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Some Notes on Dual Spherical Curves

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Abstract. In this study, by investigating one parameter spherical motion in $\mathbb{D}^3$ with two different kinds of dual indicatrice curves, we have obtained the ruled surfaces that correspond to tangent, principal normal and binormal indicatrices of the dual curve are developable. Furthermore, it can be easily seen that this study gives a link between the classical surface theory and dual spherical curves on the dual unit spheres.

1. **Introduction**

The presentation of dual spherical motion expressed with the help of dual unit vectors is based on the study of Clifford and E. Study.

Recently, dual space curves and surfaces have been extensively studied and they are powerful mathematical tools for spherical motion in $\mathbb{D}^3$. Such notions as dual numbers, dual vectors, dual angles, dual orthogonal matrices...etc. in general dual elements have been originally conceived by Clifford (1873) [1]. After him, the first applications to mechanics are due to Study (1901) [8] and he defined the mapping which is called after his name: There is a one-to-one correspondence between an oriented straight line in the Euclidean 3-space E^3 and a dual point on the surface of a dual unit sphere S^2 in the dual space $\mathbb{D}^3$ [8]. Hence, a differentiable curve on the sphere S^2 corresponds to a ruled surface in the line space in $\mathbb{R}^3$ [2, 3, 4, 5]. Here, it can be easily said that dual spherical motion is closely analogous to real spherical motion.

In this study, we present the characterizations of dual spherical curves in dual space. In the second section of this paper, we briefly give the mathematical formulations and the necessary notational conventions for the reader who is not familiar with.

In the next section, we define a dual curve in $\mathbb{D}^3$ with the assistance of unit speed curve in E^3. Then we have showed that these curves have the same parameter. Accordingly, by describing an orthonormal moving frame along dual curve, we

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give one parameter spherical motion in dual space. At that time, we have different results that have been extensively studied in [6] and [10].

In the fourth section of the study, by defining another unit speed curve with the same parameter of the first curve and also another closed spherical dual curve of class C^1 on a unit dual sphere S^1 in $\mathbb{D}^3$, different cases are investigated with taking tangent, principal normal and binormal indicatrices of these curves. In the last section, we show that the ruled surfaces that correspond to tangent, principal normal and binormal indicatrices of the other dual curve are developable. At the end of the section, the Darboux vector of this motion is calculated with the Darboux indicatrice of the dual curve.

Therefore, this study gives us a link and relation between the classical surface theory and dual spherical curves on the dual unit spheres.

2. **Basic concepts**

2.1. *Frenet frame* [3, 4]

We assume that the curve α is parametrized by arclength. Then, $\alpha'(s)$ is the unit tangent vector to the curve, which we denote by $T(s)$ Since t has constant length, $T'(s)$ will be orthogonal to $T(s)$. If $T'(s) \neq 0$ then we define principal normal

$$N(s) = \frac{T'(s)}{T'(s)} \tag{2.1}$$

vector and the curvature

$$k_1(s) = \|T'(s)\|\,. \tag{2.2}$$

So far, we have

$$T'(s) = k_1(s) \cdot T(s)\,. \tag{2.3}$$

If $k_1(s) = 0$, the principal normal vector is not defined. If $k_1(s) \neq 0$ then the binormal vector $b(s)$ is given by

$$B(s) = T(s) \times N(s)\,. \tag{2.4}$$

Then $\{T(s), N(s), B(s)\}$ form a right-handed orthonormal basis for $\mathbb{R}^3$. In summary Frenet formulas can be given as

$$T'(s) = k_1(s) \cdot N(s)\,, \tag{2.5}$$

$$N'(s) = -k_1(s) \cdot T(s) + k_2(s) \cdot B(s)\,, \tag{2.6}$$

$$B'(s) = -k_2(s) \cdot N(s)\,. \tag{2.7}$$

2.2. *Involute* [3, 4]

The orbit that is the perpendicular to the tangents of a curve is involute of this curve.

