**Journal of Informatics and Mathematical Sciences**
Vol. 9, No. 1, pp. 111–122, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Research Article

# Some Fixed Point Theorems for Expansive Mappings in Cone Pentagonal Metric Spaces

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**Abstract.** In this paper, we prove some fixed point theorems for mappings satisfying expansive conditions in non-normal cone pentagonal metric spaces. Our results extend and improve the recent results announced by Patil and Salunke [Fixed point theorems for expansion mappings in cone rectangular metric spaces, *Gen. Math. Notes* **29** (1) (2015), 30–39], Shatanawi and Awawdeh [Some fixed and coincidence point theorems for expansive maps in cone metric spaces, *Fixed Point Theory and Applications* **1** (2012), 1–10], Huang, Zhu and Wen [Fixed point theorems for expanding mappings in cone metric spaces, *Math. Reports* **14** (64) (2) (2012), 141–148], Kadelburg, Murthy and Radenovic [Common fixed points for expansive mappings in cone metric spaces, *Int. J. Math. Anal.* **5** (27) (2011), 1309–1319], Aage and Salunke [Some fixed point theorems for expansion onto mappings on cone metric spaces, *Acta Mathematica Sinica* **27** (6) (2011), 1101–1106], Kumar and Garg [Common fixed points for expansion mappings theorems in metric spaces, *Int. J. Contemp. Math. Sciences* **4** (36) (2009), 1749–1758], and many others in the literature.

**Keywords.** Cone pentagonal metric spaces; Common fixed point; Expansive maps; Weakly compatible maps

**MSC.** 47H10; 54H25

**Received:** February 1, 2017 **Accepted:** March 15, 2017



## 1. Introduction

Fixed point theory is one of the most popular tool in nonlinear analysis. The study of existence and uniqueness of fixed point of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the well known Banach contraction principle in various generalized metric spaces (e.g., see [4, 6, 7]). However, expanding mappings have enjoyed a relatively lower popularity with the results of Wang et al. [15], Daffer and Kaneko [5], Kumar and Garg [11] among others.

Long-Guang and Xian [7] introduced the concept of a cone metric space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [2, 4, 6, 13]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Garg and Agarwal [6] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Motivated and inspired by the results of [6, 12], it is our purpose in this paper to continue the study of common fixed points of a two self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [1, 14, 10, 8, 12, 11], and many others in the literature.

## 2. Preliminaries

The following definitions and Lemmas are needed in the sequel.

**Definition 2.1** (Huang and Zhang [7]). Let $E$ be a real Banach space and $P$ subset of $E$. $P$ is called a cone if and only if:

(1) $P$ is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of $P$.
In this paper, we always suppose that $E$ is a real Banach space and $P$ is a cone in $E$ with $\text{int}(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$.

**Definition 2.2** (Huang and Zhang [7]). Let $X$ be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then $\rho$ is called a cone metric on $X$, and $(X, \rho)$ is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [7]).

**Definition 2.3** (Azam, Arshad and Beg [4]). Let $X$ be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

(3) $\rho(x,y) \le \rho(x,w) + \rho(w,z) + \rho(z,y)$ for all $x,y,z \in X$ and for all distinct points $w,z \in X - \{x,y\}$ [Rectangular property].

Then $\rho$ is called a cone rectangular metric on $X$, and $(X,\rho)$ is called a cone rectangular metric space.

**Remark 2.4.** Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [4]).

**Definition 2.5** (Garg and Agarwal [6]). Let $X$ be a non empty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x,y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for $x,y \in X$;

(3) $\rho(x,y) \le \rho(x,z) + \rho(z,w) + \rho(w,u) + \rho(u,y)$ for all $x,y,z,w,u \in X$ and for all distinct points $z,w,u, \in X - \{x,y\}$ [Pentagonal property].

Then $\rho$ is called a cone pentagonal metric on $X$, and $(X,\rho)$ is called a cone pentagonal metric space.

