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Generalization of the Central Subgroup of the Nonabelian Tensor Square of a Crystallographic Group with Symmetric Point Group

Research Article

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Abstract. The central subgroup of the nonabelian tensor square of a group G, denoted by $\nabla(G)$, is a crucial tool in exploring the properties of a group. It is a normal subgroup generated by the element $g \otimes g$, for all $g \in G$. In this paper, the central subgroup of the nonabelian tensor square of a crystallographic group with symmetric point group is constructed and generalized up to finite dimension.

Keywords. Central subgroup of the nonabelian tensor square; Crystallographic group

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1. Introduction

Crystallographic groups have many interesting properties. The main focus in this paper is a crystallographic group with symmetric point group, denoted by B_2. In [1], the consistent polycyclic presentation of B_2 of dimension four, $B_2(4)$ has been constructed as follows.

$$B_2(4) = \langle a,b,l_1,l_2,l_3,l_4 \mid a^2 = l_3, b^3 = l_3^3 l_4^{-2}, b^a = b^2 l_3^{-1} l_4^2, l_1^a = l_1, l_2^a = l_1 l_2^{-1}, l_3^a = l_3, l_4^a = l_4^{-1},$$
$$l_1^b = l_2^{-1}, l_2^b = l_1 l_2^{-1}, l_3^b = l_3, l_4^b = l_4, l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i,j \le 4\rangle \tag{1.1}$$

The central subgroup of the nonabelian tensor square of a group G, denoted by $\nabla(G)$ is a normal subgroup generated by the element $g \otimes g$, for all $g \in G$. $G \otimes G$ is a group generated by the symbols $g \otimes h$, for all $g,h \in G$, subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g,h,k \in G$ where $g^h = h^{-1}gh$ [2]. Lemma 1 shows the close relationship between $\nabla(G)$ and the abelianization of the group.

Lemma 1 ([3]). *Let Gbe a group whose abelianization is finitely generated by the independent set x_iG', $i = 1,\ldots,n$. Then, $\nabla(G) = \{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \mid 1 \le i < j \le s\}$.*

In [4], the central subgroup of the nonabelian tensor square of the group $B_2(4)$ has been computed. Thus, the aim of this paper is to generalize the central subgroup of the nonabelian tensor square of the group B_2 up to dimension n.

2. Preliminaries

In this section, some basic definitions and some structural results are presented.

Definition 1 ([5], Polycyclic Presentation). *Let F_n be a free group on generators $g_1,\ldots,g_n$ and R be a set of relations of group G. The relations of a polycyclic presentation have the form $g_i^{e_i} = g_{i+1}^{x_{i,i+i}} \ldots g_n^{x_{i,n}}$ for $i \le I$, $g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}}$ for $j \le i$, $g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} \ldots g_n^{z_{i,j,n}}$ for $j \le i$ and $j \notin I$ for some $I \subseteq \{1,\ldots,n\}$, $e_i \in N$ for $i \in I$ and $x_{i,j}, y_{i,j,k} z_{i,j,k} \in Z$ for all i,j and k.*

Definition 2 ([5], Consistent Polycyclic Presentation). *Let G be a group generated by $g_1,\ldots,g_n$. The consistency of the relation in G can be determined using the consistency relations $g_k(g_j g_i) = (g_k g_j) g_i$ for $k > j > i$, $(g_i^{e_j}) g_i = g_j^{e_j^{-1}}(g_j g_i)$ for $j > i$, $j \in I$, $g_j(g_i^{e_i}) = (g_j g_i) g_i^{e_i^{-1}}$ for $j > i$, $f = \inf \in I$, $(g_i^{e_i}) g_i = g_i(g_i^{e_i})$ for $i \notin I$ and $g_j = (g_j g_i^{-1}) g_i$ for $j > i$, $i \notin I$.*

Definition 3 ([6]). *Let G be a group with presentation GR and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be*

$$\nu(G) = G, G^{\varphi} R, R^{\varphi}, {}^{x}[g,h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x\varphi}[g,h^{\varphi}], \text{ for all } x,g,h \in G.$$

Lemma 2 ([7]). *Let G be any crystallographic group of dimension n with point group P. Let $B = G \times F_m^{ab}$ where F_m^{ab} is a free abelian group of rank m. Then B is a crystallographic group of dimension $n + m$ with point group P.*

In [1] and [4], the abelianization of $B_2(4)$ and its central subgroup of the nonabelian tensor square have been determined as follows.

