Journal of Informatics and Mathematical Sciences
Vol. 10, Nos. 1 & 2, pp. 161–172, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://dx.doi.org/10.26713/jims.v10i1-2.575

http://www.rgnpublications.com

Research Article

Dual Bicomplex Fibonacci Numbers with Fibonacci and Lucas Numbers

Faik Babadağ

Department of Mathematics, Art & Science Faculty, Kırıkkale University, Ankara Yolu 7. Km, 71450 Yahşihan/Kırıkkale, Turkey
faik.babadag@kku.edu.tr

Abstract. Recently, the authors give some results about Fibonacci and Lucas numbers. In this present paper, our object introduce a detailed study of a new generation of dual bicomplex Fibonacci numbers. We define new dual vector which is called dual Fibonacci vector. We give properties of dual Fibonacci vector to expert in geometry and then we introduce some formulas, facts and properties about dual bicomplex Fibonacci numbers and variety of geometric and algebraic properties which are not generally known.

Keywords. Dual bicomplex Fibonacci numbers; Dual Fibonacci numbers; Dual Lucas numbers; Dual Fibonacci vector

MSC. 11B39; 15A24

Received: February 22, 2018 **Accepted:** March 3, 2018

1. Introduction

The defnition of dual numbers was first given by William Kingdon Clifford (1845-1879) [1] and some properties of that were studied in 1873 as a tool for his geometrical investigations. After him Eduard Study (1862-1930) used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping. The set of oriented lines in a Euclidean three-dimensional space. E^3 is one to one correspondence with the points of a dual space D^3

of triples of dual numbers [1]. Dual numbers form two dimensional commutative associative algebra over the real numbers. Also the algebra of dual numbers is a ring. Dual numbers are used as an appliance for expressing and analyzing the physical properties of rigid bodies. They are computationally efficient approach of representing rigid transforms like translation and rotation.

The remainder of the paper is organized as follows. In Section 2 we recall to some needed basic and fundamental concepts of dual numbers and dual bicomplex numbers. In Section 3 it is given a detailed study of a new generation of dual bicomplex Fibonacci numbers and in Section 4 it is given new a dual vector which is called dual Fibonacci vector and properties of this vector to expert in geometry. Finally, the conclusions and future works will be presented.

2. Preliminary

This section is devoted to some basic and fundamental concepts of dual numbers and dual bicomplex numbers. A dual number is a number of the form $\widetilde{x}=x+\varepsilon x^*$, where x and x^* are real numbers called the components of $\widetilde{x}$ and $\varepsilon=(0,1)$ arbitrary dual unit satisfy, the relation $\varepsilon^2=0$. Let us consider $D=\{\widetilde{x}\mid \widetilde{x}=x+\varepsilon x^*, x,\ x^*\in \mathrm{R}\}$. Addition of two elements in D and scalar multiplication of an element in D by a real number are defined in the usual way. D is a vector space with respect to addition and scalar multiplication. The product $\widetilde{x}\otimes\widetilde{y}$ is the element in D obtained by multiplying $x+\varepsilon x^*$ and $y+\varepsilon y^*$ as if they were polynomials and then using the relation $\varepsilon^2=0$ to simplify the result.

$$\widetilde{x}\otimes\widetilde{y}=\widetilde{x}\ \widetilde{y}=(x+\varepsilon x^*)\cdot(y+\varepsilon y^*)=xy+\varepsilon(xy^*+x^*y).$$

Then the dual number $x+\varepsilon x^*$ divided by the dual number $y+\varepsilon y^*$ provided $y\neq 0$ can be defined as

$$\frac{\widetilde{x}}{\widetilde{y}}=\frac{x+\varepsilon x^*}{y+\varepsilon y^*}=\frac{x}{y}+\varepsilon\frac{x^*y-xy^*}{y^2}.$$

A dual bicomplex number is defined in the form [2, 3],

$$\begin{aligned}\widetilde{X}&=X+\varepsilon X^*\\&=(x_0+x\ i_1+x_2\ i_2+x_3\ i_3)+\varepsilon(x_0^*+x_1^*\ i_1+x_2^*\ i_2+x_3^*\ i_3)\\&=(x_0+\varepsilon x_0^*)+(x_1+\varepsilon x_1^*)i_1+(x_2+\varepsilon x_2^*\ )i_2+(x_3+\varepsilon x_3^*)i_3\\&=\widetilde{x}_0+\widetilde{x}_1\ i_1+\widetilde{x}_2\ i_2+\widetilde{x}_3\ i_3,\end{aligned}$$

where $\widetilde{x}_0,\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3\in D$ and i_1, i_2, i_3 are the imaginary units, ε is dual unit. The dual bicomplex number $\widetilde{X}$ is constructed from eight real parameters as x_0, x_1, x_2, x_3, εx_0^*, εx_1^*, εx_2^*, εx_3^*.

Let $\widetilde{X}=X+\varepsilon X^*$ and $\widetilde{Y}=Y+\varepsilon Y^*$ be two dual bicomplex number, then the addition and subtraction are given by

$$\widetilde{X}\mp\widetilde{Y}=(X\mp Y)+\varepsilon(X^*\mp Y^*)$$

and multiplication is

$$\widetilde{X}\cdot\widetilde{Y}=XY+\varepsilon(YX^*+Y^*X).$$

The conjugates of the dual bicomplex number $\widetilde{X}$ are denoted by $\overline{\widetilde{X}}$. There are different conjugations of dual bicomplex number according to the imaginary units i_1, i_2 and i_3 as follows:

$$\left.\begin{array}{ll}
\text{(a)} & \overline{\widetilde{X}} = (x_0+\varepsilon x_0^*)-(x_1+\varepsilon x_1^*)i_1+(x_2+\varepsilon x_2^*)i_2-(x_3+\varepsilon x_3^*)i_3 \\
\text{(b)} & \overline{\widetilde{X}} = (x_0+\varepsilon x_0^*)+(x_1+\varepsilon x_1^*)i_1-(x_2+\varepsilon x_2^*)i_2-(x_3+\varepsilon x_3^*)i_3) \\
\text{(c)} & \overline{\widetilde{X}} = (x_0+\varepsilon x_0^*)-(x_1+\varepsilon x_1^*)i_1-(x_2+\varepsilon x_2^*)i_2+(x_3+\varepsilon x_3^*)i_3.
\end{array}\right\} \tag{1}$$