**Remark 2.6.** Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [6]).

**Definition 2.7.** Let $(X,\rho)$ be a cone pentagonal metric space and $S : X \to X$ be a mapping. Then $S$ is called expansive - contraction if there exists a real constant $k > 1$ such that

$$\rho(Sx,Sy) \ge k\rho(x,y), \quad \text{for all } x,y \in X.$$

Let $(X,\rho)$ be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $\rho(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to $x$, and $x$ is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$, $\rho(x_n,x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in $X$.
If every Cauchy sequence is convergent in $(X,\rho)$, then $X$ is called a complete cone pentagonal metric space.

**Lemma 2.8** (Auwalu [3]). *Let $(X,\rho)$ be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in $X$ and suppose that there is natural number $N$ such that:*

(1) *$x_n \ne x_m$ for all $n,m > N$;*

(2) $x_n, x$ *are distinct points in* $X$ *for all* $n > N$;

(3) $x_n, y$ *are distinct points in* $X$ *for all* $n > N$;

(4) $x_n \to x$ *and* $x_n \to y$ *as* $n \to \infty$.

*Then* $x = y$.

Let $T$ and $S$ be self maps of a nonempty set $X$. If $w = Tx = Sx$ for some $x \in X$, then $x$ is called a coincidence point of $T$ and $S$ and $w$ is called a point of coincidence of $T$ and $S$. Also, $T$ and $S$ are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

**Lemma 2.9** (Abbas and Jungck [2]). *Let* $T$ *and* $S$ *be weakly compatible self mappings of nonempty set* $X$. *If* $T$ *and* $S$ *have a unique point of coincidence* $w = Tx = Sx$, *then* $w$ *is the unique common fixed point of* $T$ *and* $S$.

**Lemma 2.10** (Jungck et al. [9]). *Let* $(X,\rho)$ *be a cone metric space with cone* $P$ *not necessary to be normal. Then for* $a, c, u, v, w \in E$, *we have*

(1) *If* $a \leq ka$ *and* $k \in [0,1)$, *then* $a = 0$.

(2) *If* $0 \leq u \ll c$ *for each* $0 \ll c$, *then* $u = 0$.

(3) *If* $u \leq v$ *and* $v \ll w$, *then* $u \ll w$.

(4) *If* $c \in \operatorname{int}(P)$ *and* $a_n \to 0$, *then* $\exists n_0 \in \mathbb{N} : \forall n > n_0$, $a_n \ll c$.

## 3. Main Results

In this section, we prove some fixed point theorems for expansive mappings in cone pentagonal metric space. We give an example to illustrate the results.

**Theorem 3.1.** *Let* $(X,\rho)$ *be a cone pentagonal metric space. Suppose the mappings* $S, T : X \to X$ *satisfies the condition:*

$$\rho(Sx, Sy) \geq k\rho(Tx, Ty), \ \forall x, y \in X, \tag{3.1}$$

*where* $k > 1$ *is a constant. Suppose that* $T(X) \subseteq S(X)$, *and either of* $S(X)$ *or* $T(X)$ *is complete, then* $S$ *and* $T$ *have a unique point of coincidence in* $X$. *Moreover, if* $S$ *and* $T$ *are weakly compatible, then they have a unique common fixed point in* $X$.

*Proof.* Let $x_0$ be arbitrary point in $X$. Since $T(X) \subseteq S(X)$, we can choose a point $x_1$ in $X$ such that $Tx_0 = Sx_1$. Continuing in this way, we construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$y_n = Tx_n = Sx_{n+1} \text{ for all } n = 0, 1, 2, \cdots.$$

Suppose that $y_k = y_{k-1}$ for some $k \in \mathbb{N}$, then $Sx_k = Tx_k$. Thus, $x_k$ is a coincidence point of $S$ and $T$. Hence, assume that $y_n \neq y_{n-1}$, for each $n \in \mathbb{N}$. Then using (3.1), we have that

$$\rho(y_n, y_{n-1}) = \rho(Sx_{n+1}, Sx_n)$$

$$\geq k\rho(Tx_{n+1}, Tx_n)$$
$$= k\rho(y_{n+1}, y_n).$$

Hence,

$$\rho(y_n, y_{n+1}) \leq K\rho(y_{n-1}, y_n), \text{ where } K = \frac{1}{k} \in (0,1). \tag{3.2}$$

