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Research Article

# Type-2 Fuzzy Equivalence Relation on A Groupoid under Balanced and Semibalanced Maps

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**Abstract.** In this paper we generalize the idea of balanced and semibalanced maps in type-2 fuzzy sets. The notion of type-2 fuzzy G-equivalence and G-congruence on a groupoid are introduced and some properties related to these notions have been established.

**Keywords.** Type-2 fuzzy congruence; Type-2 fuzzy semibalanced mappings; Type-2 fuzzy $f$-invariant; Type-2 fuzzy $f$-stable

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## 1. Introduction

Type-2 fuzzy sets were introduced by Zadeh [33] as an extension of type-1 fuzzy sets [32].

Type-2 fuzzy sets have membership functions as type-1 fuzzy sets. The advantage of type-2 fuzzy sets is that they are helpful in some cases where it is difficult to find the exact membership functions for a fuzzy sets. There are wide variety of applications of type-2 fuzzy sets in science and technology like computing with words [22], human resource management [13], forecasting of time-series [18], clustering [1, 27], pattern recognition [5], fuzzy logic controller [31], industrial application [7], simulation [28], neural network [4, 29], and solid transportation problem [21]. The concept of cartesian product of type-2 fuzzy sets was given by Hu et al. [16] as an extension of type-1 fuzzy sets. The properties of membership grades of type-2 fuzzy sets, set-theoretic operations of such sets have been studied by Mizumoto et al. [23, 24]. The composition of type-2 relations as an extension of type-1 sup-star composition but this formula is only for minimum type-2 t-norm has been discussed by Dubois et al. [8, 9, 10].

The motivation of this paper is to introduce semibalanced and balanced mappings in the type-2 fuzzy sets. We also discussed the type-2 fuzzy $f$-*invariant*, type-2 fuzzy $f$-*stable* and some of its properties.

A brief sketch of the paper is as follows: Section 2 introduces some basic definitions related to the concept. We have defined type-2 fuzzy G-equivalence relation, type-2 fuzzy G-preorder and type-2 fuzzy relation compatible on the groupoid in Section 3. Section 4 deals with the results of images and preimages of type-2 fuzzy equivalences and congruences on a groupoid. Section 5 describes the images and preimages of type-2 fuzzy G-equivalences and G-congruences on a groupoid.

## 2. Preliminaries

**Definition 2.1** ([20]). A type-2 fuzzy set $\tilde{A}$ defined on the universe of discourse $X$ is characterized by a membership function $\mu_{\tilde{A}}(x) : X \to F([0,1])$ and is expressed by the following set notation: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. $F([0,1])$ denotes the set of all type-1 fuzzy sets that can be defined on the set $[0,1]$. $\mu_{\tilde{A}}(x)$, itself is a type-1 fuzzy set for value of $x \in X$ and is characterized by a secondary membership function $f_x : J_x \to [0,1]$. Therefore, $\tilde{A}$ can be represented as: $\tilde{A} = \{\langle x, \{(u, f_x(u)) : u \in J_x\}\rangle : x \in X\}$, where $J_x \subseteq [0,1]$ is the set of all possible primary membership functions corresponding to an element $x$. In discrete case, a type-2 fuzzy set $\tilde{A}$ can also be expressed in the following ways:

$$\tilde{A} = \left\{\left(x, \sum_{u \in J_x} \frac{f_x(u)}{u}\right) : x \in X, u \in J_x \subseteq [0,1]\right\},$$

$$\tilde{A} = \sum_{x \in X} \frac{\sum_{u \in J_x} \frac{f_x(u)}{u}}{x}.$$

**Example 2.1.** A type-2 fuzzy set defined on a finite universal set and finite set of primary membership can be represented by a 3-dimensional picture given in Figure 1.
Let $X = \{1,2,3,4,5\}$ be the universe of discourse and suppose $J_1 = \{0.25, 0.5, 0.75, 1\} = J_2 = J_4$, $J_3 = \{0.75, 1\}$, $J_5 = \{0.25, 0.5, 1\}$ be the sets of primary membership for $x = 1,2,3,4,5$ respectively. The secondary membership function associated with $x = 1$ is represented by a fuzzy set

$$\mu_{\tilde{A}}(1) = \frac{0.3}{0.25} + \frac{0.5}{0.5} + \frac{0.7}{0.75} + \frac{0.6}{1}.$$

This secondary membership function can also be viewed through the five vertical lines at points $(1, 0.25)$, $(1, 0.5)$, $(1, 0.75)$ and $(1, 1)$ in the Figure 1. Similarly, we can define the secondary membership function for $x = 2,3,4,5$. We have shown all secondary membership functions in the following figure. Shaded portion is called the footprint of uncertainty.

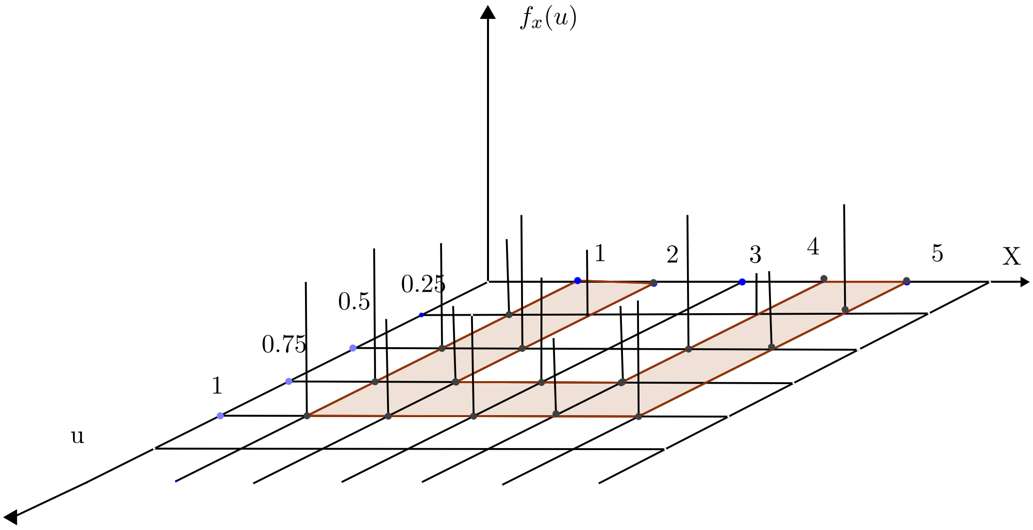


**Figure 1.** Type-2 Fuzzy Set

**Definition 2.2** ([20])**.** Uncertainty in the primary memberships of a type-2 fuzzy set consists of a bounded region that we call as the footprint of uncertainty and is denoted by $FOU(\tilde{A})$. It is defined by $FOU(\tilde{A}) = \bigcup_{x\in X} J_x$. The footprint of uncertainty in the Example 2.1 is

$$FOU(\tilde{A}) = \{0, 0.25, 0.5, 0.75, 1\}.$$

**Definition 2.3** ([20])**.** For every value $x = x'$, say, the 2-D plane whose axes are $u$ and $f_{x'}(u)$ is called the vertical slice. A secondary membership function is thus a vertical slice. It can be represented by $\mu_{\tilde{A}}(x') = \sum_{u\in J_{x'}} \frac{f_{x'}(u)}{u}$, $J_{x'} \subseteq [0,1]$, in which $0 \le f_{x'}(u) \le 1$, $x \in X$.
The domain of a secondary membership function is called the primary membership of $x$ and the amplitude of a secondary membership function is called a secondary grade. In the above equation, $f_{x'}(u)$ is a secondary grade.