The Fibonacci and Lucas sequence are given as

$$F_0=0,\ F_1=1,\ F_{n+2}=F_n+F_{n+1}$$

and

$$L_0=2,\ L_1=1,\ L_n=L_{n-1}+L_{n-2}.$$

For Fibonacci and Lucas numbers, there are many study related on Fibonacci and Lucas numbers. Dunlap [4], Vajda [5] and Hoggatt [6] defined the properties of Fibonacci and Lucas numbers and gave the relation between them. The generalized Fibonacci sequences are described by Horadam [7, 8]

$$Q_n=F_n+i_1F_{n+1}+i_2F_{n+2}+i_3F_{n+3}$$

and

$$K_n=L_n+i_1L_{n+1}+i_2L_{n+2}+i_3L_{n+3}$$

respectively, where F_n is Fibonacci number and L_n is Lucas number. Also, i_1, i_2 and i_3 are the imaginary units. Swamy [9] gave the relations of generalized Fibonacci quaternions. Iyer studied Fibonacci quaternions in [10] and obtained some other relations about Fibonacci and Lucas quaternions. In [11], Halıcı expressed the generating function and Binet formulas for these quaternions. Akyiğit, Köksal and Tosun [12] defined the split Fibonacci and split Lucas quaternions. They also gave Binet formulas and Cassini identities for these quaternions. In this paper, our object give a detailed study of a new generation of dual bicomplex Fibonacci numbers. We define new a dual vector which is called dual Fibonacci vector. We give properties of this vector to expert in geometry.

3. Dual Fibonacci Bicomplex Numbers with Fibonacci and Lucas Number Components

The nth dual Fibonacci numbers and the nth dual Lucas numbers are given by

$$\widetilde{F}_n=F_n+\varepsilon F_{n+1},$$
$$\widetilde{L}_n=L_n+\varepsilon L_{n+1},$$

respectively, a dual bicomplex Fibonacci number and a dual bicomplex Lucas number can be given with Fibonacci and Lucas number components as following formulae,

$$\left.\begin{array}{l}
\widetilde{x}_n=(F_n+\varepsilon F_{n+1})+i_1(F_{n+1}+\varepsilon F_{n+2})+i_2(F_{n+2}+\varepsilon F_{n+3})+i_3(F_{n+3}+\varepsilon F_{n+4}), \\
\widetilde{k}_n=(L_n+\varepsilon L_{n+1})+i_1(L_{n+1}+\varepsilon L_{n+2})+i_2(L_{n+2}+\varepsilon L_{n+3})+i_3(L_{n+3}+\varepsilon L_{n+4}).
\end{array}\right\} \tag{2}$$

Also, i_1, i_2 and i_3 are the imaginary units and ε is the dual unit which satisfy the following rules:

$$\left\{\begin{array}{l} i_1^2=-1,\ i_2^2=-1,\ i_3^2=1,\ \varepsilon^2=0 \\ i_1 i_2=i_2 i_1=i_3 : i_1 i_3=i_3 i_1=-i_2 : i_2 i_3=i_3 i_2=-i_1. \end{array}\right\} \tag{3}$$

The addition, subtraction and multiplication with scalar of dual bicomplex Fibonacci numbers are given by

$$\widetilde{x}_n \mp \widetilde{y}_n=(x_n \mp y_n)+\varepsilon(x_{n+1}+y_{n+1}),$$

$$\lambda\widetilde{x}_n=\lambda x_n+\lambda\varepsilon x_{n+1}.$$

From equation (1), (2) and (3), the conjugates of dual bicomplex Fibonacci numbers are defined by $\overline{\widetilde{x}}_n$ according to the imaginary units i_1, i_2 and i_3. We can be written as

$$\left.\begin{array}{lll} \text{(a)} & \overline{\widetilde{x}}_n & =(F_n+\varepsilon F_{n+1})-(F_{n+1}+\varepsilon F_{n+2})i_1+(F_{n+2}+\varepsilon F_{n+3})i_2-(F_{n+3}+\varepsilon F_{n+4})i_3 \\ & & =\widetilde{F}_n-i_1\widetilde{F}_{n+1}+i_2\widetilde{F}_{n+2}-i_3\widetilde{F}_{n+3}, \\ \text{(b)} & \overline{\widetilde{x}}_n & =(F_n+\varepsilon F_{n+1})+(F_{n+1}+\varepsilon F_{n+2})i_1-(F_{n+2}+\varepsilon F_{n+3})i_2-(F_{n+3}+\varepsilon F_{n+4})i_3 \\ & & =\widetilde{F}_n+i_1\widetilde{F}_{n+1}-i_2\widetilde{F}_{n+2}-i_3\widetilde{F}_{n+3}, \\ \text{(c)} & \overline{\widetilde{x}}_n & =(F_n+\varepsilon F_{n+1})-(F_{n+1}+\varepsilon F_{n+2})i_1-(F_{n+2}+\varepsilon F_{n+3})i_2+(F_{n+3}+\varepsilon F_{n+4})i_3 \\ & & =\widetilde{F}_n-i_1\widetilde{F}_{n+1}-i_2\widetilde{F}_{n+2}+i_3\widetilde{F}_{n+3}. \end{array}\right\} \tag{4}$$

