Journal of Informatics and Mathematical Sciences
Vol. 9, No. 1, pp. 13–26, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Research Article

On Some Classes of Invariant Submanifolds of (k,μ)-Contact Manifold

M.S. Siddesha and C.S. Bagewadi*

Department of Mathematics, Kuvempu University, Shankaraghatta 577451, Shimoga, Karnataka, India
***Corresponding author:** prof_bagewadi@yahoo.co.in

Abstract. In this paper, we study invariant submanifolds of (k,μ)-contact manifolds. We consider pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, 2-Ricci-generalized pseudoparallel submanifolds of (k,μ)-contact manifolds. Further, we search for the conditions $\mathcal{Z}(X,Y)\cdot\sigma = 0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$ on invariant submanifolds of (k,μ)-contact manifolds, where $\mathcal{Z}$ is the concircular curvature tensor.

Keywords. Invariant submanifold; (k,μ)-contact manifold; Totally geodesic; Pseudoparallel; Ricci-generalized pseudoparallel

MSC. 53B25; 53C25; 53C22; 51A15

Received: July 28, 2016 **Accepted:** January 8, 2017

1. Introduction

It is well known that the tangent space sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R_{XY}\xi = 0$, where R is the curvature tensor [7]. On the other hand, on a manifold M equipped with a Sasakian structure (ϕ,ξ,η,g), one has

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad X,Y \in \Gamma(TM). \tag{1.1}$$

As a generalization of both $R_{XY}\xi = 0$ and the Sasakian case (1.1), Blair et al. [8] introduced the class of contact metric manifolds with contact metric structures (ϕ,ξ,η,g) which satisfy

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{1.2}$$

for all $X,Y \in \Gamma(TM)$, where k and μ are real constants and $2h$ is the Lie derivative of ϕ in the direction ξ. A contact metric manifold belonging to this class is called a (k,μ)-contact manifold.

In point of applications symmetric Riemannian manifolds are the most interesting and most important and it was introduced independently by Shirokov [24] as a Riemannian manifold with covariant constant curvature tensor R, i.e., with

$$\nabla R = 0,$$

where ∇ is the Levi-Civita connection. Later, a similar development took place in the geometry of submanifolds in the space forms, where a fundamental role is played by metric tensor g (as the induced Riemannian metric) and the second fundamental form σ. Besides the Levi-Civita connection ∇ with $\nabla g = 0$, a normal connection $\nabla^{\perp}$ is also defined. The submanifolds with parallel second fundamental form, i.e., with

$$\bar{\nabla}\sigma = 0,$$

where $\bar{\nabla}$ is the pair of ∇ and $\nabla^{\perp}$, deserve special attention.

As a generalization of symmetric manifolds in 1946, Cartan introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is considered as a field of linear operators, acting on R.

A symmetric study of Riemannian semisymmetric manifolds was started first by Szabo [26, 27] and Kowalski [18], later by Boeckx, Vanhecke and others.

A semi Riemannian pair (M,g) is pseudosymmetric [13], if and only if $R\cdot R = LQ(g,R)$ holds on M, where L is a function.

Parallel submanifolds were likewise later placed in a more general class of submanifolds generalizing the notion of parallel submanifolds. Given an isometric immersion $f : M \to \widetilde{M}$, let σ be the second fundamental form and $\widetilde{\nabla}$ the van der Waerden-Bortolotti connection of M. Then Deprez defined the immersion to be semiparallel if

$$\bar{R}(X,Y)\cdot\sigma = (\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]})\cdot\sigma = 0, \tag{1.3}$$

holds for any vectors X,Y tangent to M, where $\bar{R}$ denotes the curvature tensor of the connection $\bar{\nabla}$. Semiparallel immersions have been studied by authors, see for example, [5, 12, 14–16] and [21].

An immersion satisfying the equalities

$$\bar{R}\cdot\sigma = L_1Q(g,\sigma)$$

and

$$\bar{R}\cdot\sigma = L_3Q(S,\sigma)$$

are called pseudoparallel and Ricci-generalized pseudoparallel respectively (see [3, 4]), where L_1 and L_3 are the real valued functions and for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M,g), $Q(B,T)$ is defined by [30]

$$Q(B,T)(X_1,\dots,X_K;X,Y) = -T((X\wedge_B Y)X_1,\dots,X_K) - T(X_1,\dots,X_{K-1},(X\wedge_B Y)X_K), \tag{1.4}$$

where $X \wedge_B Y$ is defined by

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \tag{1.5}$$

The study of pseudoparallel and Ricci-generalized pseudoparallel submanifolds of various manifolds were studied by several authors such as [1,2,33] and many others.

