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The Generalization of the Exterior Square of a Bieberbach Group with Symmetric Point Group

Research Article

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Abstract. The exterior square is a homological functor originated in the homotopy theory, while Bieberbach groups with symmetric point group are torsion free crystallographic groups. In this paper, the generalization of the exterior square of a Bieberbach group with symmetric point group is constructed up to finite dimension.

Keywords. Exterior square; Bieberbach group; Symmetric point group

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1. Introduction

The exterior square of a group G, $G \wedge G$ is defined as $G \wedge G = (G \otimes G)/\nabla(G)$ where $G \otimes G$ is the nonabelian tensor square of G and $\nabla(G)$ is the central subgroup of $G \otimes G$. For g and h in G, the coset $(g \otimes h)\nabla(G)$ is denoted by $g \wedge h$ [2]. $G \otimes G$ is a group generated by the symbols $g \otimes h$, for all $g,h \in G$, subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g,h,k \in G$ where $g^h = h^{-1}gh$ [1]. Meanwhile, $\nabla(G)$ is generated by the element $g \otimes g$, for all $g \in G$ [2].

In [6], the exterior square of a Bieberbach group of dimension four with symmetric point group of order six, $B_1(4)$ was computed and is given in the following theorem.

Theorem 1. *The exterior square of $B_1(4)$ is nonabelian and is given as follows:*

$$B_1(4) \wedge B_1(4) = \left\langle g_1, g_2 \dots g_5 \;\middle|\; \begin{array}{l} g_1^2 = [g_3,g_4] = [g_4,g_5] = 1, \\ [g_2,g_3] = g_3^{-1}g_5^{-1}g_4^2g_1^{-1},\ [g_2,g_4] = g_3^{-2}g_5^{-2}g_4^4, \\ [g_2,g_5] = g_3^{-1}g_4^2g_1^{-1}, [g_3,g_5] = g_1^{-1},\ [g_1,g_j] = 1 \end{array} \right\rangle$$

for $1 \le j \le 5$ *where* $g_1 = l_1 \wedge l_2$, $g_2 = a \wedge b$, $g_3 = a \wedge l_1$, $g_4 = a \wedge l_2$ *and* $g_5 = b \wedge l_2$.

In this paper, the exterior square of the group B_1 is generalized up to dimension n, denoted by $B_1(n) \wedge B_1(n)$.

2. Preparatory Results

In this section, some basic definitions and some preparatory results are presented.

Definition 1 ([4]). Let G be a group with presentation $\langle G \mid R \rangle$ and let G^φ be an isomorphic copy of G via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle G, G^\varphi \mid R, R^\varphi, {}^x[g,h^\varphi] = [{}^xg, ({}^xh)^\varphi] = {}^{x\varphi}[g,h^\varphi],\ \forall\ x,g,h \in G \rangle.$$

Theorem 2 ([3]). *Let G be a group. The map $\sigma : G \otimes G \to [G,G^\varphi] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g,h^\varphi]$ for all g, h in G is an isomorphism.*

Lemma 1 ([4]). *Let x and y be element of G such that $[x,y] = 1$. Then in $\nu(G)$, $[x,y^\varphi]$ is central in $\nu(G)$.*

In [5], the generalized presentation of the polycyclic presentation of the group B_1 has been constructed as in Lemma 2. Besides, the generalizations of the central subgroup of $B_1(n) \otimes B_1(n)$ of the group, $\nabla(B_1(n))$ and the nonabelian tensor square of the group, $B_1(n) \otimes B_1(n)$ and are also constructed in [5] as given in Theorem 4 and Theorem 5, respectively.

Lemma 2 ([5]). *The polycyclic presentation of $B_1(n)$ is consistent where*

$$B_1(n) = \langle a, b, l_1, l_2, l_3, l_4, \ldots, l_n \mid a^2 = l_3, b^3 = l_4, b^a = b^2 l_4^{-1},\ l_1^a = l_1 l_2^{-1},\ l_2^a = l_2^{-1},\ l_3^a = l_3,$$

$$l_4^a = l_4^{-1}, l_p^a = l_p, l_1^b = l_1^{-1} l_2,\ l_2^b = l_1^{-1},\ l_3^b = l_3,\ l_4^b = l_4,\ l_p^b = l_p, l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j$$

$$\text{for } 1 \le i < j \le n \text{ and } 5 \le p \le n \rangle.$$

Theorem 3 ($\nabla(B_1(n))$ [5]). *The subgroup $\nabla(B_1(n))$ is given as*

$$\nabla(B_1(n)) = \langle [a, a^\varphi], [b, b^\varphi], [l_p, l_p^\varphi], [a, b^\varphi][b, a^\varphi], [a, l_p^\varphi][l_p, a^\varphi], [b, l_p^\varphi][l_p, b^\varphi],\ [l_p, l_q^\varphi][l_q, l_p^\varphi] \rangle$$

$$\cong C_0^{\frac{(n-3)(n-2)}{2}} \times C_2^{n-3} \times C_4 \ \text{ for } \ 5 \le p < q \le n.$$