By (3.2) it follows, for all $n = 0, 1, 2, \cdots$, that

$$\rho(y_n, y_{n+1}) \leq K\rho(y_{n-1}, y_n) \leq K^2\rho(y_{n-2}, y_{n-1}) \leq \cdots \leq K^n\rho(y_0, y_1). \tag{3.3}$$

Observe that, by (3.1),

$$\begin{aligned}\rho(y_{n+1}, y_{n-1}) &= \rho(Sx_{n+2}, Sx_n)\\ &\geq k\rho(Tx_{n+2}, Tx_n)\\ &= k\rho(y_{n+2}, y_n).\end{aligned}$$

This implies that

$$k\rho(y_n, y_{n+2}) \leq \rho(y_{n-1}, y_{n+1}).$$

Hence,

$$\rho(y_n, y_{n+2}) \leq \frac{1}{k}\rho(y_{n-1}, y_n).$$

That is,

$$\rho(y_n, y_{n+2}) \leq K\rho(y_{n-1}, y_n). \tag{3.4}$$

Also, by (3.4) it follows, for all $n = 0, 1, 2, \cdots$, that

$$\rho(y_n, y_{n+2}) \leq K\rho(y_{n-1}, y_n) \leq K^2\rho(y_{n-2}, y_{n-1}) \leq \cdots \leq K^n\rho(y_0, y_1). \tag{3.5}$$

For the sequence $\{y_n\}$, we consider $\rho(y_n, y_{n+p})$ in two cases as follows:
If $p = 2k+1$, where $k \geq 1$, then by pentagonal property and (3.3), we have

$$\begin{aligned}\rho(y_n, y_{n+2k+1}) &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \rho(y_{n+2}, y_{n+3}) + \rho(y_{n+3}, y_{n+2k+1})\\ &\leq \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_{n+2}) + \rho(y_{n+2}, y_{n+3}) + \cdots\\ &\qquad + \rho(y_{n+2k-1}, y_{n+2k}) + \rho(y_{n+2k}, y_{n+2k+1})\\ &\leq K^n\rho(y_0, y_1) + K^{n+1}\rho(y_0, y_1) + K^{n+2}\rho(y_0, y_1) + \cdots\\ &\qquad + K^{n+2k-1}\rho(y_0, y_1) + K^{n+2k}\rho(y_0, y_1)\\ &\leq \frac{K^n}{1-K}\rho(y_0, y_1).\end{aligned}$$

If $p = 2k$, where $k \geq 2$, then by pentagonal property, (3.3) and (3.5), we have

$$\begin{aligned}\rho(y_n, y_{n+2k}) &\leq \rho(y_n, y_{n+2}) + \rho(y_{n+2}, y_{n+3}) + \rho(y_{n+3}, y_{n+4}) + \rho(y_{n+4}, y_{n+2k})\\ &\leq \rho(y_n, y_{n+2}) + \rho(y_{n+2}, y_{n+3}) + \rho(y_{n+3}, y_{n+4}) + \cdots\end{aligned}$$

$$
\begin{aligned}
&+\rho(y_{n+2k-2},y_{n+2k-1})+\rho(y_{n+2k-1},y_{n+2k})\\
&\le K^n\rho(y_0,y_1)+K^{n+2}\rho(y_0,y_1)+K^{n+3}\rho(y_0,y_1)+\cdots\\
&\quad +K^{n+2k-2}\rho(y_0,y_1)+K^{n+2k-1}\rho(y_0,y_1)\\
&\le \frac{K^n}{1-K}\rho(y_0,y_1).
\end{aligned}
$$

Therefore, combining the above two cases, we obtain that

$$\rho(y_n,y_{n+p})\le \frac{K^n}{1-K}\rho(y_0,y_1),\ \forall n,p\in\mathbb{N}. \tag{3.6}$$

Since $K\in(0,1)$, we get, as $n\to\infty$, $\frac{K^n}{1-K}\to 0$. Hence, for every $c\in E$ with $c\gg 0$, $\exists n_0\in\mathbb{N}$ such that

$$\rho(y_n,y_{n+p})\ll c,\ \text{for all } n\ge n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in $(X,\rho)$.