Motivated by the above studies, in this paper we consider invariant submanifolds of (k,μ)-contact manifold and prove the equivalence of totally geodesicity, pseudoparallel, 2-pseudoparallel, Ricci-genralized pseudoparallel and 2-Ricci-genralized pseudoparallel. We also consider the conditions $\mathcal{Z}(X,Y)\cdot\sigma = 0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$ on an invariant submanifold of (k,μ)-contact manifolds, where $\mathcal{Z}$ denotes the concircular curvature tensor of the submanifold.

The paper is organized as follows: In section 2, we give some preliminaries which have been used later. In section 3, we give a brief account of (k,μ)-contact manifolds and their invariant submanifolds. In section 4, we find the necessary and sufficient conditions for invariant submanifolds to be pseudoparallel and 2-pseudoparallel. Section 5 is devoted to study of Ricci-generalized pseudoparallel and 2-Ricci generalized pseudoparallel. Lastly in section 6, we prove that a $(2n+1)$-dimensional invariant submanifold M of a (k,μ)-contact manifold $\widetilde{M}$ such that the scalar curvature $r \neq 2n(2n+1)(k \pm \mu\lambda)$, the conditions $\mathcal{Z}(X,Y)\cdot\sigma = 0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$ imply that M is totally geodesic.

2. Preliminaries

Let $f:(M,g) \to (\widetilde{M},\widetilde{g})$ be an isometric immersion of an n-dimensional Riemannian manifold (M,g) into an $(n+d)$-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, $n \geq 2, d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}$, respectively. Then we have the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{2.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for any tangent vector fields X,Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be totally geodesic. The second fundamental form σ and A_N are related by $\widetilde{g}(\sigma(X,Y),N) = g(A_N X,Y)$, where g is the induced metric of $\widetilde{g}$ for any vector fields X and Y tangent to M.

The first covariant derivative of the second fundamental form σ is given by

$$\bar{\nabla}_X \sigma(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z). \tag{2.2}$$

From the Gauss and Weingarten formulas we obtain

$$(\widetilde{R}(X,Y)Z)^T = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X.$$

By (1.3), we have

$$(\bar{R}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V), \tag{2.3}$$

for all vector fields X,Y,U and V tangent to M, where $R^{\perp}(X,Y) = [\nabla^{\perp}_X,\nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}$, and $\bar{R}$

denotes the curvature tensor of $\bar{\nabla}$. Similarly, we have

$$(\bar{R}(X,Y)\cdot\bar{\nabla}\sigma)(U,V) = R^{\perp}(X,Y)(\bar{\nabla}\sigma)(U,V,W) - (\bar{\nabla}\sigma)(R(X,Y)U,V,W)$$
$$- (\bar{\nabla}\sigma)(U,R(X,Y)V,W) - (\bar{\nabla}\sigma)(U,V,R(X,Y)W), \tag{2.4}$$

for vector fields X,Y,U,V and W tangent to M, where $(\bar{\nabla}\sigma)(U,V,W)$ means $(\bar{\nabla}_U\sigma)(V,W)$ [22].

The concircular curvature tensor for $(2n+1)$-dimensional Riemannian manifold is given by [31]

$$\mathcal{Z}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{r}{2n(2n+1)}[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y], \tag{2.5}$$

where $\widetilde{r}$ is the scalar curvature of the manifold.

Similar to (2.3) and (2.4) the tensors $\mathcal{Z}(X,Y)\cdot\sigma$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma$ are defined by

$$\mathcal{Z}(X,Y)\cdot\sigma = R^{\perp}(X,Y)\sigma(U,V) - \sigma(\mathcal{Z}(X,Y)U,V) - \sigma(U,\mathcal{Z}(X,Y)V), \tag{2.6}$$

$$\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = R^{\perp}(X,Y)(\bar{\nabla}\sigma)(U,V,W) - (\bar{\nabla}\sigma)(\mathcal{Z}(X,Y)U,V,W)$$
$$- (\bar{\nabla}\sigma)(U,\mathcal{Z}(X,Y)V,W) - (\bar{\nabla}\sigma)(U,V,\mathcal{Z}(X,Y)W). \tag{2.7}$$

