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Remark on a Theorem of Prolla

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Abstract. In this paper, we obtained the form of the best approximation theorem of Prolla. Our result extends some results in literature.

1. Introduction and Preliminaries

In 1969, Ky Fan [2] established the famous best approximation theorem.

Theorem 1.1 (Ky Fan, [2]). *Let C be a nonempty, compact, convex subset of a normed linear space X. Then for any continuous mapping f from C to X, exists a point $x_0 \in C$ with*

$$\|x_0 - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.$$

This result has been generalized to other spaces X and other types of maps, see for example [4], [5], [7]. Prolla [6] and Carbone [1] obtained a form of the best approximation theorem of Ky Fan using almost affine and almost quasi-convex maps in normed vector spaces.

Definition 1.2. Let X be a normed space and C a nonempty convex subset of X.

(i) A map $g : C \to X$ is almost affine if for all $x, y \in C$ and $u \in C$

$$\|g(\lambda x + (1-\lambda)y) - u\| \le \lambda \|g(x) - u\| + (1-\lambda)\|g(y) - u\|,$$

for each λ with $0 < \lambda < 1$.

(ii) A map $g : C \to X$ is almost quasi-convex if for all $x, y \in C$ and $u \in C$

$$\|g(\lambda x + (1-\lambda)y) - u\| \le \max\{\|g(x) - u\|, \|g(y) - u\|\}$$

for each λ with $0 < \lambda < 1$.

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Theorem 1.3 (J.B. Prolla, [6]). *Let X be a normed linear space, C a nonempty convex compact subset of X, $f : C \to X$ is a continuous map and $g : C \to C$ is a continuous, almost affine, onto map. Then there exists a point $x_0 \in C$, such that*

$$\|g(x_0) - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.$$

Carbone [1] obtain the version of Theorem 1.3 using almost quasi-convex maps.

We extended results of Prolla [6] and Carbone [1].

For a nonempty subset A of X, let $\operatorname{co} A$ denote the convex hull of A.

Definition 1.4. Let C be a nonempty subset of a topological vector space X. A map $H : C \to 2^X$ is called KKM map if for every finite set $\{x_1, \ldots, x_n\} \subset C$, we have

$$\operatorname{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} H(x_k).$$

The following extension of the classical KKM principle is due to Ky Fan [3].

Theorem 1.5. [3] *Let X be a topological vector space, K be a nonempty subset of X and $H : K \to 2^X$ a map with closed values and KKM map. If $H(x)$ is compact for at least one $x \in K$ then $\bigcap_{x \in X} H(x) \neq \emptyset$.*

2. Main Result

From Theorem 1.5 we obtain the following best approximation theorem in normed spaces.

Theorem 2.1. *Let X be a normed linear space, C a nonempty convex compact subset of X, $f : C \to X$ and $g : C \to C$ continuous maps. If there exists an almost quasi-convex onto map $h : C \to C$ such that*

$$\|g(x) - f(x)\| \leq \|h(x) - f(x)\| \quad \textit{for each } x \in C, \tag{2.1}$$

then there exists a point $x_0 \in C$ such that

$$\|g(x_0) - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.$$

Proof. Let for every $y \in C$, $H : C \to 2^C$ be defined by

$$H(y) = \{x \in C : \|g(x) - f(x)\| \leq \|h(y) - f(x)\|\}.$$

We have that $H(y)$ is nonempty for all $y \in C$, because $y \in H(y)$ for all $y \in C$. Since f and g are continuous maps, then $H(y)$ is closed for all $y \in C$. Now, we show that for each finite set $\{x_1, \ldots, x_n\} \subset C$,

$$\operatorname{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} H(x_k). \tag{2.2}$$

Suppose that

$$\operatorname{co}\{x_1,\ldots,x_n\} \not\subseteq \bigcup_{k=1}^{n} H(x_k) \quad \text{for some } \{x_1,\ldots,x_n\} \subset C.$$

Then there exists $y_0 \in \operatorname{co}\{x_1,\ldots,x_n\}$ such that $y_0 \notin H(x_k)$ for each $k \in \{1,\ldots,n\}$. So, we have

$$\|g(y_0) - f(y_0)\| > \|h(x_k) - f(y_0)\| \quad \text{for each } k \in \{1,\ldots,n\}.$$

Therefore,

$$\|g(y_0) - f(y_0)\| > \max_k \|h(x_k) - f(y_0)\| \geq \|h(y_0) - f(y_0)\|.$$

This is a contradiction with condition (2.1). Hence, condition (2.2) is true for each finite $\{x_1,\ldots,x_n\} \subset C$ and a map H is KKM map. Now, from Theorem 1.5 it follows that there exists $y_0 \in C$ such that

$$y_0 \in \bigcap_{y \in C} H(y).$$

Therefore,

$$\|g(y_0) - f(y_0)\| = \inf_{x \in C} \|x - f(y_0)\|.$$

□

Example 2.2. Let $C = [0,1]$ and define maps $f, g, h : C \to C$ by

$$f(x) = 0,$$
$$h(x) = x,$$
$$g(x) = \begin{cases} x, & x \in \left[0, \frac{1}{4}\right); \\ -x + \frac{1}{2}, & x \in \left[\frac{1}{4}, \frac{1}{2}\right); \\ 2x - 1, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then map g is not almost quasi-convex and results of J.B. Prolla and A. Carbone are not applicable. Note that the maps f, g and h satisfy all hypotheses of Theorem 2.1.

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