Suppose $T(X)$ is a complete subspace of $X$, there exists a point $q\in T(X)\subseteq S(X)$ such that $\lim_{n\to\infty} y_n=\lim_{n\to\infty} Tx_n=q$ and also $\lim_{n\to\infty} Sx_{n+1}=q$, and if $S(X)$ is complete, this holds also with $q\in S(X)$.

Now, let $p\in X$ be such that $Sp=q$. Given $c\gg 0$, we choose a natural numbers $M_1,M_2,M_3$ such that $\rho(q,y_{n-1})\ll \frac{c}{4}$, $\forall\ n\ge M_1$, $\rho(y_{n-1},y_n)\ll\frac{c}{4}$, $\forall\ n\ge M_2$ and $\rho(y_n,Sp)\ll\frac{kc}{4}$, $\forall\ n\ge M_3$.
Since $y_n\ne y_m$ for $n\ne m$, by (3.1), we have that

$$
\begin{aligned}
\rho(y_n,Sp)&=\rho(Sx_{n+1},Sp)\\
&\ge k\rho(Tx_{n+1},Tp)\\
&=k\rho(y_{n+1},Tp).
\end{aligned}
$$

That is,

$$\rho(y_{n+1},Tp)\le\frac{1}{k}\rho(y_n,Sp).$$

By pentagonal property, we have that

$$
\begin{aligned}
\rho(q,Tp)&\le\rho(q,y_{n-1})+\rho(y_{n-1},y_n)+\rho(y_n,y_{n+1})+\rho(y_{n+1},Tp)\\
&\le\rho(q,y_{n-1})+\rho(y_{n-1},y_n)+\rho(y_n,y_{n+1})+\frac{1}{k}\rho(y_n,Sp)\\
&\ll\frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}=c,\ \text{for all } n\ge M,
\end{aligned}
$$

where $M:=\max\{M_1,M_2,M_3\}$. Since $c$ is arbitrary, we have $\rho(q,Tp)\ll\frac{c}{m}$, $\forall\ m\in\mathbb{N}$. Since $\frac{c}{m}\to 0$ as $m\to\infty$, we conclude $\frac{c}{m}-\rho(q,Tp)\to-\rho(q,Tp)$ as $m\to\infty$. Since $P$ is closed, $-\rho(q,Tp)\in P$. Hence $\rho(q,Tp)\in P\cap -P$. By definition of cone we get that $\rho(q,Tp)=0$, and so $Tp=q$. Hence, $Tp=Sp=q$. That is, $q$ is a point of coincidence of $S$ and $T$.

Next, we show that $q$ is unique. For suppose $q'$ be another point of coincidence of $S$ and $T$. That is,

$$Sp'=Tp'=q',\ \text{for some } p'\in X.$$

Then from (3.1), we have

$$\begin{aligned}\rho(q,q') &= \rho(Sq,Sq')\\ &\geq k\rho(Tq,Tq')\\ &= k\rho(q,q').\end{aligned}$$

So that

$$\rho(q,q') \leq \frac{1}{k}\rho(q,q'),$$

which implies that

$$\rho(q,q') = 0.$$

Hence, $q = q'$.

