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Research Article

On Some identities for Generalized Fibonacci and Lucas Sequences with Rational Subscript

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Abstract. In this paper, we exploit general techniques from matrix theory to establish some identities for generalized Fibonacci and Lucas sequences with rational subscripts of the forms $\frac{n}{2}$ and $\frac{r}{s}$. For this purpose, we consider matrix functions $X \mapsto X^{n/2}$ (resp. $X \mapsto X^{r/s}$) of two special matrices, and discuss whether the $\frac{n}{2}$ (resp. $\frac{r}{s}$) are integers or irreducible fractions.

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1. Introduction

The generalized Horadam sequence $\{W_n(a,b;p,q)\}_{n=0}^{\infty}$, or briefly $\{W_n\}$, is a recurrence sequence of order two, recursively defined by

$$W_{n+2} = pW_{n+1} - qW_n,\ W_0 = a,\ W_1 = b, \quad n \geq 0, \tag{1}$$

where a,b,p,q ($p \neq 0$ and $q \neq 0$) are arbitrary complex coefficients (see [10] and [14]).
Let $\alpha = (p+\sqrt{p^2-4q})/2$ and $\beta = (p-\sqrt{p^2-4q})/2$ be roots of equation

$$z^2 - pz + q = 0,$$

where $\sqrt{p^2-4q}$ denotes the principal square root of the complex number $\Delta = p^2-4q$, which is assumed to be nonzero. The numbers $W_n(a,b;p,q)$ given by the recurrence relation (1) can be explicitly expressed by the Binet's formula:

$$W_n = C\alpha^n + D\beta^n,$$

where $C = \frac{b-a\beta}{\alpha-\beta}$, $D = \frac{a\alpha-b}{\alpha-\beta}$ (with $p^2 \neq 4q$). In particular, in [12], Lucas shows that

$$U_n = W_n(0,1;p,q) = \frac{\alpha^n-\beta^n}{\alpha-\beta}, \quad p^2 \neq 4q, \tag{2}$$

$$V_n = W_n(2,p;p,q) = \alpha^n+\beta^n. \tag{3}$$

The numbers defined in (2) and (3) are referred to as the generalized Fibonacci and Lucas numbers, respectively. Further and detailed information may be found in [9], [10], [12], [14], [15] and [16]. Note that the generalized Fibonacci and Lucas numbers with negative subscripts are described as

$$U_{-n} = -q^{-n}U_n \text{ and } V_{-n} = q^{-n}V_n, \quad n \in \mathbb{Z}^+.$$

Generalization of the formulas (2) and (3), from an integer exponent n to a real exponent θ, has been considered by Horadam [11]. Indeed, the generalized sequences $\{U_\theta\}$ and $\{V_\theta\}$, with real subscripts, are defined by generalized Binet's formulas,

$$U_\theta = \frac{\alpha^\theta-\beta^\theta}{\alpha-\beta}, \; V_\theta = \alpha^\theta+\beta^\theta, \quad \alpha,\beta = (p \pm \sqrt{\Delta})/2, \; p^2 \neq 4q. \tag{4}$$

In this paper, we are particularly interested in providing identities for generalized Fibonacci and Lucas sequences with rational subscripts, by aid of fundamental tools from the theory of matrix functions (see [5], [8]) and [13]). Some results obtained constitute an extension of existing identities in the literature, that characterize Horadam-type sequences with integer subscripts.

To emphasize, we make use of properties of the matrix functions $A \mapsto A^{n/2}$ (resp. $A \mapsto A^{r/s}$) and $B \mapsto B^{n/2}$ (resp. $B \mapsto B^{r/s}$) (see [1], [2], [3], [5], [8] and [13]), where $n,r,s \in \mathbb{Z}$ (with $s \geq 1$) and

$$A = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix} \text{ and } B = \frac{1}{2}\begin{bmatrix} p & 1 \\ \Delta & p \end{bmatrix},$$

taking into account whether the $\frac{n}{2}$ and $\frac{r}{s}$ are integers or irreducible fractions, which is mainly involved in this work. Matrices such as A and B have been extensively exploited by several authors, in the objective to carry out identities for Horadam-type sequences, especially in the case when the subscripts are integers. See for instance [1], [2], [3], [4], [6], [7], [14], [16] and references therein.