2.3. *Mannheim partner curves in* 3*-space* [7, 9]

Let E^3 be the 3-dimensional Euclidean space with the standard inner product $\langle\, ,\rangle$. If there exists a corresponding relationship between the space curves Γ and $\Gamma_1$such that, at the corresponding points of the curves, the principal normal lines of coincides with the binormal lines of Γ_1, then Γ is called a Mannheim curve, and a Γ_1 Mannheim partner curve of Γ. The pair $\{\Gamma, \Gamma_1\}$ is said to be a Mannheim pair. From the elementary differential geometry we have the well-known characterizations of Bertrand pair. But there are rather few works on Mannheim pair. It is just known that a space curve in E^3 is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the formula $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant.

Let $\Gamma : x(s)$ be a Mannheim curve in E^3 parameterized by its arc length s and $\Gamma_1 : x_1(s_1)$ the Mannheim partner curve of with an arc length parameter s_1. Denote by $\{\alpha(s), \beta(s), \gamma(s)\}$ the Frenet frame field along $\Gamma : x(s)$, that is, $\alpha(s)$ is the tangent vector field, $\beta(s)$ the normal vector field and $\gamma(s)$ the binormal vector field of the curve Γ, respectively.

Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

2.4. *Mannheim partner curves in dual space* [7, 9]

Let $\mathbb{D}^3$ be the dual space with the standard inner product $\langle,\rangle$. If there exists a corresponding relationship between the dual space curves $\widehat{\alpha}$ and $\widehat{\beta}$ such that, at the corresponding points of the dual curves, the principal normal lines of $\widehat{\alpha}$ coincides with the binormal lines of $\widehat{\beta}$, then $\widehat{\alpha}$ is called a dual Mannheim curve, and $\widehat{\beta}$ a dual Mannheim partner curve of $\widehat{\alpha}$ The pair $\{\widehat{\alpha}, \widehat{\beta}\}$ is said to be a dual Mannheim pair.

3. One parameter spherical motion in $\mathbb{D}^3$

3.1. *First kind of dual indicatrice curve*

Let

$$\begin{aligned} &\alpha : I \to E^3 \\ &\quad s \mapsto \alpha(s) \end{aligned} \tag{3.1}$$

be unit speed curve and $\{T, N, B\}$ Frenet frame of α. T, N, B are the unit tangent, principal normal and binormal vectors respectively. With the assistance of α, we define a dual curve in $\mathbb{D}^3$. So, let us have a closed spherical dual curve $\widehat{\alpha}$ of class C^1 on a unit dual sphere S^1in $\mathbb{D}^3$. The curve α describes a closed dual spherical motion. If

$$\begin{aligned} &\widehat{\alpha} : I \to \mathbb{D}^3 \\ &\quad s \mapsto \widehat{\alpha}(s) = \alpha(s) + \varepsilon \int \alpha \wedge T ds = \int (T + \varepsilon(\alpha \wedge T)) ds \end{aligned} \tag{3.2}$$

then the curve α can be written as:

$$\widehat{\alpha} = \alpha(s) + \varepsilon\alpha^{\star}(s). \tag{3.3}$$

At this time, when we get $\alpha^{\star}(s)$ as proper integral, different geometrical approach can be given as the following.

In this case, when we have

$$\begin{aligned} \alpha : I &\to E^3 \\ s &\mapsto \alpha(s) \end{aligned} \tag{3.4}$$

the Peano direction of α and the projection of P_α can be given by [11]

$$P_\alpha = \oint_{S^1} \alpha \wedge T dt \tag{3.5}$$

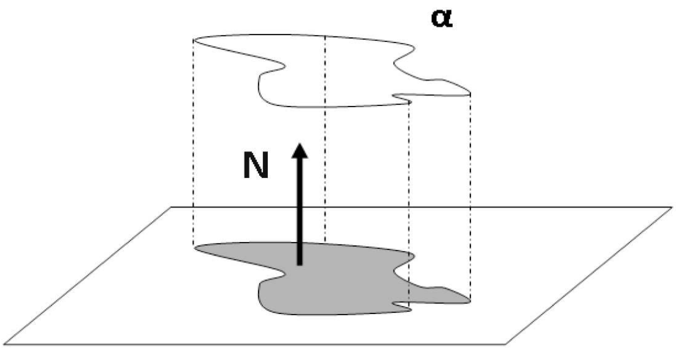

Figure 1. Projection area of closed curve α on the plane

Hence, F_α is a projection area of closed curve α on the plane that its normal is N.