**Definition 2.4** ([16])**.** Let $X$ and $Y$ be two nonempty universes. Then a type-2 fuzzy set, $R \in Map(X \times Y, Map(J,I))$ is called a type-2 fuzzy relation (T2 FR, in short) from $X$ to $Y$.

**Definition 2.5** ([16])**.** The cartesian product of two type-2 fuzzy sets is a T2 FR, which is defined for all $A \in Map(X, Map(J,I))$ and $B \in Map(Y, Map(J,I))$, as $(A \times B)(x,y) = A(x) \tilde{\wedge} B(y)$. If $J$ is bounded, then we have the following special T2 FRs, for all $x \in X$ and $y \in Y$

$$\tilde{Q}(x,y) = \begin{cases} \tilde{0}, & \text{if } x \ne y, \\ \tilde{1}, & \text{if } x = y. \end{cases}$$

**Definition 2.6** ([26])**.** Let $f$ be a mapping from the type-2 fuzzy sets $(X, Map(J,I))$ and $(Y, Map(J,I))$. If $\tilde{Q}$ is a type-2 fuzzy subset of $Y$, the inverse image $f^{-1}(\tilde{Q})$ of $\tilde{Q}$ is the type-2 fuzzy subset of $X$ defined by $f^{-1}(\tilde{Q})(x) = \tilde{Q}(f(x))$.

If $\tilde{P}$ is a type-2 fuzzy subset of $X$, the image $f(\tilde{P})$ of $\tilde{P}$ is the type-2 fuzzy subset of $Y$ defined by

$$f(\tilde{P})(y)=\begin{cases}\tilde{\sup}_{t\in f^{-1}(y)} \tilde{P}(t), & \text{if } f^{-1}(y)\neq\phi,\\ \tilde{0}, & \text{if } f^{-1}(y)=\phi, y=\dfrac{f_y(a)}{a}\in Y.\end{cases}$$

## 3. Type-2 Fuzzy G-equivalence Relation

Type-2 fuzzy reflexive, symmetric and transitive relations were discussed in [16]. We extend the definitions of reflexive relations and studied the T2FR compatible on the groupoid.

**Definition 3.1** ([16])**.** Let $\tilde{Q}$ be a T2 FR on $X$. Then $\tilde{Q}$ is said to be

(1) *reflexive* if $J$ is bounded and $\tilde{Q}(x,x)=\tilde{1}$ for all $x\in X$.

(2) *antireflexive* if $J$ is bounded and $\tilde{Q}(x,x)=\tilde{0}$ for all $x\in X$.

(3) *weakly reflexive* if $\tilde{Q}(x,y)\sqsubseteq\tilde{Q}(x,x)$ and $\tilde{Q}(y,x)\sqsubseteq\tilde{Q}(x,x)$ for all $x,y\in X$.

(4) *symmetric* if $\tilde{Q}(x,y)=\tilde{Q}(y,x)$.

(5) *antisymmetric* if $J$ is bounded and $\tilde{Q}$ satisfies $\tilde{Q}(x,y)=\tilde{0}$ or $\tilde{Q}(y,x)=\tilde{0}$ for all $x,y\in X$ $(x\neq y)$.

(6) *transitive* if $\tilde{Q}\circ\tilde{Q}\sqsubseteq\tilde{Q}$, where $\tilde{Q}\circ\tilde{Q}$ is defined by $\tilde{Q}\circ\tilde{Q}(x,y)=\tilde{\sup}_{z\in X}\{\tilde{Q}(x,z)\tilde{\wedge}\tilde{Q}(z,y)\}$.

**Definition 3.2** ([11])**.** We have given below the definition of type-2 fuzzy G-reflexive relation in addition to the already above defined definitions in [16]. $\tilde{Q}$ is said to be a type-2 fuzzy G-reflexive if

(1) $\tilde{0}<\tilde{Q}(x,x)<\tilde{1}$;

(2) $\tilde{Q}(x,y)\sqsubseteq\tilde{\inf}_{t\in X}\tilde{Q}(t,t)$ for all $x\neq y$ in $X$.

**Definition 3.3** ([11])**.** A type-2 fuzzy relation $\tilde{Q}$ in $X$ is a type-2 fuzzy G-equivalence relation in $X$ if $\tilde{Q}$ is G-reflexive, symmetric and transitive in $X$.

**Example 3.1.** There are three groups of research scholars in a research institute. A type-2 fuzzy G-equivalence relation is produced by the doctoral committees according to the level of research by the scholars. The quality of research is transformed to the following type-2 fuzzy relation whose entries are linguistic terms:

$$\tilde{Q}=\begin{pmatrix}\text{Good} & \text{Marginal} & \text{Below Average}\\ \text{Marginal} & \text{Very Good} & \text{Average}\\ \text{Below Average} & \text{Average} & \text{Outstanding}\end{pmatrix}$$

These linguistic terms can be written in the form of type-2 fuzzy sets as follows:

$$\tilde{Q} = \begin{pmatrix} \frac{0.5}{0.5}+\frac{0.6}{0.6} & \frac{0.1}{0.1}+\frac{0.2}{0.2} & \frac{0.2}{0.1}+\frac{0.3}{0.2} \\ \frac{0.1}{0.1}+\frac{0.2}{0.2} & \frac{0.7}{0.7}+\frac{0.8}{0.8} & \frac{0.1}{0.3}+\frac{0.1}{0.4} \\ \frac{0.2}{0.1}+\frac{0.3}{0.2} & \frac{0.1}{0.3}+\frac{0.1}{0.4} & \frac{0.9}{0.9}+\frac{1}{1} \end{pmatrix}$$

Clearly, (i) $\tilde{0} < \tilde{Q}(x,x) < \tilde{1}$.

(ii) $\tilde{Q}(x,y) \sqsubseteq \inf_{t\in X}\tilde{Q}(t,t)$ for all $x \neq y$ in $X$.

Therefore, $\tilde{Q}$ is a type-2 fuzzy G-reflexive relation.

Again, $\tilde{Q}(x,y) = \tilde{Q}(y,x)$ for all $x,y \in X$.

Therefore, $\tilde{Q}$ is a type-2 fuzzy symmetric relation.