The outline of this paper is as follows: In Section 2, some identities related to the generalized Fibonacci and Lucas sequences with rational subscripts of the form $\frac{n}{2}$ are given for every integer n. Section 3 is devoted to the investigation of some generalizations of the identities given in the second section, in the case of rational subscripts of the form $\frac{r}{s}$.

2. The Generalized Fibonacci and Lucas Sequences with Rational Subscript of the Form $\frac{n}{2}$

The results presented in this section are mainly based on properties of matrix functions $X \mapsto X^{n/2}$ and $X \mapsto X^n$, combined with the generalized Binet's formulas (4). Throughout this study, unless otherwise stated, we will denote by $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Let consider the scalar complex function $f^{(\ell)}(z) = z^\ell$, where ℓ is a nonzero integer number. Since A and B are nonsingular matrices, admitting two distinct eigenvalues (α and β), the function $f^{(\ell)}(z)$ is defined on the spectrum of these matrices [5]. Consequently, the matrix functions $f^{(\ell)}(A)$ and $f^{(\ell)}(B)$ are univalued and may be expressed, using the Lagrange-Sylvester interpolation polynomial [5], under the polynomial expressions

$$\begin{cases} f^{(\ell)}(A) = \frac{\alpha^\ell}{\alpha-\beta}(A-\beta I_2) + \frac{\beta^\ell}{\beta-\alpha}(A-\alpha I_2) \\ f^{(\ell)}(B) = \frac{\alpha^\ell}{\alpha-\beta}(B-\beta I_2) + \frac{\beta^\ell}{\beta-\alpha}(B-\alpha I_2), \end{cases} \tag{5}$$

where I_2 designates the 2×2 matrix identity.

Theorem 1. *For every number $\ell \in \mathbb{Z}^*$*

$$\begin{cases} f^{(\ell)}(A) = \begin{bmatrix} U_{\ell+1} & -qU_\ell \\ U_\ell & -qU_{\ell-1} \end{bmatrix} \\ f^{(\ell)}(B) = \dfrac{1}{2}\begin{bmatrix} V_\ell & U_\ell \\ \Delta U_\ell & V_\ell \end{bmatrix}. \end{cases}$$

Proof. According to formula (5), it ensues that for every $\ell \in \mathbb{Z}^*$

$$f^{(\ell)}(A) = \frac{1}{\alpha-\beta}\begin{bmatrix} \alpha^{\ell+1}-\beta^{\ell+1} & -q(\alpha^\ell-\beta^\ell) \\ \alpha^\ell-\beta^\ell & -\alpha\beta(\alpha^{\ell-1}-\beta^{\ell-1}) \end{bmatrix} = \begin{bmatrix} U_{\ell+1} & -qU_\ell \\ U_\ell & -qU_{\ell-1} \end{bmatrix}.$$

The matrix function $f^{(\ell)}(B)$ is similarly obtained using the equation (5). □

Consider now the scalar complex function $f^{(n,2)}(z) \equiv z^{n/2}$, where $n \in \mathbb{Z}^*$. When n is an even number, the function $f^{(n,2)}(z)$ is nothing else but the scalar power function $f^{(\ell)}(z) = z^\ell$ (with $n = 2\ell$) mentioned above. By contrast, when n is an odd number, $f^{(n,2)}(z)$ is a multivalued function giving rise to 2 branches. Indeed, for every nonzero complex number $z = |z|\exp[i\arg(z)]$ ($-\pi < \arg(z) \le \pi$), these branches may be characterized as follows

$$f_k^{(n,2)}(z) = \exp\left[\frac{n}{2}(\log(z) + 2ik\pi)\right] = |z|^{n/2}\exp\left[\frac{n}{2}(i\arg(z) + 2ik\pi)\right],$$

where log denotes the principal branch of the complex logarithm and $k \in \{0,1\}$. By abuse of notation, the principal branch of $f^{(n,2)}(z)$ will be denoted by $z^{n/2}$, i.e. $f_0^{(n,2)}(z) = \exp\left[\frac{n}{2}\log(z)\right] = z^{n/2}$.

Hence, for every nonzero z in $\mathbb{C}$,

$$f_k^{(n,2)}(z) = \exp(ik\pi)z^{n/2}, \tag{6}$$

where $k \in \{0,1\}$.

