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Boundedness of Parametrized Littlewood-Paley Operators with Variable Kernels on Weighted Morrey Spaces

Research Article

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Abstract. In this paper,we establish the boundedness of parameterized area integral $\mu^{\rho}_{\Omega,S}$ and the Littlewood-Paley g^*_λ function $\mu^{*,\rho}_{\Omega,\lambda}$ with variable kernel on weighted Morrey space $L^{p,k}(\omega)$.

Keywords. Parameterized Littlewood-Paley operator; Parameterized area integral; Variable kernel; Weighted Morrey space

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1. Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ equipped with normalized Lebesgue measure. Let $\Omega \in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})(r \ge 1)$ be a homogeneous function of degree zero, and

$$\int_{S^{n-1}} \Omega(x,z')d\sigma(z') = 0, \quad \text{for all } x \in \mathbb{R}^n \tag{1.1}$$

and Ω satisfies the following conditions:

(1) $\Omega(x,\lambda z) = \Omega(x,z)$, $x,z \in \mathbb{R}^n$, $\lambda > 0$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} := \sup_{x\in\mathbb{R}^n}\left(\int_{S^{n-1}} |\,\Omega(x,z')\,|^r \, d\sigma(z')\right)^{\frac{1}{r}} < \infty$, $z' = \frac{z}{|z|}$, $z \in \mathbb{R}^n \setminus \{0\}$.

The Littlewood-Paley operators are play a very important role in harmonic analysis and partial differential equations. So it is an important and interesting to study their boundedness. In 1955, the L^p boundedness of the singular integral operators with variable kernels was considered by Calderón and Zygmund [1]. Subsequently, many people have studied the boundedness of all kinds of operators with variable kernel.

For $r \geq 1$, a kernel $\Omega(x,z)$ defined as above satisfies a class of L^r-Dini($r \geq 1$) condition, if

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}(1+|\log\delta|)^{\sigma} d\delta < \infty, \tag{1.2}$$

where $\omega_r(\delta)$ is the integral modulus of continuity of order r of Ω, which is defined by

$$\omega_r(\delta) = \sup_{\substack{x\in\mathbb{R}^n \\ \|\rho\|\leq\delta}} \left(\int_{S^{n-1}} |\Omega(x,\rho z') - \Omega(x,z')|^r d\sigma(z')\right)^{\frac{1}{r}},$$

and ρ denotes the rotation in $\mathbb{R}^n$, with $\|\rho\| := \sup\limits_{x'\in S^{n-1}} \|\rho x' - x'\|$, and denote $\omega_1(\delta) = \omega(\delta)$.

The parameterized Littlewood-Paley operator $\mu^{\rho}_{\Omega,S}$ and Littlewood-Paley g^*_λ function $\mu^{*,\rho}_{\Omega,\lambda}$ are defined by

$$\mu^{\rho}_{\Omega,S}(f)(x) = \left(\int\int_{\Gamma(x)} \left|\frac{1}{t^{\rho}}\int_{|y-z|<t} \frac{\Omega(y,y-z)}{|y-z|^{n-\rho}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.3}$$

$$\mu^{*,\rho}_{\Omega,\lambda}(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^{\rho}}\int_{|y-z|<t} \frac{\Omega(y,y-z)}{|y-z|^{n-\rho}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.4}$$

where $\rho > 0$ and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

The Classic Morrey spaces $\mathscr{L}^{p,\lambda}$ were introduced by Morrey [2], which were first introduced in the study of the local character of the second order elliptic partial differential equation. In 2009, Komori-Shirai [3] defined a weighted Morrey spaces $L^{p,k}(\omega)$, and studied the boundedness of some classical operators on them. In recent years, there are fruitful results about the boundedness of Hardy-Littlewood maximal operator, C-Z operator, fractional integral operator on Morrey space and weighted Morrey space, see [4–10]. Inspired by these results, the main purpose of this paper is to establish the boundedness of parameterized area integral and Littlewood-Paley g^*_λ function with variable kernel on weighted Morrey space $L^{p,k}(\omega)$.

2. Definitions and Lemmas

Before establishing our theorems,first we introduce some necessary definitions and lemmas.