Now, $\tilde{Q}\circ\tilde{Q} = \begin{pmatrix} \frac{0.2}{0.5}+\frac{0.2}{0.6} & \frac{0.1}{0.1}+\frac{0.2}{0.2} & \frac{0.1}{0.1}+\frac{0.1}{0.2} \\ \frac{0.1}{0.1}+\frac{0.2}{0.2} & \frac{0.1}{0.7}+\frac{0.1}{0.8} & \frac{0.1}{0.3}+\frac{0.1}{0.4} \\ \frac{0.1}{0.1}+\frac{0.1}{0.2} & \frac{0.1}{0.3}+\frac{0.1}{0.4} & \frac{0.1}{0.9}+\frac{0.1}{1} \end{pmatrix}$.

Thus, $\tilde{Q}\circ\tilde{Q} \sqsubseteq \tilde{Q}$.

Hence, $\tilde{Q}$ is a type-2 fuzzy transitive relation.

Consequently, $\tilde{Q}$ is a type-2 fuzzy G-equivalence relation.

**Remark 3.1.** The type-2 fuzzy G-reflexive relation helps us to calculate type-2 fuzzy G-equivalence relation without the loss of generality of having $\tilde{Q}(x,x) = \tilde{1} = \frac{1}{1}$ on $X$.

**Definition 3.4.** If a T2 FR $\tilde{Q}$ in $X$ is a type-2 fuzzy G-reflexive and transitive, then $\tilde{Q}$ is called a type-2 fuzzy G-preorder in $X$.

**Remark 3.2.** By $\tilde{Q}^n$, $n = 1,2,3,\dots$ we mean $\tilde{Q}\circ\tilde{Q}\circ\cdots\circ\tilde{Q}$ ($n$ factors).

**Theorem 3.1.** *If $\tilde{Q}$ is a type-2 fuzzy G-preorder in $X$, then $\tilde{Q}^n = \tilde{Q}$, $n = 1,2,3,\dots$.*

*Proof.* It follows from the transitivity property that

$$\tilde{Q}\circ\tilde{Q} \sqsubseteq \tilde{Q} \tag{1}$$

On the other hand,

$$\begin{aligned} \tilde{Q}\circ\tilde{Q}(x,y) &= \tilde{\sup_{t\in X}}\{\tilde{Q}(x,t)\tilde{\wedge}\tilde{Q}(t,y)\} \\ &\sqsupseteq \tilde{Q}(x,x)\tilde{\wedge}\tilde{Q}(x,y) \end{aligned}$$

$$\Rightarrow \qquad \tilde{Q} \circ \tilde{Q}(x,y) \sqsupseteq \tilde{Q}(x,y) \ \forall \ x,y \in X.$$

Therefore,

$$\tilde{Q} \circ \tilde{Q} \sqsupseteq \tilde{Q}. \tag{2}$$

From (1) and (2), we get $\tilde{Q} \circ \tilde{Q} = \tilde{Q}$.

Similarly, $\tilde{Q} \circ \tilde{Q} \circ \tilde{Q} = \tilde{Q}$.

Therefore, $\tilde{Q}^n = \tilde{Q}$. □

**Definition 3.5.** Let $\tilde{Q}$ be a type-2 fuzzy relation on the groupoid $D$. $\tilde{Q}$ is compatible on $D$ if $\tilde{Q}(ac,bd) \geq \tilde{Q}(a,b) \tilde{\wedge} \tilde{Q}(c,d)$ for all $a,b,c,d \in D$.

A compatible type-2 fuzzy equivalence relation on a groupoid is a type-2 fuzzy congruence.

**Definition 3.6.** Let $\tilde{Q}$ be a type-2 fuzzy relation on the type-2 fuzzy set $X$ with $\tilde{0} < \tilde{\alpha} \leq \tilde{1}$. $\tilde{Q}$ is a type-2 fuzzy *stronger* G-reflexive on $X$ if $\tilde{Q}(a,a) = \tilde{\alpha}$ and $\tilde{Q}(a,b) \leq \tilde{\alpha}$ for all $a,b \in X$.

**Definition 3.7.** A type-2 fuzzy *stronger* G-reflexive, symmetric and transitive relation on $X$ is a type-2 fuzzy *stronger* G-equivalence relation on $X$.

**Remark 3.3.** A type-2 fuzzy *stronger* G-equivalence relation on $X$ is a particular case of type-2 fuzzy equivalence relation when $\tilde{\alpha} = \tilde{1}$. Every type-2 fuzzy *stronger* G-equivalence relation is a type-2 G-equivalence relation.

**Remark 3.4.** We call a compatible type-2 fuzzy *stronger* G-equivalence (G-equivalence) relation on a groupoid a *stronger* G-congruence (G-congruence).

## 4. Images and Preimages of Type-2 Fuzzy Equivalences and Congruences on a Groupoid

**Theorem 4.1.** *If $\tilde{Q}$ is a compatible type-2 fuzzy relation on the groupoid $S$ and $f$ is a groupoid homomorphism from $D \times D$ into $S \times S$, then $f^{-1}(\tilde{Q})$ is a compatible type-2 fuzzy relation on $D$.*

*Proof.* Let $a,b,c,d \in D$. Then we have,

$$\begin{aligned} f^{-1}(\tilde{Q})(ac,bd) &= \tilde{Q}(f(ac,bd)) \\ &= \tilde{Q}(f(a,b),f(c,d)) \\ &\geq \tilde{Q}(f(a,b)) \tilde{\wedge} \tilde{Q}(f(c,d)) \end{aligned}$$

$$\Rightarrow \qquad f^{-1}(\tilde{Q})(ac,bd) \geq f^{-1}(\tilde{Q})(a,b) \tilde{\wedge} f^{-1}(\tilde{Q})(c,d).$$

Hence, $f^{-1}(\tilde{Q})$ is a compatible type-2 fuzzy relation on $D$. □

**Theorem 4.2.** *If $\tilde{Q}$ is a compatible type-2 fuzzy relation on the groupoid $D$ and $f$ is a groupoid homomorphism from $D \times D$ into $S \times S$, then $f(\tilde{Q})$ is a compatible type-2 fuzzy relation on $S$.*

*Proof.* Let $u,v,w,r \in S$.

Considering the case, when either $f^{-1}(u,v)$ or $f^{-1}(w,r)$ is empty.

We have, $f(\tilde{Q})(uw,vr) \geq \tilde{0}$.

Again, taking $f^{-1}(u,v)$ and $f^{-1}(w,r)$ non empty, we get,

$$\begin{aligned}
f(\tilde{Q})(uw,vr) &= \tilde{\sup_{(x,x')\in f^{-1}(uw,vr)}} \tilde{Q}(f(x,x')) \\
&\geq \tilde{\sup_{(ac,bd)\in f^{-1}(uw,vr)}} \tilde{Q}(f(ac,bd)) \\
&\geq \tilde{\sup_{f(a,b)\cdot f(c,d)=(u,v)(w,r)}} [\tilde{Q}(a,b)\tilde{\wedge}\tilde{Q}(c,d)] \\
&= \tilde{\sup_{f(a,b)=(u,v)}} \tilde{Q}(a,b)\tilde{\wedge} \tilde{\sup_{f(c,d)=(w,r)}} \tilde{Q}(c,d) \\
&= \tilde{\sup_{(a,b)\in f^{-1}(u,v)}} \tilde{Q}(a,b)\tilde{\wedge} \tilde{\sup_{(c,d)\in f^{-1}(w,r)}} \tilde{Q}(c,d)
\end{aligned}$$

$\Rightarrow \quad f(\tilde{Q})(uw,vr) \geq f(\tilde{Q})(u,v)\tilde{\wedge} f(\tilde{Q})(v,r)$

$\Rightarrow \quad f(\tilde{Q})(uw,vr) \geq f(\tilde{Q})(u,v)\tilde{\wedge} f(\tilde{Q})(v,r)$.