Since A admits two distinct nonzero eigenvalues (α and β), it is clear that $f^{(n,2)}(z) \equiv z^{n/2}$ is defined on the spectrum of A [5]. Therefore, there exist 4 matrix functions $A \mapsto A^{n/2}$ which can be derived from the two branches of the scalar function $f^{(n,2)}(z) \equiv z^{n/2}$, defined by (6) [8]. To emphasize, all these matrix functions are primary matrix functions [8] and can be specified by the Lagrange-Sylvester interpolation polynomial [5], through the polynomial expression

$$f^{(n,2)}_{(k_1,k_2)}(A) = \frac{f^{(n,2)}_{k_1}(\alpha)}{\alpha-\beta}(A-\beta I_2) + \frac{f^{(n,2)}_{k_2}(\beta)}{\beta-\alpha}(A-\alpha I_2),$$

where $(k_1,k_2) \in \{0,1\} \times \{0,1\}$. Furthermore, since the matrix B has exactly the same spectrum as the matrix A, the previous formulas remain valid when A is substituted by B.

Theorem 2. *For every odd number $n \in \mathbb{Z}$,*

$$\begin{cases} f^{(n,2)}_{(0,0)}(A) = -f^{(n,2)}_{(1,1)}(A) = \begin{bmatrix} U_{\frac{n}{2}+1} & -qU_{\frac{n}{2}} \\ U_{\frac{n}{2}} & -qU_{\frac{n}{2}-1} \end{bmatrix} \\ f^{(n,2)}_{(0,1)}(A) = -f^{(n,2)}_{(1,0)}(A) = \frac{1}{\sqrt{\Delta}} \begin{bmatrix} V_{\frac{n}{2}+1} & -qV_{\frac{n}{2}} \\ V_{\frac{n}{2}} & -qV_{\frac{n}{2}-1} \end{bmatrix}, \end{cases}$$

and

$$\begin{cases} f^{(n,2)}_{(0,0)}(B) = -f^{(n,2)}_{(1,1)}(B) = \frac{1}{2} \begin{bmatrix} V_{\frac{n}{2}} & U_{\frac{n}{2}} \\ \Delta U_{\frac{n}{2}} & V_{\frac{n}{2}} \end{bmatrix} \\ f^{(n,2)}_{(0,1)}(B) = -f^{(n,2)}_{(1,0)}(B) = \frac{\sqrt{\Delta}}{2} \begin{bmatrix} U_{\frac{n}{2}} & \frac{1}{\Delta}V_{\frac{n}{2}} \\ V_{\frac{n}{2}} & U_{\frac{n}{2}} \end{bmatrix}. \end{cases} \tag{7}$$

Proof. Since $\alpha+\beta = p$, for every $(k_1,k_2) \in \{0,1\}^2$, we have

$$f^{(n,2)}_{(k_1,k_2)}(A) = \frac{\exp[nk_1\pi i]}{\alpha-\beta}\alpha^{n/2} \begin{bmatrix} \alpha & -q \\ 1 & -\beta \end{bmatrix} - \frac{\exp[nk_2\pi i]}{\alpha-\beta}\beta^{n/2} \begin{bmatrix} \beta & -q \\ 1 & -\alpha \end{bmatrix}.$$

In the case $k_1 = k_2 = 0$, using the generalized Binet's formula (4), we obtain

$$f^{(n,2)}_{(0,0)}(A) = \frac{1}{\alpha-\beta} \begin{bmatrix} \alpha\alpha^{n/2} - \beta\beta^{n/2} & -q\alpha^{n/2} + q\beta^{n/2} \\ \alpha^{n/2} - \beta^{n/2} & -\beta\alpha^{n/2} + \alpha\beta^{n/2} \end{bmatrix} = \begin{bmatrix} U_{\frac{n}{2}+1} & -qU_{\frac{n}{2}} \\ U_{\frac{n}{2}} & -qU_{\frac{n}{2}-1} \end{bmatrix}.$$

