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Characterization of 7-Groups with a $\mathfrak{C}_{12}$-Covering

Research Article

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Abstract. A group G is covered by a collection of its proper subgroups if it is equal to the union of the collection. A covering is called irredundant if it has no proper sub-collection which also covers G. A covering of G in which all members are maximal subgroups is called maximal. For any integer $n > 2$, a covering with n members is called an n-covering. We call the covering of G as $\mathfrak{C}_n$-covering if it is an irredundant maximal n-covering with core free intersection for G, and we call a group G a $\mathfrak{C}_n$-group if G admits a $\mathfrak{C}_n$-covering. In this paper, we completely characterize 7-groups having a maximal irredundant 12-covering with core-free intersection. From our results, it is proven that a group G is a 7-group having $\mathfrak{C}_{12}$-covering if and only if $G \cong (C_7)^3$.

Keywords. Covering group by subgroup; p-groups; Maximal irredundant covering; Core-free intersection

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1. Introduction

Let G be a finite group. If G is non-cyclic, then G can be obtained as a union of its proper subgroups. A covering C of a group G is a collection of proper subgroups of G whose union is the whole group G. We use the term n-covering for C with n members.

A covering C of G is irredundant if no proper sub-collection is also a covering for G, and is called maximal if all of its members are maximal subgroups of G. We denote the intersection of members of a maximal covering by D. A covering C of G is called core-free if the intersection $D = \bigcap_{M \in C} M$ of C is core-free in G, i.e. $D_G = \bigcap_{g \in G} g^{-1}Dg$ is the trivial subgroup of G. The covering C of G is called a $\mathfrak{C}_n$-covering whenever C is an irredundant maximal core-free n-covering for G. We say a group G is a n-group if G admits $\mathfrak{C}_n$-covering.

It is well known that there is no group that can be covered by two proper subgroups. Scorza [8] was the first to determine the structure of all groups having an irredundant 3-covering with core-free intersection.

Theorem 1.1 (See [8]). *Let $\{A_i \mid 1 \le i \le m\}$ be an irredundant covering with core-free intersection D for a group G. Then, $D = 1$ and $G \cong C_2 \times C_2$.*

In [7], Greco listed all groups with an irredundant 4-covering with core-free intersection. Also, he listed all groups with an irredundant 5-covering in which all pairwise intersection are the same. Then, in [11], Bryce et al. characterized groups with maximal irredundant 5-covering with core-free intersection completely. Specially they proved that G is a p-group if and only if G is an elementary abelian of order 16.

Abdollahi et al. [3] characterized groups with maximal irredundant 6-covering with core-free intersection. In [4], Abdollahi and Amiri listed all groups having a maximal irredundant 7-covering with core-free intersection. Ataei and Sajjad [10] characterized 5-groups with a maximal irredundant 10-covering with core-free intersection. But their result is excluded for $|G| = 5^4$. All of the above results are characterized without appealing to the theory of blocking sets.

Let n be a positive integer. We denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ of order q by $\mathrm{PG}(n,q)$. A hyperplane of $\mathrm{PG}(n,q)$ is a subspace of $\mathrm{PG}(n,q)$ having $(n-1)$-dimension. A blocking set in $\mathrm{PG}(n,q)$ is a set B of points of $\mathrm{PG}(n,q)$ that has non-empty intersection with every hyperplane. A blocking set that contains a line is called trivial. We say that a blocking set is minimal if none of its proper subsets are also blocking sets. For a blocking set B, we denote the least positive integer d such that B is contained in a d-dimensional subspace of $\mathrm{PG}(n,q)$ by $d(B)$. Thus $d(B)$ is equal to the (projective) dimension of subspace spanned by B in $\mathrm{PG}(n,q)$.

Abdollahi [1] and Abdollahi et al. [2] gave some results which clarify the relations between non-trivial minimal blocking sets of size n and $\mathfrak{C}_n$-coverings for groups. They characterized p-groups satisfying $\mathfrak{C}_n$-groups for $n \in \{7,8,9\}$ completely. Their results were derived from the theory of blocking sets. In [9], Ataei characterized nilpotent groups with $\mathfrak{C}_8$-coverings.

Here, we give a complete characterization of 7-groups having $\mathfrak{C}_{12}$-coverings.

2. Preliminaries

We quote the following propositions and lemmas that will be used in the proof later.

Proposition 2.1 (See [6]). *Let B be a minimal blocking set in* PG(2,7)*, with $|B| = n$. Then* $12 \le n \le 19$ (*example of each possible cardinality exist and there are exactly two of size* 12).