Theorem 1. *Let $\widetilde{x}_n$ be dual bicomplex Fibonacci number. Then, Fibonacci number is the 2th linear recurrence sequence. After that, we introduce the sets X_D and X'_D as fallows*

$$X_D=\{\widetilde{x}_n \mid \widetilde{x}_n=(\widetilde{F}_n,\widetilde{F}_{n+1},\widetilde{F}_{n+2},\widetilde{F}_{n+3});\ \widetilde{F}_n \text{ is nth dual Fibonacci number}\}$$

and

$$X'_D=\left\{\widetilde{\gamma}_n \mid \widetilde{\gamma}_n=\begin{bmatrix}\widetilde{z}_1 & -\widetilde{z}_2 \\ \widetilde{z}_2 & \widetilde{z}_1\end{bmatrix};(\widetilde{z}_1,\widetilde{z}_2) \text{ are dual complex Fibonacci number}\right\}.$$

Then there is an isomorphism X_D and X'_D, we can write

$$\widetilde{x}_n=(\widetilde{F}_n,\widetilde{F}_{n+1},\widetilde{F}_{n+2},\widetilde{F}_{n+3}) \to \widetilde{\gamma}_n=\begin{bmatrix}\widetilde{F}_n+i_1\widetilde{F}_{n+1} & -\widetilde{F}_{n+2}-i_1\widetilde{F}_{n+3} \\ \widetilde{F}_{n+2}+i_1\widetilde{F}_{n+3} & \widetilde{F}_n+i_1\widetilde{F}_{n+1}\end{bmatrix}.$$

Therefore, we can give

$$X'_D=\left\{\widetilde{\gamma}_n \mid \widetilde{\gamma}_n=\widetilde{z}_1\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}+\widetilde{z}_2\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix};(\widetilde{z}_1,\widetilde{z}_2) \text{ are dual complex Fibonacci number}\right\}$$

and

$$\widetilde{\gamma}_n=F_nU_1+F_{n+1}U_2+F_{n+2}U_3+F_{n+3}U_4,$$

where [13],

$$U_1=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix},\ U_2=\begin{bmatrix}i_1 & 0 \\ 0 & i_1\end{bmatrix},\ U_3=\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix},\ U_4=\begin{bmatrix}0 & -i_1 \\ i_1 & 0\end{bmatrix}.$$

Since $\det(\widetilde{\gamma}_n)\neq 0$, *the matrix $\widetilde{\gamma}_n$ has the inverse and it is in X'_D.*

Theorem 2. *Consider $\widetilde{x}_n$ and $\widetilde{k}_n$, dual bicomplex Fibonacci and Lucas numbers. Then, for $n \geq 1$, according to the imaginary units i_1, i_1 and i_3, the following relations hold:*

$$\left.\begin{array}{ll}
\text{(a)} & \widetilde{x}_n + \widetilde{x}_{n+1} = \widetilde{x}_{n+2} \\
\text{(b)} & \overline{\overline{\widetilde{X}}}_n = (\widetilde{x}_n - \widetilde{x}_{n+1}\, i_1) + i_2(\widetilde{x}_{n+2} - \widetilde{x}_{n+3} i_1) = -5\widetilde{F}_{n+3} + 2j\widetilde{L}_{n+3} \\
\text{(c)} & \overline{\widetilde{X}}_n = (\widetilde{x}_n + \widetilde{x}_{n+1}\, i_1) - i_2(\widetilde{x}_{n+2} + \widetilde{x}_{n+3} i_1) = (-3 + 2i_1)\widetilde{F}_{n+3} \\
\text{(d)} & \overline{\widetilde{X}}_n = (\widetilde{x}_n - \widetilde{x}_{n+1}\, i_1) - i_2(\widetilde{x}_{n+2} - \widetilde{x}_{n+3} i_1) = 3\widetilde{L}_{n+3}.
\end{array}\right\}$$

Proof. By using equations (2) and (4), we will obtain the proof (a), (b), (c), (d).

(a): By putting equation $F_n + F_{n+1} = F_{n+2}$, according to the imaginary unit i_1, we obtain

$$\begin{aligned}
\widetilde{x}_n + \widetilde{x}_{n+1} = (F_n + F_{n+1}) + i_1(F_{n+1} + F_{n+2}) + i_2(F_{n+2} + F_{n+3}) + i_3(F_{n+3} + F_{n+4}) \\
\varepsilon((F_{n+1} + F_{n+2}) + i_1(F_{n+2} + F_{n+3}) + i_2(F_{n+3} + F_{n+4}) + i_3(F_{n+4} + F_{n+5})) \\
= \widetilde{x}_{n+2}\,.
\end{aligned}$$

(b): By putting equations $F_{n+1} + F_{n-1} = L_n$ and $L_n - L_{n+4} = 5F_{n+2}$, according to the imaginary unit i_1, we obtain

$$\begin{aligned}
\overline{\overline{\widetilde{X}}}_n &= -[(F_n + \varepsilon F_{n+1}) + (F_{n+2} + \varepsilon F_{n+3}) - (F_{n+4} + \varepsilon F_{n+5}) - (F_{n+6} + \varepsilon F_{n+7})] \\
&\quad + 2j((F_{n+2} + \varepsilon F_{n+3}) + (F_{n+4} + \varepsilon F_{n+5})) \\
&= -\left(\widetilde{F}_n + \widetilde{F}_{n+2} - \widetilde{F}_{n+4} - \widetilde{F}_{n+6}\right) + 2j(\widetilde{F}_{n+2} + \widetilde{F}_{n+4}) \\
&= -\left(\widetilde{L}_{n+2} + \widetilde{L}_{n+4}\right) + 2j\widetilde{L}_{n+3} \\
&= -5\widetilde{F}_{n+3} + 2j\widetilde{L}_{n+3}\,.
\end{aligned}$$