Hence, $f(\tilde{Q})$ is a compatible type-2 fuzzy relation on $S$. □

The definition of semibalanced mapping in type-2 fuzzy sets is defined below:

**Definition 4.1.** Let $X$ and $Y$ be two non empty type-2 fuzzy sets. A mapping $f : X \times X \to Y \times Y$ is called a semibalanced mapping, if

(i) given $a \in X$, there exists $e \in Y$ such that $f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right) = \left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)$;

(ii) $f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right) = \left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)$ and $f\left(\frac{f_x(v)}{v},\frac{f_x(v)}{v}\right) = \left(\frac{f_y(b)}{b},\frac{f_y(b)}{b}\right)$,

implies that $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) = \left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$.

**Example 4.1.** Let $X = \{a,b,c\}$ and $Y = \{1,2,3\}$. Define the mapping $f$ from $(X \times X, Map(J,I))$ to $(Y \times Y, Map(J,I))$ as follows:

$$f\left(\frac{1}{1},\frac{0.9}{0.9}\right) = f\left(\frac{0.8}{0.8},\frac{0.9}{0.9}\right) = \left(\frac{1}{1},\frac{0.9}{0.9}\right), \quad f\left(\frac{0.9}{0.9},\frac{1}{1}\right) = f\left(\frac{0.9}{0.9},\frac{0.8}{0.8}\right) = \left(\frac{0.9}{0.9},\frac{1}{1}\right),$$
$$f\left(\frac{1}{1},\frac{0.8}{0.8}\right) = f\left(\frac{0.8}{0.8},\frac{1}{1}\right) = f\left(\frac{0.8}{0.8},\frac{0.8}{0.8}\right) = f\left(\frac{1}{1},\frac{1}{1}\right) = \left(\frac{1}{1},\frac{1}{1}\right) \text{ and } f\left(\frac{0.8}{0.8},\frac{0.8}{0.8}\right) = \left(\frac{0.8}{0.8},\frac{0.8}{0.8}\right),$$

where $X = \frac{\frac{1}{1}}{a} + \frac{\frac{0.9}{0.9}}{b} + \frac{\frac{0.8}{0.8}}{c}$ and $Y = \frac{\frac{1}{1}}{1} + \frac{\frac{0.9}{0.9}}{2} + \frac{\frac{0.8}{0.8}}{3}$, respectively. Then, $f$ is a semibalanced mapping.

**Definition 4.2.** Let $X$ and $Y$ be two non empty type-2 fuzzy sets. A mapping $f : X \times X \to Y \times Y$ is called a balanced mapping, if

(i) $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)\Rightarrow\frac{f_x(u)}{u}=\frac{f_x(v)}{v}$,

(ii) $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(b)}{b},\frac{f_y(c)}{c}\right)\Rightarrow f(\frac{f_x(v)}{v},\frac{f_x(u)}{u})=(\frac{f_y(c)}{c},\frac{f_y(b)}{b})$,

(iii) $f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)$ and $f\left(\frac{f_x(v)}{v},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(d)}{d},\frac{f_y(d)}{d}\right)$

$\Longleftrightarrow f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(d)}{d}\right)$, where $X=\frac{\frac{f_x(u)}{u}}{p}+\frac{\frac{f_x(v)}{v}}{q}$, then $p,q\in X$, primary membership of $p$ is $u$ and secondary membership is $f_x(u)$. Similarly, the primary membership of $q$ is $v$ and secondary membership is $f_x(v)$.

Again, $Y=\frac{\frac{f_y(a)}{a}}{e}+\frac{\frac{f_y(b)}{b}}{f}+\frac{\frac{f_y(c)}{c}}{g}+\frac{\frac{f_y(d)}{d}}{h}$, where $e,f,g,h\in Y$.

The following results can easily be verified.

(a) given $x=\frac{f_x(u)}{u}\in X$, then there exists $u=\frac{f_y(a)}{a}\in Y$ such that $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)$,

(b) $f$ is a one-to-one mapping from $X\times X$ into $Y\times Y$,

(c) Let $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$.

Given, $z=\frac{f_x(w)}{w}\in X$, there exists $t_{z_1}=\frac{f_y(t_z)}{t_z}\in Y$ such that $f\left(\frac{f_x(u)}{u},\frac{f_x(w)}{w}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(t_z)}{t_z}\right)$ and $f\left(\frac{f_x(w)}{w},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(t_z)}{t_z},\frac{f_y(b)}{b}\right)$.

**Example 4.2.** Let $X=\{a,b,c\}$ and $Y=\{1,2,3\}$. Define the mapping $f$ from $(X\times X,Map(J,I))$ to $(Y\times Y,Map(J,I))$ as follows:

$$f\left(\frac{0.5}{0.5},\frac{0.5}{0.5}\right)=\left(\frac{0.5}{0.5},\frac{0.5}{0.5}\right),\quad f\left(\frac{0.3}{0.3},\frac{0.3}{0.3}\right)=\left(\frac{0.3}{0.3},\frac{0.3}{0.3}\right),\quad f\left(\frac{0.2}{0.2},\frac{0.2}{0.2}\right)=\left(\frac{0.2}{0.2},\frac{0.2}{0.2}\right),$$
$$f\left(\frac{0.5}{0.5},\frac{0.3}{0.3}\right)=\left(\frac{0.5}{0.5},\frac{0.3}{0.3}\right),\quad f\left(\frac{0.3}{0.3},\frac{0.5}{0.5}\right)=\left(\frac{0.3}{0.3},\frac{0.5}{0.5}\right),\quad f\left(\frac{0.5}{0.5},\frac{0.2}{0.2}\right)=\left(\frac{0.5}{0.5},\frac{0.2}{0.2}\right),$$
$$f\left(\frac{0.2}{0.2},\frac{0.5}{0.5}\right)=\left(\frac{0.2}{0.2},\frac{0.5}{0.5}\right),\quad f\left(\frac{0.3}{0.3},\frac{0.2}{0.2}\right)=\left(\frac{0.3}{0.3},\frac{0.2}{0.2}\right),\quad f\left(\frac{0.2}{0.2},\frac{0.3}{0.3}\right)=\left(\frac{0.2}{0.2},\frac{0.3}{0.3}\right),$$

where $X=\frac{\frac{0.5}{0.5}}{a}+\frac{\frac{0.3}{0.3}}{b}+\frac{\frac{0.2}{0.2}}{c}$ and $Y=\frac{\frac{0.5}{0.5}}{1}+\frac{\frac{0.3}{0.3}}{2}+\frac{\frac{0.2}{0.2}}{3}$ respectively. Then, $f$ is a balanced mapping.