$$F_\alpha = \langle P_\alpha, N \rangle. \tag{3.6}$$

On the other hand, we can easily show that the curves α and $\widehat{\alpha}$ have the same parameter s. If we write $\widehat{\alpha}$ as:

$$\begin{aligned} \widehat{\alpha}(s) &= \alpha(s) + \varepsilon\alpha^{\star}(s) \\ &= \alpha(s) + \int \alpha \wedge T ds = \int (T + \varepsilon(\alpha \wedge T))ds \end{aligned} \tag{3.7}$$

then we can get

$$\begin{aligned} \widehat{\alpha}'(s) &= \widehat{T}(s) = \alpha'(s) + \varepsilon(\alpha \wedge T), \\ \widehat{\alpha}'(s) &= T + \varepsilon(\alpha \wedge T). \end{aligned} \tag{3.8}$$

Thus we have

$$\widehat{s} = s + \varepsilon \int_0^t \langle T, \alpha \wedge T \rangle \, dt \tag{3.9}$$

$$= s + 0 = s\,.$$

Now, we define an orthonormal moving frame along dual curve as follows in $\mathbb{D}^3$; accordingly, we can easily give dual curves in $\mathbb{D}^3$ as:

$$\widehat{\alpha}(s) = \int [T(s) + \varepsilon \alpha(s) \wedge T(s)] ds, \tag{3.10}$$

$$\widehat{\mu}(s) = \int [N(s) + \varepsilon \alpha(s) \wedge N(s)] ds, \tag{3.11}$$

$$\widehat{\gamma}(s) = \int (B(s) + \varepsilon \alpha(s) \wedge B(s)) ds. \tag{3.12}$$

And also, let the tangent, principal normal and binormal indicatrice curves of $\widehat{\alpha}$ be $\widehat{T}(s)$, $\widehat{N}(s)$ and $\widehat{B}(s)$ respectively.

$$\widehat{T}(s) = T + \varepsilon(\alpha \wedge T),$$

$$\widehat{N}(s) = N + \varepsilon(\alpha \wedge N),$$

$$\widehat{B}(s) = B + \varepsilon(\alpha \wedge B).$$

Subsequently, we can give the following theorem:

Theorem 1. *The curves $\widehat{\alpha}(s)$, $\widehat{\mu}(s)$ and $\widehat{\gamma}(s)$ are involute-evolute curve pairs.*

Proof. It can be easily seen that

$$\left\langle \widehat{\alpha}'(s), \widehat{\mu}'(s) \right\rangle = \left\langle \widehat{\alpha}'(s), \widehat{\gamma}'(s) \right\rangle = \left\langle \widehat{\mu}'(s), \widehat{\gamma}'(s) \right\rangle = 0. \tag{3.13}$$

□

Theorem 2. *The tangent, principal normal and binormal indicatrice curves of $\widehat{\alpha}(s)$ are $\widehat{T}(s)$, $\widehat{N}(s)$ and $\widehat{B}(s)$ respectively. And also, $\{\widehat{T}, \widehat{N}, \widehat{B}\}$ frame is a Blaschke Frame.*

Proof. It is clear that

$$\widehat{\alpha}(s) = \widehat{T}(s), \tag{3.14}$$

$$\widehat{N} = \frac{\dfrac{d\widehat{T}}{ds}}{\left\| \dfrac{d\widehat{T}}{ds} \right\|}$$

and

$$\widehat{B}(s) = \widehat{T} \wedge \widehat{N}.$$

□

At this time, the ruled surfaces in $\mathbb{R}^3$ correspond to dual curves $\widehat{T}$, $\widehat{N}$ and $\widehat{B}$ are the ruled surfaces that drawn by the lines T, N and B of α. That is to say, these ruled surfaces can be given as:

$$\begin{aligned}
\Phi_{\widehat{T}} &= \alpha(s) + \upsilon T(s), \qquad (3.15)\\
\Phi_{\widehat{N}} &= \alpha(s) + \upsilon N(s),\\
\Phi_{\widehat{B}} &= \alpha(s) + \upsilon B(s).
\end{aligned}$$