Theorem 4 ($B_1(n) \otimes B_1(n)$ [5]). *The nonabelian tensor square of $B_1(n)$ is nonabelian and is given as follows:*

$$B_1(n) \otimes B_1(n) = \langle g_1, g_2 \ldots g_{(n-2)^2+4} g_2^4 = g_3^2 = g_4^2 = g_t^2 = g_u^2 = [g_6, g_7] = [g_7, g_8] = 1,$$

$$[g_5, g_6] = g_6^{-1} g_8^{-1} g_7^2 g_4^{-1}, [g_5, g_7] = g_6^{-2} g_8^{-2} g_7^4,$$

$$[g_5, g_8] = g_6^{-1} g_7^2 g_4^{-1}, [g_6, g_8] = g_4^{-1}, [g_i, g_j] = [g_t, g_j]$$

$$= [g_u, g_j] = [g_v, g_j] = [g_w, g_j] = [g_x, g_j] = [g_y, g_j] = [g_z, g_j] = 1 \rangle$$

for $1 \le i \le 4,\ 1 \le j \le (n-2)^2 + 4,\ 9 \le t, u \le 2n$ *and* $2n+1 \le v, w, x, y, z \le (n-2)^2 + 4$ *where*

$$g_1 = a \otimes a,\ g_2 = b \otimes b,\ g_3 = (a \otimes b)(b \otimes a),\ g_4 = l_1 \otimes l_2, g_5 = a \otimes b,\ g_6 = a \otimes l_1,\ g_7 = a \otimes l_2,$$

$$g_8 = b \otimes l_2, g_t = b \otimes l_p,\ g_u = (b \otimes l_p)(l_p \otimes b), g_v = l_p \otimes l_p, g_w = a \otimes l_p, g_x = l_p \otimes l_q,$$

$$g_y = (a \otimes l_p)(l_p \otimes a) \text{ and } g_z = (l_p \otimes l_q)(l_q \otimes l_p) \ \text{ for } \ 5 \le p < q \le n.$$

3. Main Result

In this section, the generalization of $B_1(n) \wedge B_1(n)$ is constructed as in the following theorem.

Theorem 5 ($B_1(n) \wedge B_1(n)$). *The nonabelian exterior square of $B_1(n)$ is nonabelian and is given as follows:*

$$B_1(n) \wedge B_1(n) = \langle g_1, g_2 \ldots g_{\frac{(n-3)(n-2)}{2}+4} \mid g_1^2 = g_t^2 = [g_3, g_4] = [g_3, g_5] = 1,$$

$$[g_2, g_3] = g_3^{-1} g_5^{-1} g_4^2 g_1^{-1}, [g_2, g_4] = g_3^{-2} g_5^{-2} g_4^4, [g_2, g_5] = g_3^{-1} g_4^2 g_1^{-1},$$

$$[g_3, g_5] = g_1^{-1}, [g_1, g_j] = [g_t, g_j] = [g_w, g_j] = 1 \rangle$$

for $1 \le j \le \frac{(n-3)(n-2)}{2} + 4,\ 6 \le t \le n+1$ *and* $n+2 \le w \le \frac{(n-3)(n-2)}{2} + 4$ *where*

$$g_1 = l_1 \wedge l_2,\ g_2 = a \wedge b,\ g_3 = a \wedge l_1,\ g_4 = a \wedge l_2,\ g_5 = b \wedge l_2, g_t = b \wedge l_p$$