(d): By putting equations the identity of Fibonacci numbers and $F_{n+4} + F_n = 3F_{n+3}$, according to the imaginary unit i_2, we obtain

$$\begin{aligned}
\overline{\overline{\widetilde{X}}}_n &= [(F_n + \varepsilon F_{n+1}) - (F_{n+2} + \varepsilon F_{n+3}) + (F_{n+4} + \varepsilon F_{n+5}) - (F_{n+6} + \varepsilon F_{n+7})] \\
&\quad + 2i_2(F_{n+1} + + \varepsilon F_{n+2}) + (F_{n+5} + \varepsilon F_{n+6}) \\
&= (-3 + 6i)\widetilde{F}_{n+3}\,.
\end{aligned}$$

(d) By putting equations the identity of Fibonacci numbers and $F_{n+1} + F_{n-1} = L_n$, $L_n + L_{n+4} = 3L_{n+2}$, according to the imaginary unit i_3, we obtain

$$\begin{aligned}
\overline{\overline{\widetilde{X}}}_n &= [(F_n + \varepsilon F_{n+1}) + (F_{n+2} + \varepsilon F_{n+3}) + (F_{n+4} + \varepsilon F_{n+5}) + (F_{n+6} + \varepsilon F_{n+7})] \\
&= 3\widetilde{L}_{n+3}\,.
\end{aligned}$$

□

Lemma 3.1. *Let $\widetilde{x}_n$ be dual bicomplex Fibonacci number. Then, we can give different conjugations according to i_1, i_2 and i_3 the following relations,*

$$\left.\begin{array}{ll}
\text{(a)} & \widetilde{x}_n + \overline{\widetilde{x}}_n = 2((F_n + \varepsilon F_{n+1}) + i_2(F_{n+2} + \varepsilon F_{n+3})) = 2(\widetilde{F}_n + i_2\widetilde{F}_{n+2}) \\
\text{(b)} & \widetilde{x}_n + \overline{\overline{\widetilde{x}}}_n = 2((F_n + \varepsilon F_{n+1}) + i_1(F_{n+1} + \varepsilon F_{n+2})) = 2(\widetilde{F}_n + i_1\widetilde{F}_{n+1}) \\
\text{(d)} & \widetilde{x}_n + \overline{\overline{\widetilde{x}}}_n = 2((F_n + \varepsilon F_{n+1}) + i_3(F_{n+3} + \varepsilon F_{n+4})) = 2(\widetilde{F}_n + i_3\widetilde{F}_{n+3})
\end{array}\right\} \qquad (5)$$

Proof. The proofs of (a), (b), (c), according to the imaginary units i_1, i_2 and i_3, are clear. □

Theorem 3. *Let $\widetilde{X}_n$ and $\overline{\overline{\widetilde{X}}}_n$ be dual bicomplex Fibonacci numbers and different conjugations of dual bicomplex Fibonacci number according to the imaginary units i_1, i_2 and i_3. Then, we give the following relations between these numbers*

(a) $\widetilde{X}_n\overline{\overline{\widetilde{X}}}_n = -\widetilde{L}_{2n+3} + 2i_2\widetilde{F}_{2n+3}$

(b) $\widetilde{X}_n\overline{\widetilde{X}}_n = (-1+2i_1)\widetilde{F}_{2n+3}$

(c) $\widetilde{X}_n\overline{\overline{\widetilde{X}}}_n = 3\widetilde{F}_{2n+3} + 2i_3[(-1)^{n-1}]$

Proof. From equations (2), (3) and (4), according to the imaginary units i_1, i_2 and i_3.
(a): Using the identities of Fibonacci and Lucas numbers and equations $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}$, according to the imaginary units i_1, we see that

$$\begin{aligned}\widetilde{X}_n\overline{\overline{\widetilde{X}}}_n &= \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2 - \widetilde{F}_{n+3}^2 + 2i_2(\widetilde{F}_n\widetilde{F}_{n+2} + \widetilde{F}_{n+1}\widetilde{F}_{n+3})\\ &= F_n^2 + F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2 + 2\varepsilon([F_nF_{n+1} + F_{n+1}F_{n+2}] - [F_{n+2}F_{n+3} + F_{n+3}F_{n+4}])\\ &\quad + 2i_2((F_n + \varepsilon F_{n+1})(F_{n+2} + \varepsilon F_{n+3}) + (F_{n+1} + \varepsilon F_{n+2})(F_{n+3} + \varepsilon F_{n+4}))\\ &= (F_{2n+1} - F_{2n+5}) + \varepsilon(F_{2n+2} - F_{2n+6}) + 2i_2(F_{2n+3} + \varepsilon F_{2n+4})\\ &= -\widetilde{L}_{2n+3} + 2i_2\widetilde{F}_{2n+3}.\end{aligned}$$

(b): By using the equations $F_{n+2}F_{n-1} = F_{n+1}^2 - F_n^2$, $F_nF_{n+1} = F_{n+1}^2 + (-1)^n$, $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ and the identities of Fibonacci numbers and Lucas numbers, according to the imaginary units i_2. We prove

$$\begin{aligned}\widetilde{X}_n\overline{\widetilde{X}}_n &= \widetilde{F}_n^2 - \widetilde{F}_{n+1}^2 + \widetilde{F}_{n+2}^2 - \widetilde{F}_{n+3}^2 + 2i_1(\widetilde{F}_n\widetilde{F}_{n+1} + \widetilde{F}_{n+2}\widetilde{F}_{n+3})\\ &= -\widetilde{F}_{2n+3} + +2i_1\widetilde{F}_{2n+3}\\ &= (-1 + +2i_1)\widetilde{F}_{2n+3}.\end{aligned}$$