On the other hand, let the unit dual spheres K and $\overline{K}$ be

$$K = \{e_1, e_2, e_3\} \text{ and } \overline{K} = \{\widehat{T}, \widehat{N}, \widehat{B}\}. \qquad (3.16)$$

These are two orthonormal coordinate systems of moving unit sphere K and fixed unit dual sphere $\overline{K}$ with the same origin. At this time, one parameter dual spherical motion (dual rotation) between these dual spheres K and $\overline{K}$ can be given with $K/\overline{K}$. In this case, we can easily investigate dual spherical motion $K/\overline{K}$ with

$$\begin{bmatrix} \widehat{T'} \\ \widehat{N'} \\ \widehat{B'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2+\varepsilon \\ 0 & -k_2-\varepsilon & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \widehat{N} \\ \widehat{B} \end{bmatrix}. \qquad (3.17)$$

Besides, let the Darboux indicatrice curve of $\widehat{\alpha}(s)$ be $\widehat{W}$. According to this, the Darboux vector of this motion is:

$$\widehat{W} = (k_2+\varepsilon)\widehat{T} + k_1\widehat{B}. \qquad (3.18)$$

On the other hand, the curvatures $\widehat{k}_1(s)$ and $\widehat{k}_2(s)$ can be calculated. Let $\widehat{k}_1(s)$ and $\widehat{k}_2(s)$ be $\widehat{\kappa}(s)$ and $\widehat{\tau}(s)$ in turn. Thus,

$$\begin{aligned}
\widehat{k}_1(s) &= \widehat{\kappa}(s) \text{ and also } k_1(s) = \kappa(s), \qquad (3.19)\\
\widehat{k}_2(s) &= \widehat{\tau}(s) \text{ and also } \widehat{k}_2(s) = k_2(s) + \varepsilon = \tau(s) + \varepsilon.
\end{aligned}$$

And then

$$\frac{d\widehat{\alpha}}{d\widehat{s}} = \frac{d\widehat{\alpha}}{ds} = \widehat{T}(s), \quad \|\widehat{T}(s)\| = 1 \qquad (3.20)$$

such that $\widehat{\alpha}(s)$ is unit speed curve.

If $\widehat{\alpha}(s)$ is a normal curve such that

$$\widehat{\alpha}(s) = \lambda(s)\widehat{N}(s) + \eta(s)\widehat{B}(s) \qquad (3.21)$$

then the curvatures of dual curve $\widehat{\alpha}(s)$ are $k_1(s)$ and $k_2(s)$ as following:

$$\begin{aligned}
k_1 &= \left[c_1 \cdot \cos\left(\arctan \frac{c_1^* + sc_2}{c_1 s - c_2^*} \right) + c_2 \cdot \sin\left(\arctan \frac{c_1^* + sc_2}{c_1 s - c_2^*} \right) \right]^{-1}, \qquad (3.22)\\
k_2 &= \frac{d}{ds}\left[\arctan \frac{c_1^* + sc_2}{c_1 s - c_2^*} \right].
\end{aligned}$$

Theorem 3. *The dual curve $\widehat{\alpha}(s)$ is a normal curve if and only if $\int(\alpha \wedge T)ds$ is in rectifying plane of $\alpha(s)$.*

Proof. We know that

$$\widehat{\alpha} = \alpha + \varepsilon \alpha^* = \alpha + \varepsilon \int (\alpha \wedge T) ds, \tag{3.23}$$

$$\widehat{N} = N + \varepsilon N^* \ \text{ and } \ N^* = \alpha \wedge N,$$

$$\widehat{B} = B + \varepsilon B^* \ \text{ and } \ B^* = \alpha \wedge B$$

and also from [6]

$$\frac{1}{\widehat{k}_1} = \frac{1}{k_1} - (g(\alpha(s), N^*) + g(\alpha^*, N)), \tag{3.24}$$

$$\widehat{k}_1 = k_1.$$

Then,

$$\langle \alpha(s), N^* \rangle + \langle \alpha^*(s), N \rangle = 0, \tag{3.25}$$

$$\langle \alpha(s), \alpha \wedge N \rangle + \left\langle \int (\alpha \wedge T) ds, N \right\rangle = 0.$$