Proposition 2.2 (See [5]). *Let p be an odd prime, then $|B| \ge \frac{3}{2}(p+1)$ for the size of a non-trivial blocking set in* PG$(2,p)$.

Proposition 2.3 ([12, Theorem 1.4]). *Let B be a minimal blocking set in* PG$(3,q)$ *with $p > 3$ prime of size at most $\frac{3(p+1)}{2}+1$ is contained in a plane.*

Proposition 2.4 ([2, Proposition 2.6]). *Let p be a prime number and n be a positive integer. Then a finite p-group G is a $\mathfrak{C}_n$-group if and only if $G \cong (C_p)^{m+1}$ for some positive integer m such that* PG(m,p) *has a minimal blocking set B with $d(B) = m$ and $|B| = n$.*

Lemma 2.5 ([2, Lemma 3.2]). *Let G be a finite p-group having a $\mathfrak{C}_n$-covering $\{M_i \mid i = 1,\dots,n\}$. Then*

(a) $p \le n-1$.

(b) *If s the integer such $1 \le s \le n-2$ and $p = n-s$, then $\bigcap_{i\in S} M_i = 1$ for every subset S of* $\{1,2,\dots,n\}$ *with $|S| \ge s+1$.*

(c) *If $n = p+1$, then $G \cong (C_p)^2$.*

Lemma 2.6 ([2, Lemma 3.3]). *Let $G = (C_p)^d$ for $d \ge 2$ and p is a prime number. Suppose that G has $\mathfrak{C}_n$-coverings $\{M_i \mid i = 1,\dots,n\}$. Let $T \subseteq \{1,2,\dots,n\}$.*

(a) *If $|T| = n-p$, then $\left|\bigcap_{i\in T} M_i\right| = 1$ or p.*

(b) *If $|T| = 2$, then $\left|\bigcap_{i\in T} M_i\right| = p^{d-2}$.*

(c) *$\bigcap_{i\in S} M_i = 1$ for some T of size d.*

(d) *If $\bigcap_{i\in S} M_i = 1$ whenever $|S| = d$, then $p \le \left|\bigcap_{i\in T} M_i\right| \le n-d+1$ whenever $|T| = d-1$.*

3. 7-Groups with a $\mathfrak{C}_{12}$-Covering

In this section, we characterized 7-groups satisfying $\mathfrak{C}_{12}$-groups.

Theorem 3.1. *Let G be a 7-group. Then G is a $\mathfrak{C}_{12}$-group, if and only if $G \cong (C_7)^3$.*

Proof. Suppose that G is a 7-group. Since the Frattini subgroups of G, $\phi(G) = G'G^7 \le D$, we have D is a normal subgroup of G. Therefore $D = 1$ and G is an elementary abelian 7-group. By Lemma 2.6(b), we have

$$|G : M_i \cap M_j| = 7^2 \quad \text{for distinct } i, j \in [12]. \tag{3.1}$$

Now, from Lemma 2.5(b) we have that

$$\text{for every } S \subseteq [12] \text{ such that } |S| \ge 12 - 7 + 1 = 6, \ \bigcap_{i \in S} M_i = 1. \tag{3.2}$$

Therefore $|G| \le 7^6$. Also $|G| \ge 7^3$, since otherwise G would not have twelve distinct maximal subgroups ($|G| = 7^2$ has only eight maximal subgroups). Then, Proposition 2.3 implies the non-existence of $\mathfrak{C}_{12}$-covering for $(C_7)^4$.

Assume $|G| = 7^3$, so that $G \cong (C_7)^3$. Proposition 2.1 and Proposition 2.2 imply that there exists a blocking set of size 12. Then, Proposition 2.4 implies that $(C_7)^3$ is a $\mathfrak{C}_{12}$-group. In fact if $G = \langle a, b, c \rangle$, we obtained by GAP[13] that the set

$$F = \{\langle b,c\rangle, \langle a,c\rangle, \langle a,b\rangle, \langle a,bc\rangle, \langle a^5b,c\rangle, \langle a^5c,b\rangle, \langle a,b^4c\rangle, \langle a^5b,ac\rangle, \langle a^4b,a^5c\rangle,$$
$$\langle a^3b,a^2c\rangle, \langle a,b^5c\rangle, \langle a^4b,ac\rangle\}$$

of maximal subgroups forms a $\mathfrak{C}_{12}$-covering for G.