Therefore, $S$ and $T$ have a unique point of coincidence in $X$. If $S$ and $T$ are weakly compatible, then by Lemma 2.9, the mappings $S$ and $T$ have a unique common fixed point in $X$. This completes the proof of the theorem. □

**Theorem 3.2.** *Let $(X,\rho)$ be a cone pentagonal metric space. Suppose the mappings $S,T : X \to X$ satisfies the condition:*

$$\rho(Sx,Sy) \geq k_1\rho(Tx,Sx) + k_2\rho(Ty,Sy) + k_3\rho(Tx,Ty), \ \forall x,y \in X, \ x \neq y, \tag{3.7}$$

*where $k_1,k_2,k_3 \geq 0$ with $k_1+k_2+k_3 > 1$, $k_1 < 1, k_2 < 1$, and $k_3 > 1$. Suppose that $T(X) \subseteq S(X)$, and either of $S(X)$ or $T(X)$ is complete, then $S$ and $T$ have a unique point of coincidence in $X$. Moreover, if $S$ and $T$ are weakly compatible then $S$ and $T$ have a unique common fixed point in $X$.*

**Remark 3.3.** If we put $k_1 = k_2 = 0$ in Theorem 3.2, we have Theorem 3.1.

*Proof.* Let $x_0$ be arbitrary point in $X$. Since $T(X) \subseteq S(X)$, we can choose a point $x_1$ in $X$ such that $Tx_0 = Sx_1$. Continuing in this way, we construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$y_n = Tx_n = Sx_{n+1} \text{ for all } n = 0,1,2,\cdots.$$

Suppose that $y_k = y_{k-1}$ for some $k \in \mathbb{N}$, then $Sx_k = Tx_k$. Thus, $x_k$ is a coincidence point of $S$ and $T$. Hence, assume that $y_n \neq y_{n-1}$, for each $n \in \mathbb{N}$. Then using (3.7), we have that

$$\begin{aligned}\rho(y_n,y_{n-1}) &= \rho(Sx_{n+1},Sx_n)\\ &\geq k_1\rho(Tx_{n+1},Sx_{n+1}) + k_2\rho(Tx_n,Sx_n) + k_3\rho(Tx_{n+1},Tx_n)\\ &= k_1\rho(y_{n+1},y_n) + k_2\rho(y_n,y_{n-1}) + k_3\rho(y_{n+1},y_n),\end{aligned}$$

which implies that,

$$\rho(y_n,y_{n+1}) \leq K\rho(y_{n-1},y_n), \text{ where } K = \frac{1-k_2}{k_1+k_3} \in (0,1). \tag{3.8}$$

By (3.8) it follows, for all $n = 0,1,2,\cdots$, that

$$\rho(y_n,y_{n+1}) \leq K\rho(y_{n-1},y_n) \leq K^2\rho(y_{n-2},y_{n-1}) \leq \cdots \leq K^n\rho(y_0,y_1). \tag{3.9}$$

Observe that, by (3.7) and pentagonal property,

$$\begin{aligned}\rho(y_{n+1},y_{n-1}) &= \rho(Sx_{n+2},Sx_n)\\ &\ge k_1\rho(Tx_{n+2},Sx_{n+2})+k_2\rho(Tx_n,Sx_n)+k_3\rho(Tx_{n+2},Tx_n)\\ &= k_1\rho(y_{n+2},y_{n+1})+k_2\rho(y_n,y_{n-1})+k_3\rho(y_{n+2},y_n),\end{aligned}$$

which implies that,

$$\begin{aligned}k_3\rho(y_n,y_{n+2}) &\le \rho(y_{n-1},y_{n+1})-k_1\rho(y_{n+1},y_{n+2})-k_2\rho(y_{n-1},y_n)\\ &\le \rho(y_{n-1},y_n)+\rho(y_n,y_{n+1})+\rho(y_{n+1},y_{n+2})+\rho(y_{n+2},y_{n+1})\\ &\quad -k_1\rho(y_{n+1},y_{n+2})-k_2\rho(y_{n-1},y_n).\end{aligned}$$

Hence,

$$\rho(y_n,y_{n+2}) \le \frac{1-k_2}{k_3}\rho(y_{n-1},y_n)+\frac{1}{k_3}\rho(y_n,y_{n+1})+\frac{2-k_1}{k_3}\rho(y_{n+1},y_{n+2}).$$

That is,

$$\rho(y_n,y_{n+2}) \le \alpha\rho(y_{n-1},y_n)+\beta\rho(y_n,y_{n+1})+\gamma\rho(y_{n+1},y_{n+2}), \tag{3.10}$$

where $\alpha=\frac{1-k_2}{k_3}>0$, $\beta=\frac{1}{k_3}>0$, and $\gamma=\frac{2-k_1}{k_3}>0$.