(c): By using the equations $F_n^2 + F_{n+1}^2 = F_{2n+1}$, $F_nF_{n+1} - F_{n+2}F_{n-1} = (-1)^{n-1}$ and the identities of Fibonacci numbers and Lucas numbers, according to the imaginary units i_1. We prove

$$\begin{aligned}\widetilde{X}_n\overline{\overline{\widetilde{X}}}_n &= \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 + \widetilde{F}_{n+2}^2 + \widetilde{F}_{n+3}^2 + 2i_3(\widetilde{F}_n\widetilde{F}_{n+3} - \widetilde{F}_{n+1}\widetilde{F}_{n+2})\\ &= 3\widetilde{F}_{2n+3} + 2i_3[(-1)^{n-1}].\end{aligned}$$ □

Theorem 4. *From equations* (2), (3) *and* (4)*, let us consider dual bicomplex Fibonacci number $\widetilde{x}_n$. Then, we give the following relations of this number.*

(a) $\widetilde{X}_n\overline{\overline{\widetilde{X}}}_n - \widetilde{X}_{n-1}\overline{\overline{\widetilde{X}}}_{n-1} = -\widetilde{L}_{2n+2} + 2i_2\widetilde{F}_{2n+2}$

(b) $\widetilde{X}_n^2 = 2X_nF_n - 2L_{2n+3} + 2i_2(X_nF_{n+2} - L_{n+1})$

Proof. (a): By using $F_n^2 + F_{n+1}^2 = F_{2n+1}$, $F_{n+1} + F_{n-1} = L_n$, $L_n + L_{n+4} = 5F_n$ and the identities of Fibonacci and Lucas numbers, we see that

$$\begin{aligned}\widetilde{X}_{n-1}\overline{\overline{\widetilde{X}}}_{n-1} &= \widetilde{F}_{n-1}^2 + \widetilde{F}_n^2 - \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2 + 2i_2(\widetilde{F}_{n-1}\widetilde{F}_{n+1} + \widetilde{F}_n\widetilde{F}_{n+2})\\ &= F_{n-1}^2 + F_n^2 - F_{n+1}^2 - F_{n+2}^2 + 2\varepsilon([F_{n-1}F_n + F_nF_{n+1}] - [F_{n+1}F_{n+2} + F_{n+2}F_{n+3}])\\ &\quad + 2i_2((F_{n-1} + \varepsilon F_n)(F_{n+1} + \varepsilon F_{n+2}) + (F_n + \varepsilon F_{n+1})(F_{n+2} + \varepsilon F_{n+3}))\end{aligned}$$

$$= -\widetilde{L}_{2n+1} + 2i_2\widetilde{F}_{2n+1},$$

$$\begin{aligned}\widetilde{X}_n\overline{\widetilde{X}}_n - \widetilde{X}_{n-1}\overline{\widetilde{X}}_{n-1} &= \widetilde{F}_n^2 + \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2 - \widetilde{F}_{n+3}^2 + 2i_2(\widetilde{F}_n\widetilde{F}_{n+2} + \widetilde{F}_{n+1}\widetilde{F}_{n+3}) \\ &\quad - (\widetilde{F}_{n-1}^2 + \widetilde{F}_n^2 - \widetilde{F}_{n+1}^2 - \widetilde{F}_{n+2}^2 + 2i_2(\widetilde{F}_{n-1}\widetilde{F}_{n+1} + \widetilde{F}_n\widetilde{F}_{n+2})) \\ &= -\widetilde{L}_{2n+2} + 2i_2\widetilde{F}_{2n+2}.\end{aligned}$$

(b): By using equation (5), we have

$$\begin{aligned}X_n^2 &= X_nX_n \\ &= X_n\left[2(F_n + i_2F_{n+2}) - X_n^*\right] \\ &= X_n(2F_n + 2i_2F_{n+2}) - X_nX_n^* \\ &= 2X_n(F_n + i_2F_{n+2}) - \widetilde{L}_{2n+3} + 2i_2\widetilde{F}_{2n+3}.\end{aligned}$$

In the the same manner, according to the imaginary units i_1, i_3, we can write X_n^2. The proof is completed. □

4. Dual Fibonacci Vector with the Dual Bicomplex Fibonacci Numbers

Now, we describe a dual Fibonacci vector with vectorial part of dual bicomplex Fibonacci numbers as follow

$$\left\{\begin{aligned}\overrightarrow{\widetilde{V}}_n &= i_1\widetilde{F}_n + i_2\widetilde{F}_{n+1} + i_3\widetilde{F}_{n+2} \\ &= (\widetilde{F}_n,\ \widetilde{F}_{n+1},\ \widetilde{F}_{n+2})\end{aligned}\right\}$$

where $\overrightarrow{\widetilde{V}}_n = (\widetilde{F}_n, \widetilde{F}_{n+1}, \widetilde{F}_{n+2})$ and $\overrightarrow{\widetilde{V}}_m = (\widetilde{F}_m, \widetilde{F}_{m+1}, \widetilde{F}_{m+2})$ are dual Fibonacci vectors and $\widetilde{F}_n = F_n + \varepsilon F_{n+1}$ is nth dual Fibonacci number.