In the case $k_1 = k_2 = 1$, we have $f^{(n,2)}_{(1,1)}(A) = -f^{(n,2)}_{(0,0)}(A)$. In the case $k_1 = 0, k_2 = 1$, it follows from (4) that,

$$f^{(n,2)}_{(0,1)}(A) = \frac{1}{\alpha-\beta} \begin{bmatrix} \alpha\alpha^{n/2} + \beta\beta^{n/2} & -q\alpha^{n/2} - q\beta^{n/2} \\ \alpha^{n/2} + \beta^{n/2} & -\beta\alpha^{n/2} - \alpha\beta^{n/2} \end{bmatrix} = \frac{1}{\sqrt{\Delta}} \begin{bmatrix} V_{\frac{n}{2}+1} & -qV_{\frac{n}{2}} \\ V_{\frac{n}{2}} & -qV_{\frac{n}{2}-1} \end{bmatrix}.$$

In the case $k_1 = 1, k_2 = 0$, we have $f^{(n,2)}_{(1,0)}(A) = -f^{(n,2)}_{(0,1)}(A)$. Since the matrix B has exactly the same eigenvalues as the matrix A, the matrix functions in the (7) are obtained by doing similar calculation for the matrix B. For simplicity, we omit the details. □

Theorem 3. *For every odd number n in $\mathbb{Z}$,*

(i) $U_{\frac{n}{2}+1}U_{\frac{n}{2}-1} - U^2_{\frac{n}{2}} = \pm q^{\frac{n}{2}-1}$,

(ii) $V_{\frac{n}{2}+1}V_{\frac{n}{2}-1} - V^2_{\frac{n}{2}} = \pm \Delta q^{\frac{n}{2}-1}$,

(iii) $V^2_{\frac{n}{2}} - \Delta U^2_{\frac{n}{2}} = \pm 4q^{\frac{n}{2}}$,

(iv) $U_{n+1} = U^2_{\frac{n}{2}+1} - qU^2_{\frac{n}{2}} = \frac{1}{\Delta}(V^2_{\frac{n}{2}+1} - qV^2_{\frac{n}{2}})$,

(v) $U_n = U_{\frac{n}{2}}U_{\frac{n}{2}+1} - qU_{\frac{n}{2}-1}U_{\frac{n}{2}} = \frac{1}{\Delta}(V_{\frac{n}{2}}V_{\frac{n}{2}+1} - qV_{\frac{n}{2}-1}V_{\frac{n}{2}}) = U_{\frac{n}{2}}V_{\frac{n}{2}}$,

(vi) $V_n = \frac{1}{2}(V^2_{\frac{n}{2}} + \Delta U^2_{\frac{n}{2}})$.

Proof. • Assertions (i), (ii), and (iii): Obviously, from the Theorem 1 and Theorem 2 for every $(k_1,k_2) \in \{0,1\}^2$ we have

$$f^{(n,2)}_{(k_1,k_2)}(A) \times f^{(n,2)}_{(k_1,k_2)}(A) = A^n \tag{8}$$

and

$$f^{(n,2)}_{(k_1,k_2)}(B) \times f^{(n,2)}_{(k_1,k_2)}(B) = B^n, \tag{9}$$

for any odd integer n. Hence, $\left[\det(f^{(n,2)}_{(k_1,k_2)}(A))\right]^2 = (\det A)^n$. Therefore,

$$\left[\det(f^{(n,2)}_{(k_1,k_2)}(A))\right]^2 = q^n \quad \text{and} \quad \det(f^{(n,2)}_{(k_1,k_2)}(A)) = \exp(i\ell\pi)q^{n/2} = \pm q^{n/2}, \quad \ell \in \{0,1\}.$$

• Assertions (iv), (v), (vi): Follows directly from the identities (8) and (9). □

Let consider the matrix functions defined as

$$\mathscr{F}^{(n,2)}_I(A) = \begin{cases} f^{(\ell)}(A), & \text{with } n = 2\ell, \text{ if } n \text{ is even} \\ f^{(n,2)}_{(0,0)}(A), & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

Therefore, without lost of generality, for any integer number $n \in \mathbb{Z}$ we may write

$$\mathscr{F}^{(n,2)}_I(A) = \begin{bmatrix} U_{\frac{n}{2}+1} & -qU_{\frac{n}{2}} \\ U_{\frac{n}{2}} & -qU_{\frac{n}{2}-1} \end{bmatrix}. \tag{11}$$