Here, we get

$$\left\langle \int (\alpha \wedge T) ds, N \right\rangle = 0 \tag{3.26}$$

then

$$\left\langle \left(\int \alpha ds \right) \wedge T, N \right\rangle = 0\,. \tag{3.27}$$

In this case,

$$\det \left(\int \alpha ds, T, N \right) = 0\,. \tag{3.28}$$

Thus, $\int (\alpha \wedge T) ds$ is in rectifying plane of $\alpha(s)$.And also, we know that the dual curve $\widehat{\alpha}(s)$ is in normal plane if and only if $\widehat{\alpha}(s)$ is a dual spherical curve from [6].

Now it can be easily seen that $\int \alpha ds$ is the element of osculating plane,that is to say $\int \alpha ds \in Sp\,\{T, N\}$. Accordingly, we can give an example for this situation. If we get

$$\int \alpha ds = N(s) \tag{3.29}$$

then the value of the determinant will be zero. In this case, when we get

$$\alpha(s) = k_1 T - k_2 B \tag{3.30}$$

the curve α will be in the rectifying plane. That is to say, the curve α is rectifying curve. Also it can be easily seen that $\int \alpha ds \in Sp\,\{T, B\}$. □

Result 1. *The dual curve $\widehat{\alpha}(s)$ is a dual spherical curve if and only if $\int (\alpha \wedge T) ds$ is in rectifying plane of $\alpha(s)$.*

Theorem 4. *If the curve $\alpha : I \to E^3$ is a Salkowski curve such that the curvature k_1 is constant and k_2 is linear, then $\widehat{\alpha}$ is in rectifying plane.*

Proof. From [10], we can investigate the linearity of this curve when we have

$$\frac{\widehat{k}_2}{\widehat{k}_1} = \frac{\widehat{\tau}}{\widehat{\kappa}} = \frac{k_2+\varepsilon}{k_1} = \widehat{c}_1\widehat{s} + \widehat{c}_2. \tag{3.31}$$

Let $\widehat{c}_1$ and $\widehat{c}_2$ be

$$\begin{aligned} \widehat{c}_1 &= c_1 + \varepsilon c_1^*, \\ \widehat{c}_2 &= c_2 + \varepsilon c_2^*. \end{aligned} \tag{3.32}$$

Accordingly,

$$\begin{aligned} \frac{k_2+\varepsilon}{k_1} &= \widehat{c}_1\widehat{s} + \widehat{c}_2 = (c_1 + \varepsilon c_1^*)s + (c_2 + \varepsilon c_2^*) \\ &= c_1 s + c_2 + \varepsilon(c_1^* + c_2^*). \end{aligned} \tag{3.33}$$

Thus,

$$\frac{k_2}{k_1} = c_1 s + c_2 \quad \text{and} \quad \frac{1}{k_1} = c_1^* + c_2^* \tag{3.34}$$

Here, this shows us that k_1 is constant and k_2 is linear. Hence it can be seen that the curvature $\dfrac{k_2+\varepsilon}{k_1}$ of the Salkowski curve is also linear such that k_1 is constant and k_2 is linear. In general, the curvature $\dfrac{k_2+\varepsilon}{k_1}$ can be given as:

$$\frac{k_2+\varepsilon}{k_1} = (as+b)\cdot\frac{1}{k_1} = as + \widehat{c}\,. \tag{3.35}$$

□

4. **Second Kind of Dual Indicatrice Curve**

In this section, with the assistance of α, we can define curve β. Let $\beta(s)$ be unit speed curve and its parameter be the same as the parameter of the curve $\alpha(s)$.