Let ω be a nonnegative,locally integrable function defined on $\mathbb{R}^n$. $B = B(x_0, r_B)$ is a ball, $\omega(E)$ means measure of E, i.e.,

$$\omega(E) = \int_E \omega(x)dx; \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Definition 2.1. When $1 < p < \infty$, we say $\omega \in A_p$ if for every ball B,

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} \le C < \infty.$$

A weight function ω is said to belong to the reverse Hölder class RH_r. If there exist an $r > 1$ and a $c > 0$, such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|B|}\int_B \omega(x)^r dx\right)^{\frac{1}{r}} \le \frac{C}{|B|}\int_B \omega(x)dx, \quad \text{for every ball } B \subseteq \mathbb{R}^n$$

for all balls B denote by $\omega \in RH_r$.

Definition 2.2 ([3]). Suppose that $1 < p < \infty$, $0 < k < 1$,ω is a weight function, then weighted Morrey space is defined by

$$L^{p,k}(\omega) = \left\{ f \in L^p_{loc}(\omega) : \|f\|_{L^{p,k}_{loc}(\omega)} < \infty \right\},$$

where

$$\|f\|_{L^{p,k}_{loc}(\omega)} = \sup_B \left(\frac{1}{\omega(B)^k}\int_B |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}}.$$

Torchinsky and Wang [13] studied the weighted L^p boundedness of Marcinkiewicz integral by the method of studying the estimation of the maximum function. Using this method, Xin [14] got the weighted L^p boundedness of Littlewood-Paley operator.

Lemma 2.1 ([11]). *Suppose that $1 < p < \infty$, $\omega \in A_p$, then there exists a absolute constant $C > 0$, so that*

$$\omega(2B) \le C\omega(B), \quad \textit{for every ball } B.$$

Lemma 2.2 ([12]). *Suppose that $\omega \in RH_r$, then there exists a constant $C > 0$, so that for every ball B and any measureable set $E \subset B$,*

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{\frac{r-1}{r}}.$$

Subsequently for all $\lambda > 0$, $\omega(\lambda B) \le C\lambda^{np}\omega(B)$. C is independent of B and λ.

$L^p(\omega)$ denotes the set of measurable functions f such that

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty.$$

Lemma 2.3 ([14]). *Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies* (1.1) *and* (1.2), *$\mu^\rho_{\Omega,S}$, $\mu^{*,\rho}_{\Omega,\lambda}$ are defined by* (1.3), (1.4) *when $\rho > \frac{n}{2}$, $\lambda > 2$, $1 < p < \infty$, then there exists a constant C which is independent of f, such that*

$$M^\sharp(\mu^\rho_{\Omega,S} f)(x) \le C_p M_p f(x), \quad \textit{for all } x \in \mathbb{R}^n,$$

$$M^\sharp(\mu^{*,\rho}_{\Omega,\lambda} f)(x) \le C_p M_p f(x), \quad \textit{for all } x \in \mathbb{R}^n.$$

Lemma 2.4 ([14]). *Suppose that $\Omega \in L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})$ satisfies* (1.1) *and* (1.2)*, $\mu^\rho_{\Omega,S}$, $\mu^{*,\rho}_{\Omega,\lambda}$ are defined by* (1.3), (1.4) *when $\rho > \frac{n}{2}$, $\lambda > 2$, $1 < p < \infty$, then there exists a constant C which is independent of f, such that*

$$\|\mu^\rho_{\Omega,S}(f)\|_{L^p_\omega} \le \|f\|_{L^p_\omega}, \quad \|\mu_{\mu^{*,\rho}_{\Omega,\lambda}(f)}\|_{L^p_\omega} \le \|f\|_{L^p_\omega}.$$

3. Main Results

Now, we state our main results as follows.

Theorem 3.1. *Let $\Omega \in L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})$ is homogeneous on second variable, satisfy* (1.1)*, $\mu^\rho_{\Omega,S}$ is defined by* (1.3)*, when $0<k<1$, $\rho > \frac{n}{2}$, $1 < p < \infty$. Then there exists a constant C which is independent of f, such that*

$$\|\mu^\rho_{\Omega,S}(f)\|_{L^{p,k}(\omega)} \le \|f\|_{L^{p,k}(\omega)}.$$

Theorem 3.2. *Let $\Omega \in L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})$ is homogeneous on second variable, satisfy* (1.1)*, $\mu^{*,\rho}_{\Omega,\lambda}$ is defined by* (1.4)*, when $0<k<1$, $\rho > \frac{n}{2}$, $\lambda > 1+2\rho$, $1 < p < \infty$. Then there exists a constant C which is independent of f, such that*

$$\|\mu^{*,\rho}_{\Omega,\lambda}\|_{L^{p,k}(\omega)} \le \|f\|_{L^{p,k}(\omega)}.$$