3. (k,μ)-Contact Manifolds and their Invariant Submanifolds

A $(2n+1)$-dimensional C^{∞}-differentiable manifold $\widetilde{M}$ is said to admit an almost contact structure $(\phi,\xi,\eta,\widetilde{g})$ if it satisfies the following relations [6],

$$\phi^2 = -I + \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \ \eta\cdot\phi = 0, \tag{3.1}$$

$$\widetilde{g}(\phi X,\phi Y) = \widetilde{g}(X,Y) - \eta(X)\eta(Y),$$
$$\widetilde{g}(X,\phi Y) = -\widetilde{g}(\phi X,Y),\ \ \widetilde{g}(X,\phi X) = 0,\ \ \widetilde{g}(X,\xi) = \eta(X), \tag{3.2}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold, if it satisfies $\widetilde{g}(X,\phi Y) = d\eta(X,Y)$, for all vector fields X,Y.

We define a $(1,1)$-tensor field h by $h = \frac{1}{2}\mathcal{L}_{\xi}\phi$, where $\mathcal{L}$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $\mathrm{Tr}\cdot h = \mathrm{Tr}\cdot\phi h = 0$ and $h\xi = 0$.

A contact metric manifold satisfying (1.2) is called a (k,μ)-contact manifold. On a (k,μ)-manifold $k \le 1$, if $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition determines the curvature of $\widetilde{M}^{2n+1}$ completely [20]. In fact, for a (k,μ)-manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a (k,μ)-manifold the following relations holds [8]:

$$h^2 = (k-1)\phi^2,\quad k \le 1, \tag{3.3}$$

$$(\widetilde{\nabla}_X\phi)(Y) = \widetilde{g}(X+hX,Y)\xi - \eta(Y)(X+hX), \tag{3.4}$$

$$\widetilde{R}(X,\xi)\xi = k[\widetilde{g}(X,Y)\xi - \eta(Y)X] + \mu[\widetilde{g}(hX,Y)\xi - \eta(Y)hX], \tag{3.5}$$

$$\widetilde{R}(\xi,X)Y = k[X-\eta(X)\xi]+\mu hX, \tag{3.6}$$

$$\begin{aligned}\widetilde{S}(X,Y) &= [2(n-1)-n\mu]\widetilde{g}(X,Y)+[2(n-1)+\mu]\widetilde{g}(hX,Y)\\ &\quad +[2(1-n)+n(2k+\mu)]\eta(X)\eta(Y), \quad n\geq 1,\end{aligned} \tag{3.7}$$

$$\widetilde{S}(X,\xi) = 2nk\eta(X), \tag{3.8}$$

$$\widetilde{r} = 2n(2n-2+k-n\mu), \tag{3.9}$$

where $\widetilde{S}$ is the Ricci tensor of type $(0,2)$, Q is the Ricci operator, i.e., $\widetilde{g}(QX,Y)=\widetilde{S}(X,Y)$ and $\widetilde{r}$ is the scalar curvature of the manifold. Moreover from (2.5), we also have

$$\mathcal{Z}(\xi,Y)Z = \left(k-\frac{r}{2n(2n+1)}\right)[\widetilde{g}(Y,Z)\xi-\eta(Z)Y]+\mu[\widetilde{g}(hY,Z)\xi-\eta(Z)hY], \tag{3.10}$$

$$\mathcal{Z}(\xi,Y)\xi = \left(k-\frac{r}{2n(2n+1)}\right)[\eta(Y)\xi-Y]-\mu hY. \tag{3.11}$$

Equation (3.4) also implies that

$$\widetilde{\nabla}_X\xi = -\phi X-\phi hX, \tag{3.12}$$

Let $\widetilde{M}$ be a (k,μ)-contact manifold of dimension $(2m+1)$ and M be a submanifold of dimension $(2n+1)$. Then M is called an invariant submanifold of $\widetilde{M}$ if $\phi(TM)\subset TM$. In an invariant submanifold of a (k,μ)-contact manifold

$$\sigma(X,\xi) = 0. \tag{3.13}$$

holds, for any vector field X tangent to M [29]. The author M.M. Tripathi et al. [29] proved that

Theorem 3.1. *An invariant submanifold of* (k,μ)*-contact manifold is a* (k,μ)*-contact manifold.*

Hence the equations (3.3)-(3.9) also hold in an invariant submanifold M.