and $g_w = a \wedge l_p$ *for* $5 \le p < q \le n.$

Proof. $B_1(n)\wedge B_1(n)$ is defined as the quotient of $B_1(n)\otimes B_1(n)$ by $\nabla(B_1(n))$. Hence it is generated by the cosets $(a\otimes a)\nabla(B_1(n))$, $(b\otimes b)\nabla(B_1(n))$, $((a\otimes b)(b\otimes a))\nabla(B_1(n))$, $(l_1\otimes l_2)\nabla(B_1(n))$, $(a\otimes b)\nabla(B_1(n))$, $(a\otimes l_2)\nabla(B_1(n))$, $(l_p\otimes l_p)\nabla(B_1(n))$, $(a\otimes l_p)\nabla(B_1(n))$, $(b\otimes l_p)\nabla(B_1(n))$, $(l_p\otimes l_q)\nabla(B_1(n))$, $((a\otimes l_p)(l_p\otimes a))\nabla(B_1(n))$, $((b\otimes l_p)(l_p\otimes b))\nabla(B_1(n))$, $((l_p\otimes l_q)(l_q\otimes l_p))\nabla(B_1(n))$. Since $a\otimes a$, $b\otimes b$, $(a\otimes b)(b\otimes a)$, $l_p\otimes l_p$, $(a\otimes l_p)(l_p\otimes a)$, $(b\otimes l_p)(l_p\otimes b)$ and $(l_p\otimes l_q)(l_q\otimes l_p)$ are in $\nabla(B_1(n))$, then it can be concluded that $((a\otimes b)(b\otimes a))\nabla(B_1(n))=\nabla(B_1(n))$, $((a\otimes l_p)(l_p\otimes a))\nabla(B_1(n))=\nabla(B_1(n))$, $(a\otimes a)\nabla(B_1(n))=\nabla(B_1(n))$, $((b\otimes l_p)(l_p\otimes b))\nabla(B_1(n))=\nabla(B_1(n))(b\otimes b)\nabla(B_1(n))=\nabla(B_1(n))$, $((l_p\otimes l_q)(l_q\otimes l_p))\nabla(B_1(n))=\nabla(B_1(n))$ and $(l_p\otimes l_p)\nabla(B_1(n))=\nabla(B_1(n))$. Thus, $B_1(n)\wedge B_1(n)$ is generated by the elements as below.

$$\begin{aligned}B_1(n)\wedge B_1(n)&=\langle (l_1\otimes l_2)\nabla(B_1(n)),(a\otimes b)\nabla(B_1(n)),(a\otimes l_1)\nabla(B_1(n)),(a\otimes l_2)\nabla(B_1(n)),\\&\qquad (b\otimes l_2)\nabla(B_1(n)),\ (a\otimes l_p)\nabla(B_1(n)),(b\otimes l_p)\nabla(B_1(n)),(l_p\otimes l_q)\nabla(B_1(n))\rangle\\&=\langle l_1\wedge l_2,a\wedge b,a\wedge l_1,a\wedge l_2,b\wedge l_2,a\wedge l_p,b\wedge l_p,l_p\wedge l_q\rangle.\end{aligned}$$

By Theorem 3 and Theorem 4, both $[l_1,l_2^{\varphi}]$ and $[b,l_p^{\varphi}]$ have order 2 and $[a,b^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_2^{\varphi}]$, $[b,l_2^{\varphi}]$, $[a,l_p^{\varphi}]$ and $[l_p,l_q^{\varphi}]$ have infinite order. Since $5\le p<q\le n$, there are $n-4$ generators in terms of $[a,l_p^{\varphi}]$ and $[b,l_p^{\varphi}]$ and $\frac{(n-5)(n-4)}{2}$ generators in term of $[l_p,l_q^{\varphi}]$. Thus, there are a total of $\frac{(n-3)(n-2)}{2}+4$ generators in $B_1(n)\wedge B_1(n)$.

In Theorem 1, $B_1(4)\wedge B_1(4)$ is showed nonabelian. It follows that $B_1(n)\wedge B_1(n)$ is also nonabelian. Thus, the presentation of $B_1(n)\wedge B_1(n)$ is constructed. Let $g_1=l_1\wedge l_2$, $g_2=a\wedge b$, $g_3=a\wedge l_1$, $g_4=a\wedge l_2$, $g_5=b\wedge l_2$, $g_t=b\wedge l_p$ and $g_w=a\wedge l_p$. By Theorem 1, $g_1^2=1$ since g_1 has order 2. Also, $g_t^2=1$ since g_t has order 2. Since there are $n-4$ generators in terms of g_t, then there are $n+1$ generators included the generators in terms of g_t. Thus, $6\le t\le n+1$. By Lemma 1, g_1, g_t and g_w are central in $\nu(B_1(n))$. Hence, $[g_1,g_j]=[g_t,g_j]=[g_w,g_j]=1$ for $1\le j\le\frac{(n-3)(n-2)}{2}+4$. Since there are $\frac{(n-3)(n-2)}{2}+4$ generators in $B_1(n)\wedge B_1(n)$, then $n+2\le w\le\frac{(n-3)(n-2)}{2}+4$. Besides, by Theorem 1, $[g_2,g_3]=g_3^{-1}g_5^{-1}g_4^2g_1^{-1}$, $[g_2,g_4]=g_3^{-2}g_5^{-2}g_4^4$, $[g_2,g_5]=g_6^{-1}g_4^2g_1^{-1}$, $[g_3,g_4]=1$, $[g_3,g_5]=g_1^{-1}$ and $[g_4,g_5]=1$. Hence, the generalized presentation of $B_1(n)\wedge B_1(n)$ is showed as in Theorem 5. □

4. Conclusion

The exterior square of a Bieberbach group with symmetric point group of order six is generalized up to finite dimension. This finding can be further used to construct the generalization of other homological functors such as the Schur multiplier.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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