin jx dx \\
&= \frac{2}{\pi}\int_0^{\pi}\sum_{h=0}^{r}\sum_{k=0}^{r-h}\binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k}b^k c^h (2\cos x)^{k+2h} \sin ix \sin jx dx \\
&= \sum_{h=0}^{r}\sum_{k=0}^{r-h}\binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k}b^k c^h \frac{2}{\pi}\int_0^{\pi}(2\cos x)^{k+2h} \sin ix \sin jx dx\,.
\end{aligned}
$$

Now, we can use Lemma 4 to write the last integral in terms of the binomial coefficients. So, we have

$$[A_n^r]_{i,j} = \sum_{h=0}^{r}\sum_{k=0}^{r-h}\binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k}b^k c^h \left[\binom{k+2h}{\frac{k+2h+i-j}{2}} - \binom{k+2h}{\frac{k+2h+i+j}{2}}\right].$$

Since $\binom{k+2h}{\frac{k+2h+i-j}{2}} = \binom{k+2h}{\frac{k+2h+i+j}{2}} = 0$, when $k \equiv i+j \pmod 2$, we conclude that

$$[A_n^r]_{i,j} = \sum_{h\in H}\sum_{k\in K_h}\binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k}b^k c^h \left[\binom{k+2h}{\frac{k+2h+i-j}{2}} - \binom{k+2h}{\frac{k+2h+i+j}{2}}\right].$$

Finally, assume that $i,j > n-r+1$. Since A_n is persymmetric, A_n^r is also persymmetric. Hence $[A_n^r]_{i,j} = [A_n^r]_{n-j+1,n-i+1}$. Then, we get the result that

$$[A_n^r]_{i,j} = \sum_{h\in H}\sum_{k\in K_h}\binom{r}{h}\binom{r-h}{k}(a-2c)^{r-h-k}b^k c^h \left[\binom{k+2h}{\frac{k+2h+i-j}{2}} - \binom{k+2h}{\frac{k+2h-i-j+2n+2}{2}}\right],$$

which is desirable. □

As a final result, we can give the following corollary to our main theorem. This result is a specific version of the corollary given in [7, Corollary 4].

Corollary 6. *Let* $b \in \mathbb{C}\setminus\{0\}$. *Assume that* $r,n,i,j \in \mathbb{N}$, $n \geq 4(r-1)$ *and* $1 \leq i,\ j \leq n$. *If* $A_n = \mathrm{tridiag}_n(b,0,b)$, *then*

$$[A_n^r]_{i,j} = \begin{cases} 0 & \text{if } r \not\equiv i+j \pmod 2, \\ b^r\left[\binom{r}{\frac{r+i-j}{2}} - \binom{r}{\frac{r-i-j+2n+2}{2}}\right] & \text{if } r \equiv i+j \pmod 2 \text{ and } i,j > n-r+1, \\ b^r\left[\binom{r}{\frac{r+i-j}{2}} - \binom{r}{\frac{r+i+j}{2}}\right] & \text{otherwise.} \end{cases}$$

3. Conclusion

In [2], Arslan *et al*. obtained a general expression for the entries of the natural powers of an $n \times n$ symmetric complex pentadiagonal matrix in terms of Chebyshev polynomials. In this paper, we give a much simpler formula to compute arbitrary positive integer powers of the matrix considered by Arslan *et al*. [2] in terms of binomial coefficients.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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