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$Y\overline{Y}$ Domination in Bipartite Graphs

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Abstract. Let G be a bipartite graph. A subset S of X is called a $Y\overline{Y}$ dominating set if S is a Y-dominating set and $X-S$ is not a Y-dominating set. A subset S of X is called a minimal $Y\overline{Y}$ dominating set if any proper subset of S is not a $Y\overline{Y}$ dominating set. The minimum cardinality of a minimal $Y\overline{Y}$ dominating set is called the $Y\overline{Y}$ domination number of G and is denoted by $\gamma_{Y\overline{Y}}(G)$. In this paper some results on $Y\overline{Y}$ domination number are obtained.

1. Introduction

Let G be a graph. Let D be a dominating set of a graph G. If $\langle V-D\rangle$ is connected, D is called a non-split dominating set and if $\langle V-D\rangle$ is disconnected, then D is a split dominating set. These concepts were introduced by [1, 2] Kulli and Janakiram. In a similar fashion the concept of complementary nil domination number of a graph was introduced by [6] Tamizh Chelvam et al. We introduce the concept of $Y\overline{Y}$-dominating set in bipartite graph. Let $G=(X,Y,E)$ be a bipartite graph. The bipartite theory of graphs were introduced in [4, 5] and the parameters called X-domination number and Y-domination number were introduced. Two vertices u,v in X are X-adjacent if they are adjacent to a common vertex in Y. A subset D of X is an X-dominating set if every vertex in $X-D$ is X-adjacent to at least one vertex in D. A X-dominating set [4] S is a minimal X-dominating set if no proper subset of S is X-dominating set. The minimum cardinality of a minimal X-dominating set is called the X-domination number of G and is denoted by $\gamma_X(G)$. A subset $S\subseteq X$ which dominates all vertices in Y is called a Y-dominating set [4] of G. The Y-domination number denoted by $\gamma_Y(G)$ is the minimum cardinality of a Y-dominating set of G. A subset S of X is hyper independent [4] if there does not exist a vertex $y\in Y$ such that $N(y)\subseteq S$. The maximum cardinality of a hyper independent set of G is denoted by $\beta_h(G)$. The complement of G [3] denoted by $\overline{G}=(X,Y,E'')$ is defined as follows: (i) No two vertices in X are adjacent. (ii) No

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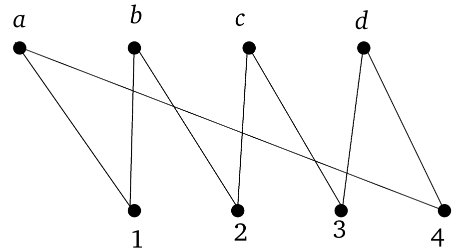
two vertices in Y are adjacent. (iii) $x \in X$ and $y \in Y$ are adjacent in $\overline{G}$ if and only if $x \in X$ and $y \in Y$ are not adjacent in G.

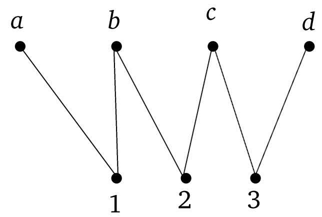
2. $Y\overline{Y}$ Dominating Set

Definition 1. A subset S of X is called a $Y\overline{Y}$ dominating set if S is a Y-dominating set and $X - S$ is not a Y-dominating set. A subset S of X is called a minimal $Y\overline{Y}$ dominating set if any proper subset of S is not a $Y\overline{Y}$ dominating set. The minimum cardinality of a minimal $Y\overline{Y}$ dominating set is called the $Y\overline{Y}$ domination number of G and is denoted by $\gamma_{Y\overline{Y}}(G)$.

Example 1. In the graph G, $S = \{b, d\}$ is a Y-dominating set but not a $Y\overline{Y}$-dominating set.



Example 2. In the graph, $S = \{b, c\}$ is a $Y\overline{Y}$-dominating set.



Observation 1. $\gamma_Y(G) \leq \gamma_{Y\overline{Y}}(G)$.

Remark 1. If Y contains a vertex of degree one then any Y-dominating set is a $Y\overline{Y}$-dominating set.

Hence, we consider bipartite graph $G = (X, Y, E)$ in which (i) every vertex in Y is of degree at least two. (ii) every vertex in X is not a full degree vertex. Vertex $x \in X$ is called a full degree vertex if x is adjacent to every vertex of Y.

Theorem 1. *A Y-dominating set S of a bipartite graph G is a $Y\overline{Y}$-dominating set of G if and only if S is not hyper independent set.*

Proof. A Y-dominating set S is such that S is not hyper independent set. There exists a $y \in Y$ such that $N(y) \subseteq S$. The vertex y is not adjacent to any vertex in $X - S$. Therefore, $X - S$ is not a Y-dominating set. Hence, S is a $Y\overline{Y}$-dominating set of G.