Definition 4.1. Consider dual Fibonacci vector, $\overrightarrow{\widetilde{V}}_n$ and $\overrightarrow{\widetilde{V}}_m$, then dot product of these vectors are defined by

$$\langle\overrightarrow{\widetilde{V}}_n, \overrightarrow{\widetilde{V}}_m\rangle = \langle\overrightarrow{F}_n, \overrightarrow{F}_m\rangle + \varepsilon\big(\langle\overrightarrow{F}_{n+1}, \overrightarrow{F}_m\rangle + \langle\overrightarrow{F}_n, \overrightarrow{F}_{m+1}\rangle\big).$$

Theorem 5. *Let $\overrightarrow{\widetilde{V}}_n$, $\overrightarrow{\widetilde{V}}_m$ be dual Fibonacci vectors, The dot product of these vectors is given as*

$$\langle\overrightarrow{\widetilde{V}}_n, \overrightarrow{\widetilde{V}}_m\rangle = \widetilde{F}_{m+n+1} + (1+\varepsilon)F_{n+2}F_{m+2} + \varepsilon L_{m+n+3}.$$

Proof. By using equations $F_{n+1}F_{m+1} + F_{n+2}F_{m+2} = F_{n+m+3}$, $F_{n+1} + F_{n-1} = L_n$, we obtain

$$\begin{aligned}\langle\overrightarrow{\widetilde{V}}_n, \overrightarrow{\widetilde{V}}_m\rangle &= \langle(\widetilde{F}_n,\ \widetilde{F}_{n+1},\ \widetilde{F}_{n+2}), (\widetilde{F}_m,\ \widetilde{F}_{m+1},\ \widetilde{F}_{m+2})\rangle \\ &= F_nF_m + F_{n+1}F_{m+1} + F_{n+2}F_{m+2} \\ &\quad + \varepsilon(F_{n+1}F_m + F_{n+2}F_{m+1} + F_{n+3}F_{m+2} + F_nF_{m+1} + F_{n+1}F_{m+2} + F_{n+2}F_{m+3}) \\ &= F_{n+2}F_{m+2} + F_{m+n+1} + \varepsilon(2F_{m+n+2} + F_{m+n+4} + F_{n+2}F_{m+2}) \\ &= \widetilde{F}_{m+n+1} + (1+\varepsilon)F_{n+2}F_{m+2} + \varepsilon L_{m+n+3}.\end{aligned}$$

□

Definition 4.2. *The permanent of a matrix $A_{n\times n}$ is defined as $p(A)$ or $per(A)$ and*

$$per(A) = \sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)}.$$

Here S_n indicates the symmetric group and permutation σ. The definition of the permanent of a matrix $A_{n\times n}$ differs from that of the determinant of a matrix $A_{n\times n}$. Since the signatures of the permutations are not taken into account [14].

Definition 4.3 (The cross product of two vector). For vectors $\overrightarrow{T}=(t_1,t_2,t_3)$ and $\overrightarrow{W}=(y_1,y_2,y_3)$ obtained by vectorial part of tessarine, the cross product is defined by using the permanent as following [14].

$$\overrightarrow{T}\wedge\overrightarrow{W}=\begin{vmatrix} i_1 & -i_2 & i_3\\ t_1 & t_2 & t_3\\ y_1 & y_2 & y_3\end{vmatrix}=-i_1[t_2y_3+t_3y_2]-i_2[t_1y_3+t_3y_1]+i_3[t_1y_2+t_2y_1].$$

Theorem 6. *Let $\overrightarrow{\widetilde{V}}_n$, $\overrightarrow{\widetilde{V}}_m$ be dual Fibonacci vectors, then by using Definition* 4.2 *and Definition* 4.3*, the cross product of these vectors is defined by*

$$\overrightarrow{\widetilde{V}}_n\wedge\overrightarrow{\widetilde{V}}_m=\overrightarrow{F}_n\wedge\overrightarrow{F}_m+\varepsilon(\overrightarrow{F}_n\wedge\overrightarrow{F}_{m+1}+\overrightarrow{F}_{n+1}\wedge\overrightarrow{F}_m). \tag{6}$$

Proof. Using the equations $L_{n+4}+L_n=3L_{n+2}$, $5F_mF_n=L_{n+m}-(-1)^mL_{n-m}$, we get

$$\begin{aligned}\overrightarrow{F}_n\wedge\overrightarrow{F}_m&=\begin{vmatrix} i_1 & -i_2 & i_3\\ F_n & F_{n+1} & F_{n+2}\\ F_m & F_{m+1} & F_{m+2}\end{vmatrix} \qquad (7)\\ &=\frac{1}{5}\{-i_1(2L_{n+m+3}+(-1)^m(L_{n-m-1}-L_{n-m+1}))\\ &\quad -i_2(2L_{n+m+2}-(-1)^m(L_{n-m-2}+L_{n-m+2}))\\ &\quad +i_3(2L_{n+m+1}+(-1)^m(L_{n-m-1}-L_{n-m+1}))\}\\ &=\frac{1}{5}\{-i_1(2L_{n+m+3}-(-1)^mL_{n-m})-i_2(2L_{n+m+2}-3(-1)^mL_{n-m})\\ &\quad +i_3(2L_{n+m+1}-(-1)^mL_{n-m})\}\end{aligned}$$

$$\begin{aligned}\overrightarrow{F}_n\wedge\overrightarrow{F}_{m+1}&=\begin{vmatrix} i_1 & -i_2 & i_3\\ F_n & F_{n+1} & F_{n+2}\\ F_{m+1} & F_{m+2} & F_{m+3}\end{vmatrix} \qquad (8)\\ &=\frac{1}{5}\{-i_1(2L_{n+m+4}-(-1)^mL_{n-m-1})-i_2(2L_{n+m+3}+3(-1)^mL_{n-m-1})\\ &\quad +i_3(2L_{n+m+2}+(-1)^mL_{n-m-1}).\}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{F}_{n+1}\times\overrightarrow{F}_m&=\begin{vmatrix} i_1 & -i_2 & i_3\\ F_{n+1} & F_{n+2} & F_{n+3}\\ F_m & F_{m+1} & F_{m+2}\end{vmatrix} \qquad (9)\\ &=\frac{1}{5}\{-i_1(2L_{n+m+4}+(-1)^m(L_{n-m+1})-i_2(2L_{n+m+3}-3(-1)^m(L_{n-m+1})\\ &\quad +i_3(2L_{n+m+2}-(-1)^m(L_{n-m+1}).\}\end{aligned}$$