Lemma 4. *For every integer numbers n and m,*

$$\mathscr{F}^{(n,2)}_I(A) \times \mathscr{F}^{(m,2)}_I(A) = \mathscr{F}^{(n+m,2)}_I(A).$$

The proof of this Lemma is based on a fundamental property of matrix functions. Indeed, since A admits two distinct eigenvalues α and β, there exists an invertible matrix Z such that

$$A = Z \times J_A \times Z^{-1} = Z \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} Z^{-1},$$

where J_A designates the Jordan normal form associated to A. In fact, the matrix $f^{(n,2)}_{(k_1,k_2)}(A)$ may be defined as

$$f^{(n,2)}_{(k_1,k_2)}(A) = Z\left(f^{(n,2)}_{(k_1,k_2)}(J_A)\right)Z^{-1} = Z \begin{bmatrix} f^{(n,2)}_{k_1}(\alpha) & 0 \\ 0 & f^{(n,2)}_{k_2}(\beta) \end{bmatrix} Z^{-1},$$

where $(k_1,k_2) \in \{0,1\}^2$ [5], [8]. Consequently, by performing $\mathscr{F}^{(n,2)}_I(A) \times \mathscr{F}^{(m,2)}_I(A) = \mathscr{F}^{(n+m,2)}_I(A)$, the desired result is obtained. For simplicity's sake, we omit the details which will appear in a similar argument below.

Theorem 5. *For any integers n and m*

$$U_{\frac{n+m}{2}+1} = U_{\frac{n}{2}+1}U_{\frac{m}{2}+1} - qU_{\frac{n}{2}}U_{\frac{m}{2}}, \qquad U_{\frac{n+m}{2}} = U_{\frac{n}{2}}U_{\frac{m}{2}+1} - qU_{\frac{n}{2}-1}U_{\frac{m}{2}}.$$

Consider now

$$\mathscr{F}_{II}^{(n,2)}(A)=\begin{cases} f^{(\ell)}(A), & \text{with } n=2\ell, \text{ if } n \text{ is even} \\ f_{(0,1)}^{(n,2)}(A), & \text{if } n \text{ is odd.} \end{cases} \tag{12}$$

Let n and m be two integers, in the purpose of carrying out similar results as in the Theorem 5, two pertinent cases have to be considered:

(i) If n and m are both odd integer, then $n+m$ is even, thus

$$\mathscr{F}_{II}^{(n,2)}(A)\times\mathscr{F}_{II}^{(m,2)}(A)=f_{(0,1)}^{(n,2)}(A)\times f_{(0,1)}^{(m,2)}(A)=Z\begin{bmatrix}\alpha^{(n+m)/2} & 0\\ 0 & \beta^{(n+m)/2}\end{bmatrix}Z^{-1}$$

$$f^{(\ell)}(A)=\begin{bmatrix}U_{\ell+1} & -qU_{\ell}\\ U_{\ell} & -qU_{\ell-1}\end{bmatrix},\quad n+m=2\ell.$$

(ii) If n is odd and m is even, then $n+m$ is odd, thus

$$\mathscr{F}_{II}^{(n,2)}(A)\times\mathscr{F}_{II}^{(m,2)}(A)=f_{(0,1)}^{(n,2)}(A)\times f^{(\ell)}(A) \text{ with } m=2\ell$$

$$=Z\times\begin{bmatrix}\alpha^{n/2}\alpha^{m/2} & 0\\ 0 & -\beta^{n/2}\beta^{m/2}\end{bmatrix}\times Z^{-1}$$

$$f_{(0,1)}^{(n+m,2)}(A)=\frac{1}{\sqrt{\Delta}}\begin{bmatrix}V_{\frac{n+m}{2}+1} & -qV_{\frac{n+m}{2}}\\ V_{\frac{n+m}{2}} & -qV_{\frac{n+m}{2}-1}\end{bmatrix}.$$

Theorem 6. *Let n and m be two integer numbers.*

(i) *If n and m are both odd, then*

(a) $\Delta U_{\frac{n+m}{2}+1}=V_{\frac{n}{2}+1}V_{\frac{m}{2}+1}-qV_{\frac{n}{2}}V_{\frac{m}{2}}$,

(b) $\Delta U_{\frac{n+m}{2}}=V_{\frac{n}{2}}V_{\frac{m}{2}+1}-qV_{\frac{n}{2}-1}V_{\frac{m}{2}}$.