**Remark 4.1.** A mapping $f:X\times X\to Y\times Y$ is called a balanced mapping if and only if it is a one-to-one semibalanced mapping.

**Theorem 4.3.** *If $f$ is a semibalanced map from type-2 fuzzy sets $X\times X$ into $Y\times Y$ and $\tilde{Q}$ is a type-2 fuzzy* stronger *G-equivalence relation on $Y$. Then, $f^{-1}(\tilde{Q})$ is a type-2* stronger *G-equivalence relation on $X$.*

*Proof.* Let $a,b \in X$ and $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$

$$\begin{aligned}
f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right) &= \tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)\right)\\
&= \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)\\
&= \tilde{\alpha}, \quad \text{for some } p \in Y.
\end{aligned}$$

Next take $a \neq b \in X$, and

$$\begin{aligned}
f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) &= \tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right)\\
&= \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) \le \tilde{\alpha}
\end{aligned}$$

$$\Rightarrow \qquad f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) \le \tilde{\alpha}.$$

Hence, $f^{-1}(\tilde{Q})$ is a type-2 fuzzy *stronger* G-reflexive relation on $X$.

Again,

$$\begin{aligned}
f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) &= \tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right)\\
&= \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)\\
&= \tilde{Q}\left(\frac{f_y(b)}{b},\frac{f_y(a)}{a}\right)\\
&= \tilde{Q}\left(f\left(\frac{f_x(v)}{v},\frac{f_x(u)}{u}\right)\right)
\end{aligned}$$

$$\Rightarrow \qquad f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) = f^{-1}(\tilde{Q})\left(\frac{f_x(v)}{v},\frac{f_x(u)}{u}\right).$$

Further,

$$\begin{aligned}
(f^{-1}(\tilde{Q})\circ f^{-1}(\tilde{Q}))\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) &= \tilde{\sup_{c\in X}}\left\{f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(w)}{w}\right)\tilde{\wedge} f^{-1}(\tilde{Q})\left(\frac{f_x(w)}{w},\frac{f_x(v)}{v}\right)\right\}\\
&= \tilde{\sup_{c\in X}}\left\{\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(w)}{w}\right)\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_x(w)}{w},\frac{f_x(v)}{v}\right)\right)\right\}\\
&= \tilde{\sup_{c\in X}}\left\{\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(t_z)}{t_z}\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_y(t_z)}{t_z},\frac{f_y(b)}{b}\right)\right)\right\}\\
&\le \tilde{\sup_{g\in Y}}\left\{\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(c)}{c}\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_y(c)}{c},\frac{f_y(b)}{b}\right)\right)\right\}\\
&= (\tilde{Q}\circ\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)\\
&\sqsubseteq \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right), \quad \text{by transitivity of } \tilde{Q}
\end{aligned}$$

$$= \tilde{Q}\left(f\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right)\right)$$
$$= f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right)$$
$$\Rightarrow \quad (f^{-1}(\tilde{Q}) \circ f^{-1}(\tilde{Q}))\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) \sqsubseteq f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right).$$

Therefore, $f^{-1}(\tilde{Q})$ is a type-2 symmetric and transitive relation on $X$.
Consequently, $f^{-1}(\tilde{Q})$ is a type-2 *stronger* G-equivalence relation on $X$. □

**Theorem 4.4.** *If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-congruence relation on the groupoid $S$ and $f$ is a groupoid homomorphism from $D \times D$ into $S \times S$, which is an semibalanced map, then $f^{-1}(\tilde{Q})$ is a* stronger *G-congruence on $D$.*

*Proof.* It follows from the above Theorems 4.1 and 4.3. □

**Definition 4.3.** Let $f$ be a map from type-2 fuzzy sets $X \times X$ into $Y \times Y$. A type-2 fuzzy relation $\tilde{Q}$ on $X$ is *$f$-invariant* if $f\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = f\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v_1)}{v_1}\right)$ implies that $\tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v_1)}{v_1}\right)$.

A type-2 fuzzy relation $\tilde{Q}$ on $X$ is *weakly $f$-invariant* if $f\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = f\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v)}{v}\right)$ implies that $\tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v)}{v}\right)$.

**Remark 4.2.** If $\tilde{Q}$ is $f$-invariant, then $\tilde{Q}$ is weakly $f$-invariant, but not conversely.

**Theorem 4.5.** *Let $f$ be a semibalanced map from type-2 fuzzy sets $X \times X$ into $Y \times Y$. If $\tilde{Q}$ is a* weakly $f$-invariant *type-2 fuzzy symmetric relation on $X$ with $\tilde{Q} \circ \tilde{Q} = \tilde{Q}$, then $\tilde{Q}$ is* $f$-invariant.

*Proof.* Let $a, b, c, b_1, c_1 \in X$ and $e, f \in Y$.
Given, $\tilde{Q}$ is *weakly $f$-invariant*, so we get,

$$\tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(w)}{w}\right) = \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(w)}{w}\right),$$
$$\tilde{Q}\left(\frac{f_x(w)}{w}, \frac{f_x(v)}{v}\right) = \tilde{Q}\left(\frac{f_x(w)}{w}, \frac{f_x(v_1)}{v_1}\right).$$

Now,

$$\tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = \tilde{Q} \circ \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right)$$
$$= \tilde{\sup_{c \in X}}\left\{\tilde{Q}\left(f\left(\frac{f_x(u)}{u}, \frac{f_x(w)}{w}\right)\right) \tilde{\wedge} \tilde{Q}\left(f\left(\frac{f_x(w)}{w}, \frac{f_x(v)}{v}\right)\right)\right\}$$
$$= \tilde{\sup_{c \in X}}\left\{\tilde{Q}\left(f\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(w)}{w}\right)\right) \tilde{\wedge} \tilde{Q}\left(f\left(\frac{f_x(w)}{w}, \frac{f_x(v_1)}{v_1}\right)\right)\right\}$$
$$= \tilde{Q} \circ \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v_1)}{v_1}\right)$$

$$= \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v_1)}{v_1}\right)$$

$$\Rightarrow \qquad \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) = \tilde{Q}\left(\frac{f_x(u_1)}{u_1}, \frac{f_x(v_1)}{v_1}\right).$$

Hence, $\tilde{Q}$ is $f$*-invariant*. □

**Theorem 4.6.** *Let $f$ be a semibalanced map from type-2 fuzzy sets $X \times X$ into $Y \times Y$. If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-equivalence (G-equivalence) relation on $X$ which is* weakly $f$-invariant*, then $\tilde{Q}$ is* $f$-invariant.