Lemma 3 ([1]). *The abelianization $B_2(4)$ is generated by $aB_2(4)'$ of infinite order, $l_2B_2(4)'$ of order 3 and $l_4B_2(4)'$ of order 2. In symbols, we write $B_2(4)^{ab} \cong \langle aB_2(4), l_2B_2(4), l_4B_2(4)\rangle \cong C_0 \times C_2 \times C_3$.*

Theorem 1 ([4]). *The subgroup $\nabla(B_2(4))$ is given as*

$$\nabla(B_2(4)) = \langle [a,a^\varphi],[l_2,l_2^\varphi],[l_4,l_4^\varphi],[a,l_2^\varphi][l_2,a^\varphi],[a,l_4^\varphi][l_4,a^\varphi]\rangle \cong C_0 \times C_2 \times C_3^2 \times C_4.$$

Lemma 4 ([3]). *Let G be a group with elements x and y such that $[x,y]=1$. Then,*

(i) $[x^n,(y^m)^\varphi][y^m,(x^n)^\varphi] = ([x,y^\varphi][y,(x^\varphi)])^{nm}$,

(ii) *If $g_1 \in G'$ or $g_2 \in G'$, then $[g_1,g_2^\varphi]^{-1} = [g_2,g_2^\varphi]$.*

3. Main Result

In this section, the central subgroup of the nonabelian tensor square of B_2 is generalized up to finite dimension. First, the generalized polycyclic presentation of B_2 is constructed as follows.

Lemma 5. *The polycyclic presentation of $B_2(n)$ is consistent where*

$$\begin{aligned} B_2(n) = \langle a,b,l_1,l_2,l_3,l_4 \mid a^2 = l_3, b^3 = l_3^3 l_4^{-2}, b^a = b^2 l_3^{-1} l_4^2, l_1^a = l_1, l_2^a = l_1 l_2^{-1}, l_3^a = l_3,\\ l_4^a = l_4^{-1}, l_p^a = l_p, l_1^b = l_2^{-1}, l_2^b = l_1 l_2^{-1}, l_3^b = l_3, l_4^b = l_4, l_p^b = l_p, {}_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j\\ \text{for } 1 \le i < j \le n \text{ and } 5 \le p \le n\rangle \end{aligned} \quad (3.1)$$

Proof. By Lemma 2, $B_2(n) = B_2(4) \times F_{n-4}^{ab}$ for $n \le 4$ where $B_2(4)$ has the presentation as in (1.1) and F_{n-4}^{ab} is free abelian of rank $n-4$ which is generated by $l_5, l_6, \ldots, l_n$ and l_p commutes with all elements in $B_2(n)$ for $5 \le p \le n$. Thus, $l_p^a = l_p$, $l_p^b = l_p$, $l_p^{l_1} = l_p$, $l_p^{l_1} = l_p$, $l_p^{l_3} = l_p$ and $l_p^{l_4}$ for $5 \le p \le n$. Therefore, $B_2(n)$ has the polycyclic presentation as in (3.1) which satisfies all the relations as given in Definition 2. □

Next, the generalization of the abelianization of the group B_2 is presented as follows.

Lemma 6. *The abelianization of $B_2(n)$,*

$$B_2(n)^{ab} = \langle aB_2(n)', l_2B_2(n)', l_4B_2(n)', l_pB_2(n)'\rangle \cong C_0^{n-3} \times C_2 \times C_3 \quad \textit{for } 5 \le p \le n.$$

Proof. The abelianization of $B_1(n)^{ab}$ is generated by $aB_2(n)'$, $bB_2(n)'$, $l_2B_2(n)'$, $l_3B_2(n)'$, $l_4B_2(n)'$ and $l_pB_2(n)'$ for $5 \le p \le n$. By Lemma 3, the independent cosets are $aB_2(n)'$, $l_2B_2(n)'$ and $l_4B_2(n)'$. Also, $l_pB_1(n)'$ is independent of other coset. Hence, it can be concluded that $B_2(n)^{ab} = \langle aB_2(n)', l_2B_2(n)', l_4B_2(n)', l_pB_2(n)'\rangle$. By Lemma 3, $aB_2(n)'$ is of infinite order, $l_2B_2(n)'$

is of order 3 and $l_4B_2(n)'$ is of order 2. Besides, $l_pB_2(n)'$ is showed to have infinite order since there is no l_p^r in $B_2(n)'$ for any integer r. Since $5 \le p \le n$, then there are $n-4$ cosets in term of $l_pB_2(n)'$. Therefore, $B_2(n)^{ab} \cong C_0 \times C_2 \times C_3 \times C_0^{n-4} = C_0^{n-3} \times C_2 \times C_3$. □

Then, the construction of $\nabla(B_2(n))$ is showed as in the following theorem.