Conversely, let S be a $Y\overline{Y}$-dominating set. That is, S is a Y-dominating set and $X - S$ is not a Y-dominating set. There exists $y \in Y$ not adjacent to any vertex in

$X-S$. Equivalently, there exists $y \in Y$ such that $N(y) \subseteq S$. Therefore, S is not a hyper independent set. □

Theorem 2. *A subset S of X is a $Y\overline{Y}$-dominating set if and only if* (i) *$X-S$ is hyper independent set* (ii) *S is not hyper independent set.*

Proof. Let $S \subseteq X$ be a $Y\overline{Y}$-dominating set. Then S is a Y-dominating set. By Theorem 1, S is not hyper independent set. Every $y \in Y$ is adjacent to at least one vertex of S. That is $N(y) \nsubseteq X-S$, $\forall y \in Y$. Therefore, $X-S$ is a hyper independent set.

Conversely, a subset S of X satisfies conditions (i) and (ii). Since $X-S$ is hyper independent set, for every $y \in Y$, $N(y) \nsubseteq X-S$. Therefore, every vertex $y \in Y$ is adjacent to a vertex of S. Hence, S is a Y-dominating set. By condition (ii) and by theorem 1, S is a $Y\overline{Y}$-dominating set of G. □

Proposition 1. *Let G be a graph, every $\gamma_{Y\overline{Y}}$-set intersects with every γ_Y-set of G.*

Proof. Let D be a $\gamma_{Y\overline{Y}}$-set and D_1 be a γ_Y-set of G. Suppose that $D \cap D_1 = \phi$, then $D_1 \subseteq X-D$, $X-D$ contains a Y-dominating set D_1. Therefore, $X-D$ itself is a Y-dominating set, which is a contradiction. □

Theorem 3. *Let D be a $Y\overline{Y}$- dominating set of a graph G. Then D is minimal if and only if for each $u \in D$ one of the following conditions is satisfied:*

(i) *u has a private neighbour.*
(ii) *$X-(D-\{u\})$ is a Y-dominating set of G.*

Proof. Suppose D is a minimal $Y\overline{Y}$-dominating set of G. Then $D-\{u\}$ is not a $Y\overline{Y}$-dominating set. That is $D-\{u\}$ is not a Y-dominating set or $X-(D-\{u\})$ is a Y-dominating set. If $(X-(D-\{u\}))$ is a Y-dominating set of G, we get (ii). If $D-\{u\}$ is not a Y-dominating set, there exists $y \in Y$ not adjacent to any vertex in $D-\{u\}$ but adjacent to u. Hence, u has a private neighbour, condition (i) holds.

Conversely, assume conditions (i) and (ii) hold. Let D be a $Y\overline{Y}$-dominating set of G. By condition (i) $u \in S$ has a private neighbour. Then $D-\{u\}$ is not a Y-dominating set. Therefore, D is a minimal $Y\overline{Y}$-dominating set. For some $u \in D$, $X-(D-\{u\})$ is a Y-dominating set of G, then $D-\{u\}$ is not a $Y\overline{Y}$-dominating set of G. Hence, D is a minimal $Y\overline{Y}$-dominating set of G. □

3. **Bounds for $Y\overline{Y}$-domination number**

Theorem 4. *For any graph G with $p \geq 2$, $\gamma_{Y\overline{Y}}(G) \leq p-1$.*

Proof. Every vertex in X is not a full degree vertex. Therefore, there exists a vertex $x \in X$ with degree less than $|Y|$. Let the vertex be x. Then, $X-\{x\}$ is a $Y\overline{Y}$-dominating set of G. Therefore, $\gamma_{Y\overline{Y}}(G) \leq |X-\{x\}| = p-1$. □

Let $\delta_X(G)$ denote the minimum number of edges incident with vertices of Y.

Theorem 5. *For any graph G, $\delta_X(G) \leq \gamma_{Y\overline{Y}}(G) \leq \gamma_Y(G) + \delta_X(G) - 1$.*

Proof. Let S be a $\gamma_{Y\overline{Y}}$-set of G. Since $X - S$ is not a Y-dominating set of G, there exists a vertex $y \in Y$ such that $N(y) \subseteq S$. That is, $\delta_X(G) \leq |N(y)| \leq |S|$ and hence $\delta_X(G) \leq \gamma_{Y\overline{Y}}(G)$. Let D be a γ_Y-dominating set of G. Let $y \in Y$ be a vertex such that $d_X(y) = \delta_X(G)$. Then at least one vertex $x_1 \in N(y)$ is in D. Now $D_1 = D \cup (N(y) - \{x_1\})$ is a $Y\overline{Y}$-dominating set. Hence, $\gamma_{Y\overline{Y}}(G) \leq |D| + |N(y)| - 1 \leq \gamma_Y(G) + \delta_X(G) - 1$. □

4. Particular values of $Y\overline{Y}$-domination number

Theorem 6. *If G is a connected graph, then $\gamma_{Y\overline{Y}}(G) = p - 1$ if and only if $\delta_X(G) = p - 1$.*

Proof. Suppose $\gamma_{Y\overline{Y}}(G) = p - 1$. Let us assume $\delta_X(G) \leq p - 2$. Then there exists a vertex $y \in Y$ not adjacent to two vertices x_1, x_2. Then, $X - \{x_1, x_2\}$ is a $Y\overline{Y}$-dominating set. Therefore, $\gamma_{Y\overline{Y}}(G) \leq p - 2$, a contradiction. Therefore, $\delta_X(G) = p - 1$.