For the sequence $\{y_n\}$, we consider $\rho(y_n,y_{n+p})$ in two cases as follows:
If $p=2k+1$, where $k\ge 1$, then by pentagonal property and (3.9), we have

$$\begin{aligned}\rho(y_n,y_{n+2k+1}) &\le \rho(y_n,y_{n+1})+\rho(y_{n+1},y_{n+2})+\rho(y_{n+2},y_{n+3})+\rho(y_{n+3},y_{n+2k+1})\\ &\le \rho(y_n,y_{n+1})+\rho(y_{n+1},y_{n+2})+\rho(y_{n+2},y_{n+3})+\cdots\\ &\quad +\rho(y_{n+2k-1},y_{n+2k})+\rho(y_{n+2k},y_{n+2k+1})\\ &\le K^n\rho(y_0,y_1)+K^{n+1}\rho(y_0,y_1)+K^{n+2}\rho(y_0,y_1)+\cdots\\ &\quad +K^{n+2k-1}\rho(y_0,y_1)+K^{n+2k}\rho(y_0,y_1)\\ &\le \frac{K^n}{1-K}\rho(y_0,y_1).\end{aligned}$$

If $p=2k$, where $k\ge 2$, then by pentagonal property, (3.9) and (3.10), we have

$$\begin{aligned}\rho(y_n,y_{n+2k}) &\le \rho(y_n,y_{n+2})+\rho(y_{n+2},y_{n+3})+\rho(y_{n+3},y_{n+4})+\rho(y_{n+4},y_{n+2k})\\ &\le \alpha\rho(y_{n-1},y_n)+\beta\rho(y_n,y_{n+1})+\gamma\rho(y_{n+1},y_{n+2})+\rho(y_{n+2},y_{n+3})\\ &\quad +\rho(y_{n+3},y_{n+4})+\cdots+\rho(y_{n+2k-2},y_{n+2k-1})+\rho(y_{n+2k-1},y_{n+2k})\\ &\le \alpha K^{n-1}\rho(y_0,y_1)+\beta K^n\rho(y_0,y_1)+\gamma K^{n+1}\rho(y_0,y_1)+K^{n+2}\rho(y_0,y_1)\\ &\quad +K^{n+3}\rho(y_0,y_1)+\cdots+K^{n+2k-2}\rho(y_0,y_1)+K^{n+2k-1}\rho(y_0,y_1)\\ &\le \alpha K^{n-1}\rho(y_0,y_1)+\beta K^n\rho(y_0,y_1)+\gamma K^{n+1}\rho(y_0,y_1)+\frac{K^{n+2}}{1-K}\rho(y_0,y_1).\end{aligned}$$

Since $\alpha,\beta,\gamma>0$, $K\in(0,1)$, we obtain that $\alpha K^{n-1}\rho(y_0,y_1)\to 0$, $\beta K^{n}\rho(y_0,y_1)\to 0$, $\gamma K^{n+1}\rho(y_0,y_1)\to 0$, $\frac{K^n}{1-K}\rho(y_0,y_1)\to 0$, $\frac{K^{n+2}}{1-K}\rho(y_0,y_1)\to 0$, as $n\to\infty$. Hence, for every $c\in E$ with $c\gg 0$, $\exists n_0\in\mathbb{N}$ such that

$$\rho(y_n,y_{n+p})\ll c, \text{ for all } n\geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in $(X,\rho)$.
Suppose $T(X)$ is a complete subspace of $X$. Then there exists $q\in T(X)\subseteq S(X)$ such that $\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty} Tx_n=q$ and also $\lim\limits_{n\to\infty} Sx_{n+1}=q$, and if $S(X)$ is complete, this holds also with $q\in S(X)$.