(ii) *If n is odd and m is even*

(a) $V_{\frac{n+m}{2}+1}=V_{\frac{n}{2}+1}U_{\frac{m}{2}+1}-qV_{\frac{n}{2}}U_{\frac{m}{2}}$,

(b) $V_{\frac{n+m}{2}}=V_{\frac{n}{2}}U_{\frac{m}{2}+1}-qV_{\frac{n}{2}-1}U_{\frac{m}{2}}$.

The results related to $f_{(1,0)}(A)$ and $f_{(1,1)}(A)$ are automatically covered by the above study, i.e., the investigation of these branches does not lead to new identities. Furthermore, some existing results in literature occur when n and m are both even. See for example [7], [9], [10], [14], [15], [16] and references therein.

Finally, we underline that if the matrix functions defined in (10), (11), and (12) are evaluated by substituting A by B, other identities can be obtained.

3. The Generalized Fibonacci and Lucas Sequences with Arbitrary Rational Subscript

Consider the scalar complex function $f^{(r,s)}(z)\equiv z^{r/s}$, where $(r,s)\in\mathbb{Z}^*\times\mathbb{N}^*$, such that $\frac{r}{s}$ is an irreducible fraction, i.e., $gcd(r,s)=1$. Recall that the matrices A and B are nonsingular with the same minimal polynomial $M_A(z) = M_B(z) = (z-\alpha)(z-\beta)$. Accordingly, there exist s^2 primary matrix function $A \mapsto A^{r/s}$, that may be determined by the expression

$$f^{(r,s)}_{(k_1,k_2)}(A) = \frac{\exp\left[\frac{2ik_1r\pi}{s}\right]\alpha^{r/s}}{\alpha-\beta}(A-\beta I_2) + \frac{\exp\left[\frac{2ik_2r\pi}{s}\right]\beta^{r/s}}{\beta-\alpha}(A-\alpha I_2), \tag{13}$$

where $k_1, k_2 \in \Re(s) = \{0,\dots,s-1\}$. Thus,

$$f^{(r,s)}_{(k_1,k_2)}(A) = \begin{bmatrix} \frac{K_1\alpha^{\frac{r}{s}+1}-K_2\beta^{\frac{r}{s}+1}}{\alpha-\beta} & -q\left(\frac{K_1\alpha^{\frac{r}{s}}-K_2\beta^{\frac{r}{s}}}{\alpha-\beta}\right) \\ \frac{K_1\alpha^{\frac{r}{s}}-K_2\beta^{\frac{r}{s}}}{\alpha-\beta} & -q\left(\frac{K_1\alpha^{\frac{r}{s}-1}-K_2\beta^{\frac{r}{s}-1}}{\alpha-\beta}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{K_1+K_2}{2}U_{\frac{r}{s}+1} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}+1} & -q\left(\frac{K_1+K_2}{2}U_{\frac{r}{s}} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}}\right) \\ \frac{K_1+K_2}{2}U_{\frac{r}{s}} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}} & -q\left(\frac{K_1+K_2}{2}U_{\frac{r}{s}-1} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}-1}\right) \end{bmatrix},$$

where $K_1 = \exp\left[\frac{2ik_1r\pi}{s}\right]$ and $K_2 = \exp\left[\frac{2ik_2r\pi}{s}\right]$.

Similarly, there exist s^2 primary matrix function $B \mapsto B^{r/s}$, that can be defined by the formula (13), i.e., by substituting A by B.

Theorem 7. *Let $r \in \mathbb{Z}^*$,and $s \in \mathbb{N}^*$ such that $\frac{r}{s}$ is an irreducible fraction, then*

$$f^{(r,s)}_{(k_1,k_2)}(A) = \begin{bmatrix} \frac{K_1+K_2}{2}U_{\frac{r}{s}+1} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}+1} & -q\left(\frac{K_1+K_2}{2}U_{\frac{r}{s}} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}}\right) \\ \frac{K_1+K_2}{2}U_{\frac{r}{s}} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}} & -q\left(\frac{K_1+K_2}{2}U_{\frac{r}{s}-1} + \frac{K_1-K_2}{2\sqrt{\Delta}}V_{\frac{r}{s}-1}\right) \end{bmatrix},$$