4. Proofs of Theoremsn

Proof of Theorem 3.1. For every ball $B \subset \mathbb{R}^n$, let $f = f_1 + f_2$, where $f_1 = f_{\chi_{2B}}$, χ_{2B} denotes the characteristic function of $2B$. Then

$$\begin{aligned}
&\frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B |\mu^\rho_{\Omega,S}(f)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B |\mu^\rho_{\Omega,S}(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} + \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B |\mu^\rho_{\Omega,S}(f_2)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&= I_1 + I_2. \qquad (4.1)
\end{aligned}$$

Then by Lemma 2.1 and Lemma 2.4,

$$\begin{aligned}
I_1 &\le \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_{\mathbb{R}^n} |(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&= \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_{2B} |(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&\le C\|f\|_{L^{p,k}(\omega)}\frac{\omega(2B)^{\frac{k}{p}}}{\omega(B)^{\frac{k}{p}}} \\
&\le C\|f\|_{L^{p,k}(\omega)}. \qquad (4.2)
\end{aligned}$$

To estimate I_2, we first estimate $\mu^{\rho}_{\Omega,S}(f_2)(x)$ for $x \in 2B$.

$$\left|\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right| = \left|\frac{1}{t^{\rho}}\int_{(2B)^C\cap\{z:|y-z|<t\}}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|$$
$$\leq \left|\frac{1}{t^{\rho}}\sum_{j=1}^{\infty}\int_{(2^{j+1}B\setminus 2^jB)\cap\{z:|y-z|<t\}}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|. \quad (4.3)$$

Suppose that $B = B(y, r_B)$, for all $x \in B$, $y \in \Gamma(x)$, $z \in 2^{j+1}B \setminus 2^jB$, then there exists $y_0 \in B(x,t)\cap B(z,t)$, such that

$$2t \geq |x-y_0| + |y_0 - z| \geq |x-z| \geq |z-y| - |x-y| \geq 2^j r_B - r_B \geq 2^{j-1} r_B.$$

According to (4.3),

$$|\mu^{\rho}_{\Omega,S}(f_2)(x)|$$
$$\leq C\left(\int_{2^{j-2}r_B}^{\infty}\int_{|y-x|<t}\left|\frac{1}{t^{\rho}}\int_{(2^{j+1}B\setminus 2^jB)}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}$$
$$\leq C\left(\int_{2^{j-2}r_B}^{\infty}\int_{|y-x|<t}\left|\frac{1}{t^{\rho}}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{\frac{(n-\rho)}{n}}}\int_{(2^{j+1}B\setminus 2^jB)}\Omega(y,y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \quad (4.4)$$

If $\Omega \in L^{\infty}(\mathbb{R}) \times L^2(S^{n-1})$

$$\int_{(2^{j+1}B\setminus 2^jB)}|\Omega(y,y-z)|\,|f(z)|dz$$
$$\leq \left(\int_{(2^{j+1}B\setminus 2^jB)}|\Omega(y,y-z)|^2dz\right)^{\frac{1}{2}}\left(\int_{(2^{j+1}B\setminus 2^jB)}|f(z)|^2dz\right)^{\frac{1}{2}}$$
$$\leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\left(\int_{2^{j+1}B}|f(z)|^2dz\right)^{\frac{1}{2}}. \quad (4.5)$$

According to (4.4), (4.5)

$$|\mu^{\rho}_{\Omega,S}(f_2)(x)| \leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{\frac{(n-\rho)}{n}}}$$
$$\cdot\left(\int_{2^{j+1}B}|f(z)|^2dz\right)^{\frac{1}{2}}\left(\int_{(2^{j+1}B\setminus 2^jB)}\left(\int_{|y-x|<t}dy\right)\frac{dt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}$$
$$\leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{\frac{(n-\rho)}{n}}}\left(\int_{2^{j+1}B}|f(z)|^2dz\right)^{\frac{1}{2}}\frac{1}{|2^{j+1}B|^{\frac{\rho}{n}}}$$
$$= C\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B}|f(z)|^2dz\right)^{\frac{1}{2}}.$$