*Proof.* The result is deduced from the above Theorems 3.1 and 4.5. □

**Theorem 4.7.** *Let $f$ be a semibalanced map from $X \times X$ onto $Y \times Y$. If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-equivalence relation on $X$, which is* weakly $f$-invariant*, then $f(\tilde{Q})$ is a type-2 fuzzy* stronger *G-equivalence relation on $Y$.*

*Proof.* Given that $f$ is an onto semibalanced map, there exists, $a, a' \in X$ s.t.

$$f\left(\frac{f_x(u)}{u}, \frac{f_x(u')}{u'}\right) = \left(\frac{f_y(a)}{a}, \frac{f_y(a)}{a}\right) = f\left(\frac{f_x(u)}{u}, \frac{f_x(u)}{u}\right).$$

By Theorem 4.6, $\tilde{Q}$ is $f$*-invariant*.

Then,

$$\begin{aligned} f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(a)}{a}\right) &= \tilde{\sup_{(x,x')\in f^{-1}(e,e)}} \tilde{Q}\left(\frac{f_x(v)}{v}, \frac{f_x(v')}{v'}\right) \\ &= \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(u)}{u}\right). \\ &= \tilde{\alpha} \end{aligned}$$

If $e, f \in Y$, then there exists $b, c \in X$ s.t.

$$f\left(\frac{f_x(p)}{p}, \frac{f_x(q)}{q}\right) = \left(\frac{f_y(b)}{b}, \frac{f_y(c)}{c}\right),$$

and

$$f\left(\frac{f_x(q)}{q}, \frac{f_x(p)}{p}\right) = \left(\frac{f_y(c)}{c}, \frac{f_y(b)}{b}\right).$$

Now,

$$\begin{aligned} f(\tilde{Q})\left(\frac{f_y(b)}{b}, \frac{f_y(c)}{c}\right) &= \tilde{\sup_{(d,d')\in f^{-1}(g,h)}} \tilde{Q}\left(\frac{f_x(r)}{r}, \frac{f_x(s)}{s}\right) \\ &= \tilde{Q}\left(\frac{f_x(p)}{p}, \frac{f_x(q)}{q}\right) \\ &\le \tilde{\alpha} \end{aligned}$$

$$\Rightarrow \qquad f(\tilde{Q})\left(\frac{f_y(b)}{b}, \frac{f_y(c)}{c}\right) \le \tilde{\alpha}$$

Thus, $f(\tilde{Q})$ is type-2 fuzzy *stronger* G-reflexive relation on $Y$.

$$\begin{aligned}
f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) &= \tilde{\sup_{(d,d')\in f^{-1}(g,h)}} \tilde{Q}\left(\frac{f_x(r)}{r},\frac{f_x(s)}{s}\right)\\
&= \tilde{Q}\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\\
&= \tilde{Q}\left(\frac{f_x(v)}{v},\frac{f_x(u)}{u}\right)\\
&= f(\tilde{Q})\left(\frac{f_y(b)}{b},\frac{f_y(a)}{a}\right).
\end{aligned}$$

Hence, $f(\tilde{Q})$ is a type-2 fuzzy symmetric relation on $Y$.

$$\begin{aligned}
(f(\tilde{Q})\circ f(\tilde{Q}))\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) &= \tilde{\sup_{w'\in Y}}\left\{f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(d)}{d}\right)\tilde{\wedge} f(\tilde{Q})\left(\frac{f_y(d)}{d},\frac{f_y(b)}{b}\right)\right\}\\
&= \tilde{\sup_{e'\in X}}\left\{\tilde{Q}\left(\frac{f_x(u)}{u},\frac{f_x(t)}{t}\right)\tilde{\wedge}\tilde{Q}\left(\frac{f_x(t)}{t},\frac{f_x(v)}{v}\right)\right\}\\
&= (\tilde{Q}\circ\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\\
&\sqsubseteq \tilde{Q}\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\\
&= f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)
\end{aligned}$$

$$\Rightarrow \qquad (f(\tilde{Q})\circ f(\tilde{Q}))\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) \sqsubseteq f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right).$$

Therefore, $f(\tilde{Q})$ is a type-2 fuzzy transitive relation in $Y$.

Consequently, $f(\tilde{Q})$ is a type-2 fuzzy *stronger* G-equivalence relation in $Y$. □

**Corollary 4.1.** *Let $f$ be a balanced map from $X\times X$ onto $Y\times Y$. If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-equivalence relation on $X$, then $f(\tilde{Q})$ is a type-2 fuzzy* stronger *G-equivalence relation on $Y$.*

**Theorem 4.8.** *Let $f$ be a semibalanced map and a groupiod homomorphism from $D\times D$ onto $S\times S$. If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-congruence relation on $D$, which is* weakly $f$-invariant, *then $f(\tilde{Q})$ is a* stronger *G-congruence on $S$.*

*Proof.* The theorem can be proved by using the above Theorems 4.2 and 4.7. □

**Corollary 4.2.** *Let $f$ be a balanced map and a groupiod homomorphism from $D\times D$ onto $S\times S$. If $\tilde{Q}$ is a type-2 fuzzy* stronger *G-congruence relation on $D$, which is* weakly $f$-invariant, *then $f(\tilde{Q})$ is a* stronger *G-congruence on $S$.*

## 5. Images and Preimages of Type-2 Fuzzy G-equivalences and G-Congruences on A Groupoid

**Theorem 5.1.** *Let $f$ be a semibalanced map from $X\times X$ onto $Y\times Y$. If $\tilde{Q}$ is a type-2 fuzzy G-equivalence relation on $X$, which is* weakly $f$-invariant, *then $f(\tilde{Q})$ is a type-2 fuzzy G-equivalence* *relation on* $Y$ *with* $\tilde{\delta}(f(\tilde{Q})) = \tilde{\delta}(\tilde{Q})$.

*Proof.* Given, $f$ is a semibalanced map from $X \times X$ onto $Y \times Y$.
If $e, f \in Y$, then there exists $b, c \in X$ s.t.

$$f\left(\frac{f_x(u)}{u}, \frac{f_x(u)}{u}\right) = \left(\frac{f_y(a)}{a}, \frac{f_y(a)}{a}\right) = f\left(\frac{f_x(u)}{u}, \frac{f_x(u')}{u'}\right)$$

and

$$f\left(\frac{f_x(q)}{q}, \frac{f_x(p)}{p}\right) = \left(\frac{f_y(c)}{c}, \frac{f_y(b)}{b}\right).$$

We then have,

$$\begin{aligned} f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(a)}{a}\right) &= \tilde{\sup_{(x,x')\in f^{-1}(e,e)}} \tilde{Q}\left(\frac{f_x(v)}{v}, \frac{f_x(v')}{v'}\right) \\ &= \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(u)}{u}\right) \\ &> \tilde{0} \end{aligned}$$

and

$$\begin{aligned} f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(b)}{b}\right) &= \tilde{\sup_{(d,d')\in f^{-1}(g,h)}} \tilde{Q}\left(\frac{f_x(r)}{r}, \frac{f_x(s)}{s}\right) \\ &= \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) \\ &\leq \tilde{\delta}(\tilde{Q}) \\ &= \tilde{\inf_{a\in X}} \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(u)}{u}\right) \\ &= \tilde{\inf_{z\in Y}} f(\tilde{Q})\left(\frac{f_y(p)}{p}, \frac{f_y(p)}{p}\right) \\ &= \tilde{\delta}(f(\tilde{Q})) \end{aligned}$$

$$\Rightarrow \qquad f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(b)}{b}\right) \leq \tilde{\delta}(f(\tilde{Q}))$$