Substituting the equations (7), (8) and (9) in equation (6), using the identities of Fibonacci and

Lucas numbers, we have

$$
\begin{aligned}
&= \frac{1}{5}\{-i_1(2\widetilde{L}_{n+m+3} + (-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+4}) \\
&\quad - i_2(\widetilde{L}_{n+m+2} - 3(-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+3}) \\
&\quad + i_3(\widetilde{L}_{n+m+1} + (-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+2}).\}
\end{aligned} \tag{10}
$$

□

Lemma 4.4. *Consider dual Fibonacci vector* $\overrightarrow{\widetilde{V}}_n$. *Then,* $\overrightarrow{\widetilde{V}}_n$ *is a dual unit Fibonacci vector if and only if* $L_{2n+2} = \frac{5-(-1)^{n+1}}{2}$.

Proof. Using the equations $5F_mF_n = L_{n+m} - (-1)^m L_{n-m}$, we have

$$
\begin{aligned}
\|\overrightarrow{\widetilde{V}}_n\|^{\frac{1}{2}} &= F_n^2 + F_{n+1}^2 + F_{n+2}^2 \\
&= F_nF_n + F_{n+1}F_{n+1} + F_{n+2}F_{n+2} \\
&= \frac{1}{5}\big(L_{2n} + L_{2n+2} + L_{2n+4} + (-1)^{n+1}\big) \\
&= \frac{1}{5}\big(2L_{2n+2} + (-1)^{n+1}\big).
\end{aligned}
$$

Since $\|\overrightarrow{\widetilde{V}}_n\| = 1$, we obtain $L_{2n+2} = \frac{5-(-1)^{n+1}}{2}$. □

Theorem 7. *Get two units of dual Fibonacci Vector,* $\overrightarrow{\widetilde{V}}_n$ *and* $\overrightarrow{\widetilde{V}}_m$. *Then the dual bicomplex number product of these vectors is given as*

$$
\begin{aligned}
\overrightarrow{\widetilde{V}}_n \times \overrightarrow{\widetilde{V}}_m &= \big[\langle \vec{F}_n, \vec{F}_m\rangle + \varepsilon\big(\langle \vec{F}_{n+1}, \vec{F}_m\rangle + \langle \vec{F}_n, \vec{F}_{m+1}\rangle\big)\big] \\
&\quad + \vec{F}_n \Lambda \vec{F}_m + \varepsilon(\vec{F}_n \Lambda \vec{F}_{m+1} + \vec{F}_{n+1}\Lambda \vec{F}_m).
\end{aligned} \tag{11}
$$

Proof. Substituting the equations (6) and (10) in equation (11), using the identities of dual Fibonacci numbers and dual Lucas numbers, we obtain

$$
\begin{aligned}
&= \widetilde{L}_{m+n+4} + \widetilde{F}_{m+n+1} - (-1)^{m+2} L_{n-m}(1+\varepsilon) \\
&\quad + \frac{1}{5}\{-i_1(2\widetilde{L}_{n+m+3} + (-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+4}) \\
&\quad - i_2(\widetilde{L}_{n+m+2} - 3(-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+3}) \\
&\quad + i_3(\widetilde{L}_{n+m+1} + (-1)^m L_{n-m}(1+\varepsilon) + 2\varepsilon L_{n+m+2})\}.
\end{aligned}
$$

□

Definition 4.5 (Study Mapping)**.** Let us consider a dual unit vector $\vec{V} = \vec{u} + \varepsilon \vec{u}^*$. Then consider the equation of oriented line which is related to $\vec{V}$, $\vec{\gamma} = \vec{u} \times \vec{u}^* + w\vec{u}$, where $t \in I \subset R$ is a parameter.

Proof. Denote a dual unit vector $\vec{V}$, then the oriented line related to $\vec{V}$ is in direction of $\vec{u}$ as Study mapping, any two point M and X are on line [15]. Let us consider origin which is point O, then we can write

$$\overrightarrow{OX} = \overrightarrow{OM} + \overrightarrow{MX}$$

$$\overrightarrow{OX} = \overrightarrow{OM} + t\overrightarrow{u}.$$

Let a point X be on the line vectors $\overrightarrow{u}$ and $\overrightarrow{u}^*$, then we can write

$$\overrightarrow{u}^* = \overrightarrow{OX} \times \overrightarrow{u}. \tag{12}$$

By using equation (12), we obtain

$$\begin{aligned}\overrightarrow{u} \times \overrightarrow{u}^* &= \overrightarrow{u} \times (\overrightarrow{OX} \times \overrightarrow{u}) \\ &= \langle \overrightarrow{u}, \overrightarrow{u}\rangle \overrightarrow{OX} - \langle \overrightarrow{u}, \overrightarrow{OX}\rangle \overrightarrow{u}.\end{aligned}$$

In equation (12), if we write the last equation, we obtain

$$\overrightarrow{OM} = \overrightarrow{u} \times \overrightarrow{u}^* + (\langle \overrightarrow{u}, \overrightarrow{OX}\rangle + t)\overrightarrow{u}.$$

Let us consider $\langle \overrightarrow{u}, \overrightarrow{OX}\rangle + t = w$ then if a new parameter is γ, we can write the result as:

$$\overrightarrow{\gamma} = \overrightarrow{u} \times \overrightarrow{u}^* + w\overrightarrow{u}. \tag{13}$$