Using the Hölder inequality and $\omega \in A_p$, $1 < p < \infty$,

$$\begin{aligned}
|\mu^{\rho}_{\Omega,S}(f_2)(x)| &\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|} \\
&\quad\cdot\left(\int_{2^{j+1}B}|f(z)|^p\omega(z)dz\right)^{\frac{1}{p}}\left(\int_{2^{j+1}B}\omega(z)^{-\frac{2}{p-2}}dz\right)^{\frac{p-2}{2p}} \\
&\le C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}\|f\|_{L^{p,k}(\omega)} \\
&\quad\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^{\frac{1}{p}}}\omega(2^{j+1}B)^{\frac{k}{p}}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\omega(z)dz\right)^{-\frac{1}{p}} \\
&\le C\|f\|_{L^{p,k}(\omega)}\sum_{n=1}^{\infty}\omega(2^{j+1}B)^{\frac{(k-1)}{p}}. \qquad (4.6)
\end{aligned}$$

So

$$I_2 \le C\|f\|_{L^{p,k}(\omega)}\sum_{j=1}^{\infty}\frac{\omega(B)^{\frac{(1-k)}{p}}}{\omega(2^{j+1}B)^{\frac{(1-k)}{p}}}.$$

Then by Lemma 2.2,

$$\frac{\omega(B)}{\omega(2^{j+1}B)} \le C\left(\frac{|B|}{|2^{j+1}B|}\right)^{\frac{(r-1)(1-k)}{r}} \le C\left(\frac{1}{2^{jn}}\right)^{\frac{(r-1)}{r}}.$$

So

$$I_2 \le C\|f\|_{L^{p,k}(\omega)}\sum_{j=1}^{\infty}\left(\frac{1}{2^{jn}}\right)^{\frac{(r-1)(1-k)}{pr}} \le C\|f\|_{L^{p,k}(\omega)}. \qquad (4.7)$$

Due to the $\frac{(r-1)(1-k)}{pr} > 0$, according to (4.2), (4.7)

$$\|\mu^{\rho}_{\Omega,S}(f)\|_{L^p_\omega} \le C\|f\|_{L^{p,k}(\omega)}. \qquad \square$$

Proof of Theorem 3.2. For every ball $B \in \mathbb{R}^n$, let $f = f_1 + f_2$, where $f_1 = f_{\chi_{2B}}$, χ_{2B} denotes the characteristic function of $2B$. Then

$$\begin{aligned}
&\frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^{*,\rho}_{\Omega,\lambda}(f)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^{*,\rho}_{\Omega,\lambda}(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} + \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^{*,\rho}_{\Omega,\lambda}(f_2)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&= J_1 + J_2. \qquad (4.8)
\end{aligned}$$

Then by Lemma 2.1 and Lemma 2.4,

$$\begin{aligned}
J_1 &\le \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_{\mathbb{R}^n}|(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&= \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_{2B}|(f_1)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&\le C\|f\|_{L^{p,k}(\omega)}\frac{\omega(2B)^{\frac{k}{p}}}{\omega(B)^{\frac{k}{p}}} \\
&\le C\|f\|_{L^{p,k}(\omega)}.
\end{aligned} \tag{4.9}$$

The follows is to proof J_2.

$$\begin{aligned}
\mu^{*,\rho}_{\Omega,\lambda}(f_2)(x)^2 &= \int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+1}} \\
&\quad+\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t<|x-y|<2^jt}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+1}} \\
&\le C(\mu^\rho_{\Omega,S}(f_2)(x)^2+\sum_{j=1}^\infty 2^{-j\lambda n}\mu^\rho_{\Omega,S,2^j}(f_2)(x)^2).
\end{aligned}$$

So

$$\begin{aligned}
J_2 &\le \frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^\rho_{\Omega,S}(f_2)(x)|^p\omega(x)dx\right)^{\frac{1}{p}}+\sum_{j=1}^\infty 2^{-\frac{j\lambda n}{2}}\frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^\rho_{\Omega,S,2^j}(f_2)(x)|^p\omega(x)dx\right)^{\frac{1}{p}} \\
&= J_{21}+J_{22}.
\end{aligned}$$