and

$$f^{(r,s)}_{(k_1,k_2)}(B) = \frac{1}{2}\begin{bmatrix} \sqrt{\Delta}\frac{K_1-K_s}{2}U_{\frac{r}{s}} + \frac{K_1+K_s}{2}V_{\frac{r}{s}} & \frac{K_1+K_s}{2}U_{\frac{r}{s}} + \frac{K_1-K_s}{2\sqrt{\Delta}}V_{\frac{r}{s}} \\ \Delta\frac{K_1+K_s}{2}U_{\frac{r}{s}} + \sqrt{\Delta}\frac{K_1-K_s}{2}V_{\frac{r}{s}} & \sqrt{\Delta}\frac{K_1-K_s}{2}U_{\frac{r}{s}} + \frac{K_1+K_s}{2}V_{\frac{r}{s}} \end{bmatrix},$$

where $K_1 = \exp\left[\frac{2ik_1r\pi}{s}\right]$ and $K_2 = \exp\left[\frac{2ik_2r\pi}{s}\right]$ and $k_1, k_2 \in \Re(s) = \{0,\dots,s-1\}$.

In the remainder of this section, we will focus on the principal branches of the previous matrix function:

$$f^{(r,s)}_{(0,0)}(A) = \begin{bmatrix} U_{\frac{r}{s}+1} & -qU_{\frac{r}{s}} \\ U_{\frac{r}{s}} & -qU_{\frac{r}{s}-1} \end{bmatrix} \quad \text{and} \quad f^{(r,s)}_{(0,0)}(B) = \frac{1}{2}\begin{bmatrix} V_{\frac{r}{s}} & U_{\frac{r}{s}} \\ \Delta U_{\frac{r}{s}} & V_{\frac{r}{s}} \end{bmatrix}.$$

Let $r_1, r_2 \in \mathbb{Z}^*$, and $s \in \mathbb{N}^*$. Then, it can be easily shown that:

(i) If $\frac{r_1}{s}$, $\frac{r_2}{s}$, and $\frac{r_1+r_2}{s}$ are all irreducible fractions, then

$$f^{(r_1,s)}_{(0,0)}(A) \times f^{(r_2,s)}_{(0,0)}(A) = f^{(r_1+r_2,s)}_{(0,0)}(A), \qquad f^{(r_1,s)}_{(0,0)}(B) \times f^{(r_2,s)}_{(0,0)}(B) = f^{(r_1+r_2,s)}_{(0,0)}(B).$$

(ii) If $\frac{r_1}{s}$ is any irreducible fraction and $\ell = \frac{r_2}{s} \in \mathbb{N}^*$, then

$$f^{(r_1,s)}_{(0,0)}(A) \times f^{(r_2,s)}(A) = f^{(r_1+r_2,s)}_{(0,0)}(A),$$

$$\begin{bmatrix} U_{\frac{r_1}{s}+1}U_{\frac{r_2}{s}+1} - qU_{\frac{r_1}{s}}U_{\frac{r_2}{s}} & -qU_{\frac{r_1}{s}+1}U_{\frac{r_2}{s}} + q^2U_{\frac{r_1}{s}}U_{\frac{r_2}{s}-1} \\ U_{\frac{r_1}{s}}U_{\frac{r_2}{s}+1} - qU_{\frac{r_1}{s}-1}U_{\frac{r_2}{s}} & -qU_{\frac{r_1}{s}}U_{\frac{r_2}{s}} + q^2U_{\frac{r_1}{s}-1}U_{\frac{r_2}{s}-1} \end{bmatrix} = \begin{bmatrix} U_{\frac{r_1+r_2}{s}+1} & -qU_{\frac{r_1+r_2}{s}} \\ U_{\frac{r_1+r_2}{s}} & -qU_{\frac{r_1+r_2}{s}-1} \end{bmatrix},$$