$$\begin{aligned} \beta :& I \to E^3 \\ & s \mapsto \beta(s) \end{aligned} \tag{4.1}$$

Similarly, we define another dual curve $\widehat{\beta}$ in $\mathbb{D}^3$. At this time, we have to say that $\{T, N, B\}$ frame is Frenet Frame of α. Thus, let us have another closed spherical dual curve $\widehat{\beta}$ of class C^1 on a unit dual sphere S^1 in $\mathbb{D}^3$. The curve β describes a closed dual spherical motion if

$$\begin{aligned} \widehat{\beta} :& I \to \mathbb{D}^3 \\ & s \mapsto \widehat{\beta}(s) = \alpha(s) + \varepsilon\int \beta\wedge T ds = \int (T + \varepsilon(\beta\wedge T))ds \end{aligned} \tag{4.2}$$

In that case, taking $\overline{T}$, $\overline{N}$ and $\overline{B}$ be tangent, principal normal and binormal indicatrices of the other dual curve $\widehat{\beta} = \int (T + \varepsilon(\beta \wedge T))ds$, we get

$$\begin{aligned} \overline{T} &= T + \varepsilon\beta \wedge T, \\ \overline{N} &= N + \varepsilon\beta \wedge N, \\ \overline{B} &= B + \varepsilon\beta \wedge B. \end{aligned} \tag{4.3}$$

and also ruled surfaces that correspond to these indicatrices of the other dual curve $\widehat{\beta}$ can be given as:

$$\begin{aligned} \Phi_{\overline{T}} &= \beta + vT, \\ \Phi_{\overline{N}} &= \beta + vN, \\ \Phi_{\overline{B}} &= \beta + vB. \end{aligned} \tag{4.4}$$

On the other hand, although we have investigated above that the arc parameters are equal to each other, the frame that occurred is not the Blaschke Frame.

Now, we can investigate the developability of the ruled surfaces that correspond to $\overline{T}$, $\overline{N}$ and $\overline{B}$ tangent, principal normal and binormal indicatrices of the dual curve $\widehat{\beta}$. Firstly, if $\Phi_{\overline{T}} = \beta + vT$ then

$$\begin{aligned} P_{\overline{T}} &= \det(\beta', T, T') \\ &= \det(\beta', T, k_1 N) \\ &= k_1 \det(\beta', T, N). \end{aligned} \tag{4.5}$$

Thus, $\beta' \in Sp\{T, N\}$ if and only if the ruled surface $\Phi_{\overline{T}}$ that corresponds to $\overline{T}$ is developable. On the other hand, if $\beta' = \lambda B$, then the ruled surface is not developable.

Subsequently, if $\Phi_{\overline{N}} = \beta + vN$ then

$$\begin{aligned} P_{\overline{N}} &= \det(\beta', N, N') \\ &= \det(\beta', N, -k_1 T + k_2 B) \\ &= -k_1 \det(\beta', N, T) + k_2 \det(\beta', N, B) \\ &= k_1\lambda_3 + k_2\lambda_1. \end{aligned} \tag{4.6}$$

Therefore, the ruled surface that corresponds to $\overline{N}$ is developable if and only if

$$\begin{gathered} k_1\lambda_3 + k_2\lambda_1 = 0, \\ \frac{k_1}{k_2} = -\frac{\lambda_1}{\lambda_3}. \end{gathered} \tag{4.7}$$

After that, it can be easily seen that if $\Phi_{\overline{B}} = \beta + vB$ then

$$\begin{aligned} P_{\overline{B}} &= \det(\beta', B, B') \\ &= \det(\lambda_1 T + \lambda_2 N + \lambda_3 B, B, -k_2 N) \\ &= -k_2 \det(\lambda_1 T + \lambda_2 N + \lambda_3 B, B, N) \end{aligned} \tag{4.8}$$

$$= -k_2(-\lambda_1)$$
$$= \lambda_1 k_2 .$$

As stated above, the ruled surface that corresponds to $\overline{B}$ is developable if and only if $\beta' \in Sp\{T,N\}$ or the curve α is planar. Accordingly, we can investigate different cases:

4.1. $\beta(s) = \alpha(s)$

In this case, we can get the same frame and results as 3.1.