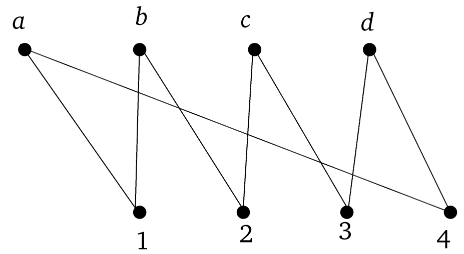
Conversely, $\delta_X(G) \leq \gamma_{Y\overline{Y}}(G) \leq p - 1$. Therefore, $p - 1 \leq \gamma_{Y\overline{Y}}(G) \leq p - 1$. Hence, $\gamma_{Y\overline{Y}}(G) = p - 1$. □

Corollary 1. $\gamma_{Y\overline{Y}}(K_{m,n} - e) = m - 1$ *and* $\gamma_{Y\overline{Y}}(\overline{mK_2}) = m - 1$

Theorem 7. *For any graph G, if $\gamma_Y(G) = 1$ and $\delta_X(G) = 2$ then $\gamma_{Y\overline{Y}}(G) = 2$.*

Proof. $\gamma_Y(G) = 1$ and $\delta_X(G) = 2$ in theorem:5, we get $\gamma_{Y\overline{Y}}(G) = 2$. □

Remark 2. Converse of the above need not be true. Consider the graph



$S = \{b, c\}$ is a $Y\overline{Y}$-dominating set. $\delta_X(G) = 2$ and $\gamma_Y(G) = 2$.

5. Bipartite theory of $Y\overline{Y}$-dominating set

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is said to be a cnd-set of a graph G if it is dominating set and its complement $V - S$ is not a dominating set. The minimum cardinality of a cnd-set is called the [6]complementary nil domination number of G and is denoted by $\gamma_{cnd}(G)$.

Theorem 8. *For any graph G, $\gamma_{Y\overline{Y}}(VV^+(G)) = \gamma_{cnd}(G)$.*

Proof. Let S be a $\gamma_{Y\overline{Y}}$_set of $VV^+(G) = (X, Y, E)$. Then S is a Y-dominating set in $VV^+(G)$ and $X - S$ is not a Y-dominating set in $VV^+(G)$. In G, S is a dominating set and $X - S$ is not a dominating set. That is S is complementary nil dominating set. Hence, $\gamma_{cnd}(G) \leq |S| = \gamma_{Y\overline{Y}}(VV^+(G))$.

Conversely, let D be a γ_{cnd}-set of G. Then D is a dominating set of G and $V - D$ is not a dominating set of G. In the graph $VV^+(G)$, D is a Y-dominating set and $X - D$ is not a Y-dominating set. Therefore, $\gamma_{Y\overline{Y}}(VV^+(G)) \leq |D| = \gamma_{cnd}(G)$. □

A set $S \subseteq V$ is said to be a cntd-set of a graph G if it is total dominating set and its complement $V - S$ is not a total dominating set. The minimum cardinality of a cntd-set is called the complementary nil total domination number of G and is denoted by $\gamma_{cntd}(G)$.

Theorem 9. *For any graph G, $\gamma_{Y\overline{Y}}(VV(G)) = \gamma_{cntd}(G)$.*

Proof. Let S be a $\gamma_{Y\overline{Y}-}$set of $VV(G) = (X, Y, E)$. Then S is a Y-dominating set in $VV(G)$ and $X - S$ is not a Y-dominating set in $VV(G)$. In G, S is a total dominating set and $X - S$ is not a total dominating set. That is S is complementary nil total dominating set. Hence, $\gamma_{cntd}(G) \leq |S| = \gamma_{Y\overline{Y}}(VV(G))$.

Conversely, let D be a γ_{cntd}-set of G. Then D is a total dominating set of G and $V - D$ is not a total dominating set of G. In the graph $VV(G)$, D is a Y-dominating set and $X - D$ is not a Y-dominating set. Therefore, $\gamma_{Y\overline{Y}}(VV(G)) \leq |D| = \gamma_{cntd}(G)$. □

References

[1] V.R. Kulli and B. Janakiram, The split domination number of a graph, Graph theory notes of New York, XXXII (1997), 16–19.

[2] V.R. Kulli and Janakiram, The nonsplit domination number of a graph, Indian J. Pure Appl. Math. **31**(4) (2000), 441–447.

[3] Sampathkumar and L. Pusphalatha, Generalized complements of a graph, Indian J. Pure Appl. Math. **29**(6) (1989), 635–639.

[4] S. Hedetniemi and R. Laskar, A Bipartite theory of graphs I, Congressus Numerantium **55** (December 1986), 5–14.

[5] S. Hedetniemi and R. Laskar, A Bipartite theory of graphs II, Congressus Numerantium **64** (November 1988), 137–146.

[6] T. Tamizh Chelvam and S. Robinson Chellathurai, Complementary nil domination number of a graph, Tamkang Journal of Mathematics **40**(2) (2009), 165–172.

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