Now, let $p\in X$ be such that $Sp=q$. Given $c\gg 0$, we choose a natural numbers $M_1,M_2,M_3$ such that $\rho(q,y_{n-1})\ll\frac{c}{4}$, $\forall n\geq M_1$, $\rho(y_{n-1},y_n)\ll\frac{c}{4}$, $\forall n\geq M_2$ and $\rho(y_n,Sp)\ll\frac{k_3c}{4}$, $\forall n\geq M_3$.
Since $y_n\neq y_m$ for $n\neq m$, by (3.7), we have that

$$\begin{aligned}d(y_n,Sp)&=d(Sx_{n+1},Sp)\\&\geq k_1\rho(Tx_{n+1},Sx_{n+1})+k_2\rho(Tp,Sp)+k_3\rho(Tx_{n+1},Tp)\\&\geq k_1\rho(y_{n+1},y_n)+k_2\rho(Tp,Sp)+k_3\rho(y_{n+1},Tp)\\&\geq k_3\rho(y_{n+1},Tp),\end{aligned}$$

which implies that,

$$\rho(y_{n+1},Tp)\leq\frac{1}{k_3}\rho(y_n,Sp).$$

By pentagonal property, we have that

$$\begin{aligned}\rho(q,Tp)&\leq\rho(q,y_{n-1})+\rho(y_{n-1},y_n)+\rho(y_n,y_{n+1})+\rho(y_{n+1},Tp)\\&\leq\rho(q,y_{n-1})+\rho(y_{n-1},y_n)+\rho(y_n,y_{n+1})+\frac{1}{k_3}\rho(y_n,Sp)\\&\ll\frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4}=c, \text{ for all } n\geq M,\end{aligned}$$

where $M:=\max\{M_1,M_2,M_3\}$. Since $c$ is arbitrary, we have $\rho(q,Tp)\ll\frac{c}{m}$, $\forall m\in\mathbb{N}$. Since $\frac{c}{m}\to 0$ as $m\to\infty$, we conclude $\frac{c}{m}-\rho(q,Tp)\to-\rho(q,Tp)$ as $m\to\infty$. Since $P$ is closed, $-\rho(q,Tp)\in P$. Hence $\rho(q,Tp)\in P\cap -P$. By definition of cone we get that $\rho(q,Tp)=0$, and so $Tp=q$. Hence, $Tp=Sp=q$. That is, $q$ is a point of coincidence of $S$ and $T$.

Next, we show that $q$ is unique. For suppose $q'$ be another point of coincidence of $S$ and $T$. That is,

$$Sp'=Tp'=q', \text{ for some } p'\in X.$$

Then from (3.7), we have

$$\begin{aligned}\rho(q,q')&=\rho(Sq,Sq')\\&\geq k_1\rho(Tq,Sq)+k_2\rho(Tq',Sq')+k_3\rho(Tq,Tq')\\&\geq k_1\rho(q,q)+k_2\rho(q',q')+k_3\rho(q,q')\end{aligned}$$

$$\geq k_3\rho(q,q').$$

So that

$$\rho(q,q') \leq \frac{1}{k_3}\rho(q,q'),$$

which implies that

$$\rho(q,q') = 0.$$

Hence, $q = q'$.
Therefore, $S$ and $T$ have a unique point of coincidence in $X$. If $S$ and $T$ are weakly compatible, then by Lemma 2.9, the mappings $S$ and $T$ have a unique common fixed point in $X$. This completes the proof of the theorem. □

The following example illustrates the result of Theorem 3.2.