4.2. $\beta(s) = T(s)$

In this case, the dual frame is

$$\begin{aligned} \overline{T} &= T, \\ \overline{N} &= N + \varepsilon B, \\ \overline{B} &= B - \varepsilon N. \end{aligned} \tag{4.9}$$

The ruled surfaces that correspond to $\overline{T}$, $\overline{N}$ and $\overline{B}$ can be given as:

$$\Phi_{\overline{T}} = T + \upsilon T, \tag{4.10}$$
$$\Phi_{\overline{N}} = N + \upsilon N, \tag{4.11}$$
$$\Phi_{\overline{B}} = B + \upsilon B. \tag{4.12}$$

Theorem 5. *From* (4.10)*, the ruled surface is coni; also from* (4.11) *and* (4.12)*, it can be easily seen that the ruled surfaces are developable.*

Proof. It is clear that

$$\begin{aligned} P_{\overline{T}} &= \frac{\det(T',T,T')}{\langle T',T'\rangle} = 0, \\ P_{\overline{N}} &= \frac{\det(T',N,N')}{\langle N',N'\rangle} = 0, \\ P_{\overline{B}} &= \frac{\det(T',B,B')}{\langle B',B'\rangle} = 0. \end{aligned} \tag{4.13}$$

Thus, the ruled surfaces that correspond to $\overline{T}$, $\overline{N}$ and $\overline{B}$ are developable. □

Subsequently, from (4.9), dual spherical motion of this case can be calculated with:

$$\begin{bmatrix} \overline{T'} \\ \overline{N'} \\ \overline{B'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -\varepsilon k_1 \\ -k_1 & 0 & k_2 \\ \varepsilon k_1 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix} \tag{4.14}$$

Besides, the Darboux indicatrice curve of $\widehat{\beta}(s)$ be $\overline{W}$ for this case. According to this, the Darboux vector of this motion is:

$$\overline{W} = -k_2\overline{T} - \varepsilon k_1\overline{N} - k_1\overline{B} . \tag{4.15}$$

Thus,

$$\overline{W} \wedge \overline{T} = -\varepsilon k_1 \overline{B} + k_1 \overline{N} = \overline{T'}, \tag{4.16}$$
$$\overline{W} \wedge \overline{N} = -k_2 \overline{B} - k_1 \overline{T} = \overline{N'},$$
$$\overline{W} \wedge \overline{B} = -k_2 \overline{N} + \varepsilon k_1 \overline{T} = \overline{B'}.$$

4.3. $\beta(s) = N(s)$

In this case, if we take $\beta(s) = N(s)$ then we have the dual frame as following:

$$\overline{T} = T - \varepsilon B, \tag{4.17}$$
$$\overline{N} = N,$$
$$\overline{B} = B + \varepsilon T.$$

Thus, the ruled surfaces that correspond to $\overline{T}$, $\overline{N}$ and $\overline{B}$ are

$$\overset{\bullet}{\Phi}_{\overline{T}} = N + \upsilon B, \tag{4.18}$$
$$\overset{\bullet}{\Phi}_{\overline{N}} = N + \upsilon N, \tag{4.19}$$
$$\overset{\bullet}{\Phi}_{\overline{B}} = N + \upsilon T. \tag{4.20}$$

From (4.18), it can be seen that

$$\overset{\bullet}{P}_{\overline{T}} = \frac{\det(N', B, B')}{\langle B', B' \rangle} \tag{4.21}$$
$$= \frac{\det(-k_1 T + k_2 B, B, -k_2 N)}{k_2^2} = -\frac{k_1}{k_2} = \text{constant}.$$

Result 2. *The curve α is helix if and only if $\overset{\bullet}{P}_{\overline{T}}$ is constant. At the same time, from* (4.19)*, we can easily say that the surface is coni. Similarly, from* (4.20)

$$\overset{\bullet}{P}_{\overline{T}} = \frac{\det(N', T, T')}{\langle T', T' \rangle} \tag{4.22}$$
$$= \frac{\det(-k_1 T + k_2 B, T, -k_1 N)}{k_1^2} = \frac{k_2}{k_1} = constant.$$