and

$$f^{(r_1,s)}_{(0,0)}(B)\times f^{(r_2,s)}(B)=f^{(r_1+r_2,s)}_{(0,0)}(B),$$

$$\frac{1}{2}\begin{bmatrix} V_{\frac{r_1}{s}}V_{\frac{r_2}{s}}+\Delta U_{\frac{r_1}{s}}U_{\frac{r_2}{s}} & V_{\frac{r_1}{s}}U_{\frac{r_2}{s}}+U_{\frac{r_1}{s}}V_{\frac{r_2}{s}} \\ \Delta\left(U_{\frac{r_1}{s}}V_{\frac{r_2}{s}}+V_{\frac{r_1}{s}}U_{\frac{r_2}{s}}\right) & \Delta U_{\frac{r_1}{s}}U_{\frac{r_2}{s}}+V_{\frac{r_1}{s}}V_{\frac{r_2}{s}} \end{bmatrix}=\begin{bmatrix} V_{\frac{r_1+r_2}{s}} & U_{\frac{r_1+r_2}{s}} \\ \Delta U_{\frac{r_1+r_2}{s}} & V_{\frac{r_1+r_2}{s}} \end{bmatrix}.$$

The following theorem summarizes the previous discussion.

Theorem 8. *Let consider* $r_1,r_2,\in\mathbb{Z}^*$, *and* $s\in\mathbb{N}^*$.

(i) *If* $\frac{r_1}{s}$, $\frac{r_2}{s}$, *and* $\frac{r_1+r_2}{s}$ *are irreducible fractions, then*

(a) $U_{\frac{r_1+r_2}{s}+1}=U_{\frac{r_1}{s}+1}U_{\frac{r_2}{s}+1}-qU_{\frac{r_1}{s}}U_{\frac{r_2}{s}}$

(b) $U_{\frac{r_1+r_2}{s}-1}=U_{\frac{r_1}{s}}U_{\frac{r_2}{s}}-qU_{\frac{r_1}{s}-1}U_{\frac{r_2}{s}-1}$

(c) $U_{\frac{r_1+r_2}{s}}=U_{\frac{r_1}{s}}U_{\frac{r_2}{s}+1}-qU_{\frac{r_1}{s}-1}U_{\frac{r_2}{s}}$

(d) $U_{\frac{r_1+r_2}{s}}=U_{\frac{r_1}{s}+1}U_{\frac{r_2}{s}}-qU_{\frac{r_1}{s}}U_{\frac{r_2}{s}-1}$

(e) $U_{\frac{r_1+r_2}{s}}=\frac{1}{2}\left(V_{\frac{r_1}{s}}U_{\frac{r_2}{s}}+U_{\frac{r_1}{s}}V_{\frac{r_2}{s}}\right)$

(f) $V_{\frac{r_1+r_2}{2}}=\frac{1}{2}\left(V_{\frac{r_1}{s}}V_{\frac{r_2}{s}}+\Delta U_{\frac{r_1}{s}}U_{\frac{r_2}{s}}\right)$

(ii) *If* $\frac{r_1}{s}$ *is any irreducible fraction and* $\ell=\frac{r_2}{s}\in\mathbb{N}^*$, *then*

(a) $U_{\frac{r_1}{s}+\ell+1}=U_{\frac{r_1}{s}+1}U_{\ell+1}-qU_{\frac{r_1}{s}}U_{\ell}$

(b) $U_{\frac{r_1}{s}+\ell}=U_{\frac{r_1}{s}+1}U_{\ell}-qU_{\frac{r_1}{s}}U_{\ell-1}$

(c) $U_{\frac{r_1}{s}+\ell}=U_{\frac{r_1}{s}}U_{\ell+1}-qU_{\frac{r_1}{s}-1}U_{\ell}$

(d) $U_{\frac{r_1}{s}+\ell-1}=U_{\frac{r_1}{s}}U_{\ell}-qU_{\frac{r_1}{s}-1}U_{\ell-1}$

(e) $U_{\frac{r_1}{s}+\ell}=\frac{1}{2}\left(V_{\frac{r_1}{s}}U_{\ell}+U_{\frac{r_1}{s}}V_{\ell}\right)$

(f) $V_{\frac{r_1}{s}+\ell}=\frac{1}{2}\left(V_{\frac{r_1}{s}}V_{\ell}+\Delta U_{\frac{r_1}{s}}U_{\ell}\right)$

(g) $U_{\frac{r_1}{s}+\ell+1}=U_{\frac{r_1}{s}+1}U_{\ell+1}-qU_{\frac{r_1}{s}}U_{\ell}$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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