**Example.** Let $X = \{1,2,3,4,5\}$, $E = \mathbb{R}^2$ and $P = \{(x,y) : x,y \geq 0\}$ is a cone in $E$. Define $\rho : X \times X \to E$ as follows:

$$\rho(x,x) = 0, \quad \forall\ x \in X;$$

$$\rho(1,2) = \rho(2,1) = (4,8);$$

$$\rho(1,3) = \rho(3,1) = \rho(3,4) = \rho(4,3) = \rho(2,4) = \rho(4,2) = (1,2);$$

$$\rho(1,5) = \rho(5,1) = \rho(2,5) = \rho(5,2) = \rho(3,5) = \rho(5,3) = \rho(4,5) = \rho(5,4) = (3,6).$$

Then $(X,\rho)$ is a complete cone pentagonal metric space, but $(X,\rho)$ is not a cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned}(4,8) = \rho(1,2) &> \rho(1,3)+\rho(3,4)+\rho(4,2)\\ &= (1,2)+(1,2)+(1,2)\\ &= (3,6),\ \text{as } (4,8)-(3,6) = (1,2) \in P.\end{aligned}$$

Define a mapping $S,T : X \to X$ as follows:

$$S(x) = x,\ \forall x \in X.$$

$$T(x) = \begin{cases} 4, & \text{if } x \neq 5; \\ 2, & \text{if } x = 5. \end{cases}$$

Clearly $T(X) \subseteq S(X)$, and the mappings $S$ and $T$ are weakly compatible. Hence, all the conditions of Theorem 3.2 holds for all $x,y \in X$, where $k_3 \in (1,2]$, $k_1 = 0$, $k_2 = 0$, and $4 \in X$ is the unique common fixed point of the mappings $S$ and $T$.

Now as corollaries, we recover, extend and generalize the recent results of [1, 14, 10, 8, 12, 11], and many others in the literature, to a more general cone pentagonal metric space.

**Corollary 3.4.** *Let $(X,\rho)$ be a complete cone pentagonal metric space. Suppose the mappings*

*$S,T:X\to X$ satisfies the condition:*

$$\rho(Sx,Sy)\geq k\big(\rho(Tx,Sx)+\rho(Ty,Sy)\big),\ \forall x,y\in X,\ x\neq y, \tag{3.11}$$

*where $k\in(1/2,1)$. Suppose that $T(X)\subseteq S(X)$, and either of $S(X)$ or $T(X)$ is complete, then $S$ and $T$ have a unique point of coincidence in $X$. Moreover, if $S$ and $T$ are weakly compatible then $S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Putting $k_1=k_2$ and $k_3=0$ in Theorem 3.2. The result follows. □

**Corollary 3.5.** *Let $(X,\rho)$ be a complete cone pentagonal metric space and let $S:X\to X$ be onto mapping which satisfies the condition:*

$$\rho(Sx,Sy)\geq k_1\rho(x,Sx)+k_2\rho(y,Sy)+k_3\rho(x,y),\ \forall x,y\in X,\ x\neq y,$$

*where $k_1,k_2,k_3\geq 0$ with $k_1+k_2+k_3>1$, $k_1<1,k_2<1$, and $k_3>1$. Then $S$ have a unique fixed point in $X$.*

*Proof.* Putting $T=I$ in Theorem 3.2. This completes the proof. □

**Corollary 3.6.** *Let $(X,\rho)$ be a complete cone pentagonal metric space and let $S:X\to X$ be surjective which satisfies the condition:*

$$\rho(Sx,Sy)\geq k\big(\rho(x,Sx)+\rho(y,Sy)\big),\ \forall x,y\in X,\ x\neq y,$$

*where $k\in(1/2,1)$. Then $S$ have a unique fixed point in $X$.*

*Proof.* Putting $T=I$ in Corollary 3.4. This completes the proof. □

**Corollary 3.7.** *Let $(X,\rho)$ be a cone pentagonal metric space and let $S:X\to X$ be onto mapping which satisfies:*

$$\rho(Sx,Sy)\geq k\rho(x,y),\ \forall x,y\in X,$$

*where $k>1$ is a constant. Then $S$ has a unique fixed point in $X$.*

*Proof.* Putting $T=I$ and $k_1=k_2=0$ in Theorem 3.2. The result follows. □

## 4. Conclusion

In this paper, we prove existence of common fixed points for two self mappings satisfying expansive conditions in non-normal cone pentagonal metric spaces. The established results extend and improve recent results obtained by many authors. We give an example to elucidate our results.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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