Result 3. *The curve α is helix if and only if $\overset{\bullet}{P}_{\overline{B}}$ is constant.*

Result 4. *It can be easily calculated that $\overset{\bullet}{P}_{\overline{T}} \cdot \overset{\bullet}{P}_{\overline{B}} = -1$*

Subsequently, from (4.17), dual spherical motion of this case can be calculated with:

$$\begin{bmatrix} \overline{T'} \\ \overline{N'} \\ \overline{B'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 + \varepsilon k_2 & 0 \\ -k_1 - \varepsilon k_2 & 0 & k_2 - \varepsilon k_1 \\ 0 & -k_2 + \varepsilon k_1 & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix}. \tag{4.23}$$

Besides, the Darboux indicatrice curve of $\widehat{\beta}(s)$ be $\widetilde{W}$ for this case. According to this, the Darboux vector of this motion is:

$$\widetilde{W} = (-k_2 + \varepsilon k_1)\overline{T} + (k_1 + \varepsilon k_2)\overline{B}\,. \tag{4.24}$$

Thus,

$$\begin{aligned} \widetilde{W} \wedge \overline{T} &= (-k_1 - \varepsilon k_2)\overline{N} = \overline{T'}, \\ \widetilde{W} \wedge \overline{N} &= (-k_2 + \varepsilon k_1)\overline{B} + (-k_1 - \varepsilon k_2)\overline{T} = \overline{N'}, \\ \widetilde{W} \wedge \overline{B} &= (-k_2 + \varepsilon k_1)\overline{N} = \overline{T'}\,. \end{aligned} \tag{4.25}$$

4.4. $\beta(s) = B(s)$

In this case, if we take $\beta(s) = B(s)$ then we have the dual frame as:

$$\begin{aligned} \overline{T} &= T + \varepsilon N, \\ \overline{N} &= N - \varepsilon T, \\ \overline{B} &= B \end{aligned} \tag{4.26}$$

and then we can get the ruled surfaces of $\overline{T}$, $\overline{N}$ and $\overline{B}$ as:

$$\breve{\Phi}_{\overline{T}} = B + vN, \tag{4.27}$$

$$\breve{\Phi}_{\overline{N}} = B + vT, \tag{4.28}$$

$$\breve{\Phi}_{\overline{B}} = B + vB. \tag{4.29}$$

From (4.27), (4.28) and (4.29), it can be easily seen that

$$\breve{P}_{\overline{T}} = \breve{P}_{\overline{N}} = \breve{P}_{\overline{B}} = 0. \tag{4.30}$$

Thus, the ruled surfaces are developable. Accordingly, from (4.26), dual spherical motion of this case can be calculated with:

$$\begin{bmatrix} \overline{T'} \\ \overline{N'} \\ \overline{B'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ -\varepsilon k_2 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix}. \tag{4.31}$$

Besides, the Darboux indicatrice curve of $\widehat{\beta}(s)$ be $\overleftrightarrow{W}$ for this case. According to this, the Darboux vector of this motion is:

$$\overleftrightarrow{W} = k_2\overline{T} - \varepsilon k_2\overline{N} + k_1\overline{B}. \tag{4.32}$$

Thus,

$$\begin{aligned} \overleftrightarrow{W} \wedge \overline{T} &= k_1\overline{N} + \varepsilon k_2\overline{B} = \overline{T'}, \\ \overleftrightarrow{W} \wedge \overline{N} &= -k_1\overline{T} + k_2\overline{B} = \overline{N'}, \\ \overleftrightarrow{W} \wedge \overline{B} &= -\varepsilon k_2\overline{T} - k_2\overline{N} = \overline{B'}. \end{aligned} \tag{4.33}$$

5. **Conclusions**

The starting point of this study is to define two different dual spherical curves with the same parameter. We have developed this approach with giving one parameter spherical motion in dual space. Accordingly, different cases and results have occurred. Some of these cases have showed us that we have obtained different Darboux vectors for each motion.

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