by the Proof of Theorem 3.1,

$$J_{21}\le C\|f\|_{L^{p,k}(\omega)}. \tag{4.10}$$

$$\begin{aligned}
\left|\mu^\rho_{\Omega,S,2^j}(f_2)(x)\right| &= \left(\int\int_{\Gamma_{2^j}(x)}\left|\frac{1}{t^\rho}\int_{(2B)^C\cap\{z:|y-z|<t\}}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\le \left(\int\int_{\Gamma_{2^j}(x)}\left|\sum_{i=1}^\infty\int_{(2^{i+1}B\setminus 2^iB)\cap\{z:|y-z|<t\}}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Suppose that $B=B(y,r_B)$, for all $x\in B$, $y\in\Gamma_{2^j}(x)$, $z\in 2^{i+1}B\setminus 2^iB$, then there exists $y_0\in B(x,2^jt)\cap B(z,t)$,such that

$$t+2^jt\ge|x-y_0|+|y_0-z|\ge|x-z|\ge|z-y|-|x-y|\ge 2^ir_B-r_B\ge 2^{i-1}r_B.$$

So $2^{j+1}t \geq 2^{i-1}r_B$,

$$\begin{aligned}
&|\mu^{\rho}_{\Omega,S,2^j}(f_2)(x)| \\
&\leq C\left(\int_{2^{i-2-j}r_B}^{\infty}\int_{|y-x|<2^jt}\left|\frac{1}{t^{\rho}}\sum_{i=1}^{\infty}\int_{(2^{i+1}B\setminus 2^iB)}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_2(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{2^{i-j-2}r_B}^{\infty}\int_{|y-x|<2^jt}\left|\sum_{i=1}^{\infty}\frac{1}{|2^{i+1}B|^{\frac{(n-\rho)}{n}}}\int_{(2^{i+1}B\setminus 2^iB)}\Omega(y,y-z)f(z)dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}}.
\end{aligned}$$

If $\Omega\in L^{\infty}(\mathbb{R})\times L^2(S^{n-1})$, using the similar method of (4.5) and (4.6), the Hölder inequality and $\omega\in A_p$, $1<p<\infty$,

$$\begin{aligned}
|\mu^{\rho}_{\Omega,S,2^j}(f_2)(x)| &\leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{i=1}^{\infty}\frac{1}{|2^{i+1}B|^{\frac{(n-\rho)}{n}}} \\
&\quad\cdot\left(\int_{2^{j+1}B}|f(z)|^2dz\right)^{\frac{1}{2}}\left(\int_{2^{i-2-j}r_B}^{\infty}\left(\int_{|y-x|<2^jt}dy\right)\frac{dt}{t^{n+2\rho+1}}\right)^{\frac{1}{2}} \\
&\leq C2^{jn/2+jn\rho}\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\sum_{i=1}^{\infty}\frac{1}{|2^{i+1}B|}\left(\int_{2^{i+1}B}|f(z)|^2dz\right)^{\frac{1}{2}} \\
&\leq C2^{jn/2+jn\rho}\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\|f\|_{L^{p,k}(\omega)}\left(\int_{2^{i+1}B}\omega(z)dz\right)^{-\frac{1}{p}} \\
&\leq C2^{jn/2+jn\rho}\|\Omega\|_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}\|f\|_{L^{p,k}(\omega)}\sum_{i=1}^{\infty}\omega(2^{i+1}B)^{\frac{(k-1)}{p}}.
\end{aligned} \tag{4.11}$$

So

$$J_2\leq C\sum_{j=1}^{\infty}2^{-\frac{j\lambda n}{2}}2^{jn/2+jn\rho}\|f\|_{L^{p,k}(\omega)}\sum_{i=1}^{\infty}\frac{\omega(B)^{\frac{(1-k)}{p}}}{\omega(2^{i+1}B)^{\frac{(1-k)}{p}}}.$$

Similar to the proof of Theorem 3.1

$$J_2\leq C\sum_{j=1}^{\infty}2^{-\frac{j\lambda n}{2}}2^{jn/2+jn\rho}\|f\|_{L^{p,k}(\omega)} \tag{4.12}$$

According to (4.9), (4.12)

$$\frac{1}{\omega(B)^{\frac{k}{p}}}\left(\int_B|\mu^{*,\rho}_{\Omega,\lambda}(f)(x)|^p\omega(x)dx\right)^{\frac{1}{p}}\leq C\left(1+\sum_{j=1}^{\infty}2^{-\frac{j\lambda n}{2}}2^{jn/2+jn\rho}\right)\|f\|_{L^{p,k}(\omega)}$$

for $\lambda>1+2\rho$,

$$\|\mu^{*,\rho}_{\Omega,\lambda}f\|_{L^{p,k}(\omega)}\leq\|f\|_{L^{p,k}(\omega)}. \qquad \square$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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