Theorem 2. *The subgroup* $\nabla(B_2(n))$ *is given as*

$$\begin{aligned}\nabla(B_2(n)) = \langle &[a,a^\varphi],[l_2,l_2^\varphi],[l_4,l_4^\varphi],[l_p,l_p^\varphi],[a,l_2^\varphi][l_2,a^\varphi],[a,l_4^\varphi][l_4,a^\varphi],[a,l_p^\varphi][l_p,a^\varphi],\\ &[l_2,l_p^\varphi][l_p,l_2^\varphi],[l_4,l_p^\varphi][l_p,l_4^\varphi],[l_p,l_q^\varphi][l_q,l_p^\varphi]\rangle\\ \cong\ & C_0^{\frac{(n-3)(n-2)}{2}} \times C_2^{n-3} \times C_3^{n-2} \times C_4 \quad \textit{for } 5 \le p < q \le n.\end{aligned}$$

Proof. By Lemma **??**, $B_1(n)^{ab}$ is generated by $aB_2(n)'$, $l_2B_2(n)'$, $l_4B_2(n)'$, and $l_pB_2(n)'$ for $5 \le p \le n$. Thus, by Lemma 1, $\nabla(B_1(n)) = \langle[a,a^\varphi]$, $[l_2,l_2^\varphi]$, $[l_4,l_4^\varphi]$, $[l_p,l_p^\varphi]$, $[a,l_2^\varphi][l_2,a^\varphi]$, $[a,l_4^\varphi][l_4,a^\varphi]$, $[a,l_p^\varphi][l_p,a^\varphi]$, $[l_2,l_p^\varphi][l_p,l_2^\varphi]$, $[l_4,l_p^\varphi][l_p,l_4^\varphi]$, $[l_p,l_q^\varphi][l_q,l_p^\varphi]\rangle$ for $5 \le p < q \le n$.

By Theorem 1, $[a,a^\varphi]$ has infinite order, $[l_4,l_4^\varphi]$ has order 4, $[a,l_4^\varphi]$ $[l_4,a^\varphi]$ has order 2, and both $[a,l_2^\varphi][l_2,a^\varphi]$ and $[l_2,l_2^\varphi]$ have order 3. By Lemma 6(i) and (ii), it can be concluded that $[l_2,l_p^\varphi][l_p,l_2^\varphi]$ has order 3 since $([l_2,l_p^\varphi][l_p,l_2^\varphi])^3 = [l_2^3,l_p^\varphi][l_p,l_2^{3\varphi}] = [l_2^3,l_p^\varphi][l_2^3,l_p^\varphi]^{-1} = 1$. Similarly, $[l_4,l_p^\varphi][l_p,l_4^\varphi]$ has order 2. Next, suppose that the order of $[a,l_p^\varphi][l_p,a^\varphi]$ is finite, then $[a^r,l_p^{s\varphi}][l_p^s,a^{r\varphi}] = ([a,l_p^\varphi][l_p,a^\varphi])^{rs} = 1$ for any integers r and s. Thus, $[l_p^s,a^{r\varphi}] = [a^r,l_p^{s\varphi}]^{-1}$. However, this is not true since there is no a^r and l_p^s in $B_2(n)'$. Therefore, $[a,l_p^\varphi][l_p,a^\varphi]$ has infinite order. Using the similar argument, $[l_p,l_q^\varphi][l_q,l_p^\varphi]$ and $[l_p,l_p^\varphi]$ also have infinite order.

Since $5 \le p < q \le n$, then there are $n-4$ generators in terms of $[l_p,l_p^\varphi]$, $[a,l_p^\varphi][l_p,a^\varphi]$, $[l_2,l_p^\varphi][l_p,l_2^\varphi]$ and $[l_4,l_p^\varphi][l_p,l_4^\varphi]$ and $\frac{(n-5)(n-4)}{2}$ generators in term of $[l_p,l_q^\varphi][l_q,l_p^\varphi]$. Hence, $\nabla(B_2(n)) \cong C_0 \times C_3 \times C_4 \times C_0^{n-4} \times C_3 \times C_2 \times C_0^{n-4} \times C_3^{n-4} \times C_2^{n-4} \times C_0^{\frac{(n-5)(n-4)}{2}} = C_0^{\frac{(n-3)(n-2)}{2}} \times C_2^{n-3} \times C_3^{n-2} \times C_4$. □

4. Conclusion

In this paper, the generalization of the central subgroup of the nonabelian tensor square of a crystallographic group with symmetric point group, $B_2(n)$ is constructed up to finite dimension n. Besides, the generalized polycyclic presentation and the generalized abelianization of the group are also presented.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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