Thus, $f(\tilde{Q})$ is a type-2 fuzzy G-reflexive relation on $Y$ with $\tilde{\delta}(\tilde{Q}) = \tilde{\delta}(f(\tilde{Q}))$. Again,

$$\begin{aligned} f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(b)}{b}\right) &= \tilde{\sup_{(d,d')\in f^{-1}(g,h)}} \tilde{Q}\left(\frac{f_x(r)}{r}, \frac{f_x(s)}{s}\right) \\ &= \tilde{Q}\left(\frac{f_x(u)}{u}, \frac{f_x(v)}{v}\right) \\ &= \tilde{Q}\left(\frac{f_x(v)}{v}, \frac{f_x(u)}{u}\right) \end{aligned}$$

$$\Rightarrow \qquad f(\tilde{Q})\left(\frac{f_y(a)}{a}, \frac{f_y(b)}{b}\right) = f(\tilde{Q})\left(\frac{f_y(b)}{b}, \frac{f_y(a)}{a}\right).$$

Hence, $f(\tilde{Q})$ is a type-2 fuzzy symmetric relation on $Y$.

Further,

$$(f(\tilde{Q})\circ f(\tilde{Q}))\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)=\tilde{\sup_{w'\in Y}}\left\{f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(d)}{d}\right)\tilde{\wedge}f(\tilde{Q})\left(\frac{f_y(d)}{d},\frac{f_y(b)}{b}\right)\right\}$$

$$=\tilde{\sup_{e'\in X}}\left\{\tilde{Q}\left(\frac{f_x(u)}{u},\frac{f_x(t)}{t}\right)\tilde{\wedge}\tilde{Q}\left(\frac{f_x(t)}{t},\frac{f_x(v)}{v}\right)\right\}$$

$$=(\tilde{Q}\circ\tilde{Q})(\frac{f_x(u)}{u},\frac{f_x(v)}{v})$$

$$\sqsubseteq\tilde{Q}\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)$$

$$=f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$$

$$\Rightarrow\qquad (f(\tilde{Q})\circ f(\tilde{Q}))\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)\sqsubseteq f(\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right).$$

Therefore, $f(\tilde{Q})$ is a type-2 fuzzy transitive relation in $Y$.
Consequently, $f(\tilde{Q})$ is a type-2 fuzzy G-equivalence relation on $Y$. □

**Corollary 5.1.** *Let $f$ be a balanced map from $X\times X$ onto $Y\times Y$. If $\tilde{Q}$ is a type-2 fuzzy G-equivalence relation on $X$, then $f(\tilde{Q})$ is a type-2 fuzzy G-equivalence relation on $Y$ with $\tilde{\delta}(f(\tilde{Q}))=\tilde{\delta}(\tilde{Q})$.*

**Theorem 5.2.** *If $f$ is a groupoid homomorphism and a semibalanced map from $D\times D$ onto $S\times S$. If $\tilde{Q}$ is a type-2 fuzzy G-congruence relation on $D$, which is* weakly $f$-invariant, *then $f(\tilde{Q})$ is a G-congruence on $S$ with $\tilde{\delta}(f(\tilde{Q}))=\tilde{\delta}(\tilde{Q})$.*

*Proof.* The proof is similar to the above Theorems 4.2 and 5.1. □

**Corollary 5.2.** *If $f$ is a groupoid homomorphism and a balanced map from $D\times D$ onto $S\times S$. If $\tilde{Q}$ is a type-2 fuzzy G-congruence relation on $D$, which is* weakly $f$-invariant, *then $f(\tilde{Q})$ is a G-congruence on $S$ with $\tilde{\delta}(f(\tilde{Q}))=\tilde{\delta}(\tilde{Q})$.*

**Definition 5.1.** Let $\tilde{Q}$ be a type-2 fuzzy relation on $Y$ and let $f$ be a map from $X\times X$ into $Y\times Y$. We say $\tilde{Q}$ is $f$-*stable*, if given that $a\neq b\in X$, and $e\in Y$ such that

$$f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)$$

implies that

$$\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right)\leq\tilde{Q}\left(f\left(\frac{f_x(w)}{w},\frac{f_x(w)}{w}\right)\right),\quad \text{for all } c\in X.$$

**Example 5.1.** Consider a mapping $f$ from $(X\times X,Map(J,I))$ to $(Y\times Y,Map(J,I))$ of Example 4.1. We have defined the two type-2 fuzzy relations $\tilde{Q}$ and $\tilde{R}$ on $Y$ as follows:

$$\tilde{Q}\left(\frac{1}{1},\frac{1}{1}\right)=\frac{0.5}{0.5}+\frac{0.6}{0.6},\quad \tilde{Q}\left(\frac{0.9}{0.9},\frac{0.9}{0.9}\right)=\tilde{Q}\left(\frac{0.8}{0.8},\frac{0.8}{0.8}\right)=\frac{0.7}{0.7}+\frac{0.8}{0.8},\quad \tilde{Q}\left(\frac{1}{1},\frac{0.9}{0.9}\right)=\frac{0.3}{0.3}+\frac{0.4}{0.4}$$

and

$$\tilde{R}\left(\frac{1}{1},\frac{1}{1}\right)=\frac{0.1}{0.1}+\frac{0.2}{0.2},$$
$$\tilde{R}\left(\frac{1}{1},\frac{0.9}{0.9}\right)=\tilde{R}\left(\frac{1}{1},\frac{0.8}{0.8}\right)=\tilde{R}\left(\frac{0.9}{0.9},\frac{0.9}{0.9}\right)=\frac{0.4}{0.4}+\frac{0.5}{0.5},$$
$$\tilde{R}\left(\frac{0.9}{0.9},\frac{1}{1}\right)=\tilde{R}\left(\frac{0.8}{0.8},\frac{1}{1}\right)=\tilde{R}\left(\frac{0.9}{0.9},\frac{0.8}{0.8}\right)=\tilde{R}\left(\frac{0.8}{0.8},\frac{0.9}{0.9}\right)=\frac{0.1}{0.1}+\frac{0.2}{0.2}.$$

We see that $\tilde{Q}\left(f\left(\frac{1}{1},\frac{0.8}{0.8}\right)\right)=\tilde{Q}\left(\frac{1}{1},\frac{1}{1}\right)=\frac{0.5}{0.5}+\frac{0.6}{0.6}\le\tilde{Q}\left(f\left(\frac{1}{1},\frac{1}{1}\right)\right)$ and $\tilde{R}\left(f\left(\frac{1}{1},\frac{0.8}{0.8}\right)\right)=\frac{0.1}{0.1}+\frac{0.2}{0.2}\le \tilde{R}\left(f\left(\frac{1}{1},\frac{1}{1}\right)\right)$. Hence, $\tilde{Q}$ and $\tilde{R}$ are $f$-*stable*. $\tilde{Q}$ is G-equivalence relation while $\tilde{R}$ is not a G-equivalence relation.