□

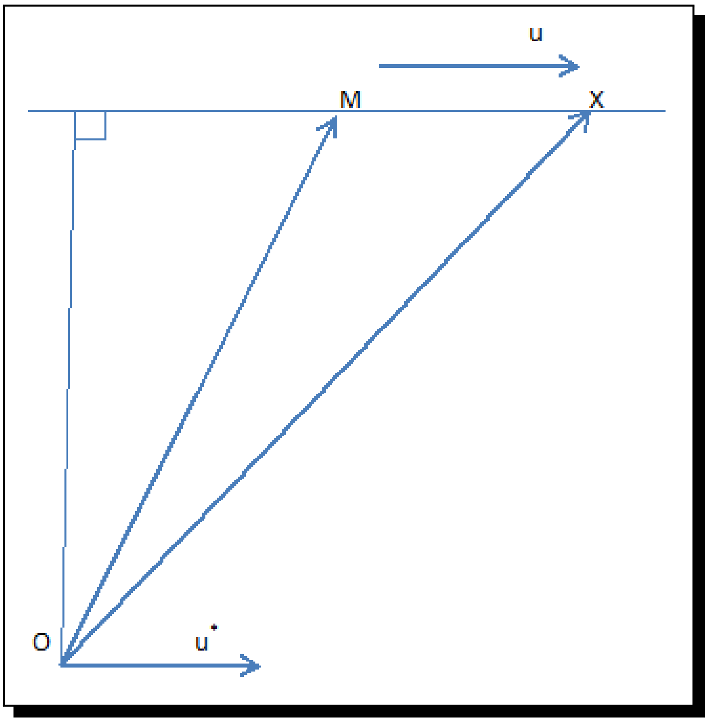

Figure 1. DStudy Mapping.

Corollary 4.6. *Let us denote dual Fibonacci bicomplex unit vector* $\overrightarrow{\tilde{\tilde{V}}}_{o_n}$. *A unit vector* $\overrightarrow{\tilde{\tilde{V}}}_{o_n}$ *is terminal point* $\overrightarrow{\tilde{\tilde{V}}}_n = \overrightarrow{\tilde{F}}_n = \overrightarrow{F}_n + \varepsilon\overrightarrow{F}_{n+1}$ *on the dual unit sphere. Then the equation of oriented line which is according to* $\overrightarrow{\tilde{F}}_n$ *is given as*

$$\overrightarrow{\gamma} = \overrightarrow{u} \times \overrightarrow{u}^* + w\overrightarrow{u}$$

$$\overrightarrow{\gamma} = 2i_1(L_{2n+4} + (-1)^n) + i_2(2L_{2n+3} - 3(-1)^n) + i_3(2L_{2n+4} - 3(-1)^n) + w\overrightarrow{F}_n,$$

where $w \in I \subset R$ *is a parameter.*

Proof. We can obtain the equation of oriented line by using equation (13), which is related to $\overrightarrow{\widetilde{F}}_n$ as following:

$$\overrightarrow{\gamma} = \overrightarrow{F}_n \times \overrightarrow{F}_{n+1} + w\overrightarrow{F}_n.$$

By using Definition 4.3 and Definition 4.4, the identities of Fibonacci numbers and equation $5F_mF_n = L_{n+m} - (-1)^m L_{n-m}$, we obtain

$$\begin{aligned}\overrightarrow{u} \times \overrightarrow{u}^* &= \overrightarrow{F}_n \times \overrightarrow{F}_{m+1}\\ &= \begin{vmatrix} -i_1 & -i_2 & i_3 \\ F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{m+3} \end{vmatrix}\\ &= 2i_1(L_{2n+4} + (-1)^n) - i_2(2L_{2n+3} - 3(-1)^n) + i_3(2L_{2n+4} - 3(-1)^n).\end{aligned}$$

In equation (13), if we write last equation. This completes the procession. □

5. Conclusion

bicomplex numbers that are emerged as a generalization of the best known dual bicomplex numbers. We gave the dual vector which is called dual Fibonacci vector for these dual bicomplex numbers. By using dual Fibonacci vector, we get some results of this vector to expert in geometry such as the cross product of two vector and study mapping.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] W. Guggenheimer, *Differential Geometry*, McGraw-Hill, New York (1963).

[2] D. Rochon and M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, *Anal Univ. Oradea. Fasc. Mathematics* **11** (2004), 71 – 110.

[3] F. Babadag, The real matrices forms of the bicomplex numbers and homothetic exponential motions, *Journal of Advances in Mathematics* **8** (2014), 1401 – 1406.

[4] R.A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific (1997).

[5] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Ellis Horwood Limited Publ., England (1989).

[6] E. Verner and Hoggatt, Jr., *Fibonacci and Lucas Numbers*, The Fibonacci Association (1969).

[7] A.F. Horadam, *A generalized Fibonacci sequence*, *American Math. Monthly* **68** (1961), 455 – 459.

[8] A.F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, *American Math. Monthly* **70** (1963), 289 – 291.

[9] M.N. Swamy, On generalized Fibonacci quaternions, *The Fibonacci Quarterly* **5** (1973), 547 – 550.

[10] M.R. Iyer, Some results on Fibonacci quaternions, *The Fibonacci Quarterly* **7**(2) (1969), 201 – 210.

[11] S. Halıcı, On Fibonacci quaternions, *Adv. in Appl. Clifford Algebras* **22** (2012), 321 – 327.

[12] M. Akyiğit, H.H. Köksal and M. Tosun, Split Fibonacci quaternions, *Adv. in Appl. Clifford Algebras* **23** (2013), 535 – 545.

[13] L. Feng, Decomposition of some type of Quatenions matrices, *Linear and Multilinear Algebra* **58**(4) (2010), 431 – 444.

[14] D. Tascı, *Linear Algebra*, 3, Academic Pres, Selcuk University, 142 – 155 (2005).

[15] H.H. Hacısalihoğlu, *Motion Geometry and Quaternions Theory*, G.Ü. Fen Edebiyat Fakültesi, Ankara (1983).