Now, let $|G| = 7^5$. Then Lemma 2.6 implies that $\left|\bigcap_{i \in T} M_i\right| = 1$ for at least one $T \in [12]^5$. Therefore, we assume that there exist $S \in [12]^5$ such that $\left|\bigcap_{i \in S} M_i\right| = 1$. Since, the covering is irredundant, therefore there exist $j \in [12]$ such that for all $L \in [12]^5$, $N = \bigcap_{i \in L} M_i \not\le M_j$. Therefore, $7^5 = |G| = \left|G : \bigcap_{i \in 1}^{6} M_i\right| = |G:N||G:M_j| = |G:N|7$, $|G:N| = 7^4$, $|N| = 7$, which is a contradiction by $\left|\bigcup_{i=1}^{5} M_i\right| = 1$.

Then, we assume that $|G| = 7^6$. Lemma 2.6(d) implies that

$$\left|\bigcap_{i \in T} M_i\right| = 7 \quad \text{for every } T \in [12]^5. \tag{3.3}$$

Then by (3.1) we have that $|M_i \cap M_j| = 7^4$ for distinct $i, j \in [12]$ and so for every $K \in [12]^3$, we have $\left|\bigcap_{i \in K} M_i\right| = 7^3$ or 7^4. Now we prove that $\left|\bigcap_{i \in K} M_i\right| = 7^3$ for all $K \in [12]^3$. Suppose for contradiction, that there exist $L \in [12]^3$ such that $\left|\bigcap_{i\in L} M_i\right| = 7^4$. Let $L' \in [12]^3$ such that $L \cap L' = \phi$. Then it follows from (3.1) and (3.2) that $\left|\bigcap_{i\in L\bigcup L''} M_i\right| = \left|\bigcap_{i\in L'\bigcup L''} M_i\right| = 1$ for every L'' is a proper subgroup of L of size 2. Since $|L'' \cup L'| = 5$, it follows that $|G| \le 7^5$, which is a contradiction. Therefore, we conclude

$$\left|\bigcap_{i\in K} M_i\right| = 3^5 \quad \text{for all } K \in [12]^3. \tag{3.4}$$

By (3.1), we have $\left|\bigcap_{i\in T} M_i\right| \in \{7^2, 7^3\}$ for all $T \in [12]^4$, we prove that $\left|\bigcap_{i\in T} M_i\right| = 7^2$ for all $T \in [12]^4$. Suppose for a contradiction, that there exists $L \in [12]^4$ such that $\left|\bigcap_{i\in L} M_i\right| = 7^3$. Let $L' \in [12]^2$ such that $L \cap L' = \phi$. Then (3.1) and (3.3) imply that $\left|\bigcap_{i\in L\bigcup L''} M_i\right| = \left|\bigcap_{i\in L'\bigcup L''} M_i\right| = 1$ for every $L'' \subset L$ of size 3. Since $|L'' \cup L'| = 5$, it follows that $|G| \le 7^5$, which is a contradiction. Therefore

$$\left|\bigcap_{i\in T} M_i\right| = 7^2 \quad \text{for all } T \in [12]^4. \tag{3.5}$$

Now using (3.1) until (3.5), it follows from the inclusion-exclusion principle that $\left|\bigcup_{i=1}^{12} M_i\right| = \binom{12}{1}7^5 - \binom{12}{2}7^4 + \binom{12}{3}7^3 - \binom{12}{4}7^2 + \binom{12}{5}7 - \binom{12}{6} + \binom{12}{7} - \binom{12}{8} + \binom{12}{9} - \binom{12}{10} + \binom{12}{11} - \binom{12}{12} = 99505$, which is not 7^6, the final contradiction. □

4. Conclusion

The only 7-group that can be covered by twelve irredundant maximal subgroups with core-free intersection is $C_7 \times C_7 \times C_7$. By using GAP[13] we obtain that if $C_7 \times C_7 \times C_7 = \langle a,b,c\rangle$, then the set $F = \{\langle b,c\rangle, \langle a,c\rangle, \langle a,b\rangle, \langle a,bc\rangle, \langle a^5b,c\rangle, \langle a^5c,b\rangle, \langle a,b^4c\rangle, \langle a^5b,ac\rangle, \langle a^4b,a^5c\rangle, \langle a^3b,a^2c\rangle, \langle a,b^5c\rangle, \langle a^4b,ac\rangle\}$ is one of the collections of maximal subgroups of $C_7 \times C_7 \times C_7$ that satisfy the $\mathfrak{C}_{12}$-covering.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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