**Theorem 5.3.** *Let $f$ be a semibalanced map from $X\times X$ into $Y\times Y$. If $\tilde{Q}$ is a type-2 fuzzy G-equivalence relation on $Y$, which is $f$-*stable*. Then $f^{-1}(\tilde{Q})$ is a type-2 fuzzy G-equivalence relation on $X$ with $\tilde{\delta}(f^{-1}(\tilde{Q}))\ge\tilde{\delta}(\tilde{Q})$. Further, if $f$ is onto, then $\tilde{\delta}(f^{-1}(\tilde{Q}))=\tilde{\delta}(\tilde{Q})$.*

*Proof.* Let $a\in X$. Then,

$$\begin{aligned} f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)&=\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)\right)\\ &=\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)\\ &\ge\tilde{0},\quad \text{for some } p\in Y. \end{aligned}$$

Next we consider $a\ne b\in X$, then $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)=\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$.

$f$-*stablility* is applied when $\frac{f_y(a)}{a}=\frac{f_y(b)}{b}$.

Therefore,

$$\begin{aligned} f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)&=\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right)\\ &\le\tilde{\delta}(\tilde{Q})\\ &=\tilde{\inf_{e\in Y}}\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)\\ \Rightarrow\qquad f^{-1}(\tilde{Q})(\frac{f_x(u)}{u},\frac{f_x(v)}{v})&\le\tilde{\inf_{e\in Y}}\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(a)}{a}\right)\\ &\le\tilde{\inf_{a\in X}}\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)\right)\\ &=\tilde{\inf_{a\in X}}f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(u)}{u}\right)\\ &=\tilde{\delta}(f^{-1}(\tilde{Q}))\\ \Rightarrow\qquad f^{-1}(\tilde{Q})(\frac{f_x(u)}{u},\frac{f_x(v)}{v})&\le\tilde{\delta}(f^{-1}(\tilde{Q})). \end{aligned}$$

Thus, $f^{-1}(\tilde{Q})$ is a type-2 fuzzy G-reflexive relation on $X$ with $\tilde{\delta}(f^{-1}(\tilde{Q}))\ge\tilde{\delta}(\tilde{Q})$.

Again,

$$\begin{aligned} f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) &= \tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right) \\ &= \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) \\ &= \tilde{Q}\left(\frac{f_y(b)}{b},\frac{f_y(a)}{a}\right) \\ &= \tilde{Q}\left(f\left(\frac{f_x(v)}{v},\frac{f_x(u)}{u}\right)\right) \end{aligned}$$

$$\Rightarrow \qquad f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) = f^{-1}(\tilde{Q})\left(\frac{f_x(v)}{v},\frac{f_x(u)}{u}\right).$$

Hence, $f^{-1}(\tilde{Q})$ is a type-2 fuzzy symmetric relation on $X$.

Finally, let $a,b \in X$ and $f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) = \left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right)$

$$\begin{aligned} (f^{-1}(\tilde{Q})\circ f^{-1}(\tilde{Q}))\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) &= \tilde{\sup_{c\in X}}\left\{f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(w)}{w}\right)\tilde{\wedge} f^{-1}(\tilde{Q})\left(\frac{f_x(w)}{w},\frac{f_x(v)}{v}\right)\right\} \\ &= \tilde{\sup_{c\in X}}\left\{\tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(w)}{w}\right)\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_x(w)}{w},\frac{f_x(v)}{v}\right)\right)\right\} \\ &= \tilde{\sup_{c\in X}}\left\{\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(t_z)}{t_z}\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_y(t_z)}{t_z},\frac{f_y(b)}{b}\right)\right)\right\} \\ &\le \tilde{\sup_{g\in Y}}\left\{\tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(c)}{c}\right)\tilde{\wedge}\tilde{Q}\left(f\left(\frac{f_y(c)}{c},\frac{f_y(b)}{b}\right)\right)\right\} \\ &= (\tilde{Q}\circ\tilde{Q})\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) \\ &\sqsubseteq \tilde{Q}\left(\frac{f_y(a)}{a},\frac{f_y(b)}{b}\right) \\ &= \tilde{Q}\left(f\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right)\right) \\ &= f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) \end{aligned}$$

$$\Rightarrow \qquad (f^{-1}(\tilde{Q})\circ f^{-1}(\tilde{Q}))\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right) \sqsubseteq f^{-1}(\tilde{Q})\left(\frac{f_x(u)}{u},\frac{f_x(v)}{v}\right).$$

Hence, $f^{-1}(\tilde{Q})$ is a type-2 fuzzy transitive relation on $X$.

Consequently, $f^{-1}(\tilde{Q})$ is a type-2 fuzzy G-equivalence relation on $X$. □

**Corollary 5.3.** *Let $f$ be a balanced map from $X \times X$ into $Y \times Y$. If $\tilde{Q}$ is a type-2 fuzzy G-equivalence relation on $Y$, which is $f$-stable, then $f^{-1}(\tilde{Q})$ is a type-2 fuzzy G-equivalence on $X$ with $\tilde{\delta}(f^{-1}(\tilde{Q})) \ge \tilde{\delta}(\tilde{Q})$. Further, if $f$ is onto, then $\tilde{\delta}(f^{-1}(\tilde{Q})) = \tilde{\delta}(\tilde{Q})$.*

*Proof.* The proof is similar to the above theorem. □

**Theorem 5.4.** *Let $f$ be a groupoid homomorphism and a semibalanced map from $D \times D$ into $S \times S$. If $\tilde{Q}$ is a type-2 fuzzy G-congruence relation on $Y$, then $f^{-1}(\tilde{Q})$ is a G-congruence on $X$ with $\tilde{\delta}(f^{-1}(\tilde{Q})) \geq \tilde{\delta}(\tilde{Q})$. Further, if $f$ is onto, then $\tilde{\delta}(f^{-1}(\tilde{Q})) = \tilde{\delta}(\tilde{Q})$.*

*Proof.* The proof is similar to Theorems 4.1 and 5.3. □

**Corollary 5.4.** *Let $f$ be a groupoid homomorphism and balanced map from $D \times D$ into $S \times S$. If $\tilde{Q}$ is a type-2 fuzzy G-congruence relation on $Y$, then $f^{-1}(\tilde{Q})$ is a G-congruence on $X$ with $\tilde{\delta}(f^{-1}(\tilde{Q})) \geq \tilde{\delta}(\tilde{Q})$. Further, if $f$ is onto, then $\tilde{\delta}(f^{-1}(\tilde{Q})) = \tilde{\delta}(\tilde{Q})$.*

## 6. Conclusion

We have obtained basic results of type-2 fuzzy G-equivalences and G-congruences under the semibalanced mappings. We have pointed out that the role of isomorphisms in classical algebraic structure is analogous to that of semibalanced mappings for type-2 fuzzy G-equivalences and G-congruences in this present note.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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