4. Pseudoparallel Invariant Submanifolds of (k,μ)-Contact Manifolds

This section deals with pseudoparallel and 2-pseudoparallel invariant submanifolds of (k,μ)-contact manifolds.

Definition 4.1. An immersion is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudoparallel with respect to Levi-Civita connection $\bar{\nabla}$, respectively, if

(1) $\bar{R}\cdot\sigma$ and $Q(g,\sigma)$,
(2) $\bar{R}\cdot\bar{\nabla}\sigma$ and $Q(g,\bar{\nabla}\sigma)$,
(3) $\bar{R}\cdot\sigma$ and $Q(S,\sigma)$,
(4) $\bar{R}\cdot\bar{\nabla}\sigma$ and $Q(S,\bar{\nabla}\sigma)$

are linearly dependent. Equivalently these are expressed by the following equations:

$$\bar{R}\cdot\sigma = L_1Q(g,\sigma),$$

$$\bar{R}\cdot\bar{\nabla}\sigma = L_2 Q(g,\bar{\nabla}\sigma),$$

$$\bar{R}\cdot\sigma = L_3 Q(S,\sigma),$$

$$\bar{R}\cdot\bar{\nabla}\sigma = L_4 Q(S,\bar{\nabla}\sigma)$$

where $\bar{R}$ denotes the curvature tensor with respect to connection $\bar{\nabla}$ and L_1, L_2, L_3, L_4 are the functions defined on $U_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $U_2 = \{x \in M : \bar{\nabla}\sigma(x) \neq g(x)\}$, $U_3 = \{x \in M : \sigma(x) \neq S(x)\}$ and $U_4 = \{x \in M : \bar{\nabla}\sigma(x) \neq S(x)\}$ respectively.

To prove the results, we use the following results:

Lemma 4.1 ([28]). *It is known that if (M,ϕ,ξ,η,g) is a contact Riemannian manifold and ξ belongs to the (k,μ)-nullity distribution, then $k \le 1$. If $k < 1$, then (M,ϕ,ξ,η,g) admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ defined by the eigen spaces of h, where $\lambda = \sqrt{1-k}$. Further, if $X \in D(\lambda)$, then $hX = \lambda X$ and if $X \in D(-\lambda)$, then $hX = -\lambda X$.*

Proposition 4.1 ([5]). *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then M is totally geodesic if and only if M is semiparallel provided that $k \neq \pm\mu\lambda$.*

Theorem 4.1. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then M is pseudoparallel if and only if M is totally geodesic provided $L_1 \neq (k \pm \mu\lambda)$.*

Proof. Let M be pseudoparallel, then we have

$$(\bar{R}(X,Y)\cdot\sigma)(U,V) = L_1 Q(g,\sigma)(X,Y,U,V).$$

Then from (1.4), (1.5) and (2.3), we get

$$R^{\perp}(X,Y)(\sigma(U,V)) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V)$$
$$= L_1\{-g(V,X)\sigma(U,Y) + g(U,X)\sigma(V,Y) - g(V,Y)\sigma(U,X) + g(U,Y)\sigma(V,X)\}. \quad (4.1)$$

Setting $V = \xi = Y$ in (4.1) and using (3.13), we get

$$\sigma(U,R(X,\xi)\xi) = L_1\sigma(U,X).$$

Making use of equations (3.6) and (3.13), we have

$$\sigma(U,kX + \mu hX) = L_1\sigma(U,X).$$

By virtue of Lemma (4.1)

$$(k \pm \mu\lambda - L_1)\sigma(U,X) = 0,$$

which implies $\sigma(U,X) = 0$, provided $L_1 \neq (k \pm \mu\lambda)$.

The converse part holds trivially. This proves the theorem. □

Theorem 4.2. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then M is 2-pseudoparallel if and only if M is totally geodesic provided $L_2 \neq (k \pm \mu\lambda)$.*

Proof. Let M be 2-pseudoparallel, then we have

$$\bar{R}(X,Y)\cdot\bar{\nabla}\sigma(U,V,W) = L_2 Q(g,\bar{\nabla}\sigma)(U,V,W,X,Y).$$

Put $X = V = \xi$ and in view of (1.4) and (2.4), we have

$$\begin{aligned} &R^{\perp}(\xi,Y)(\bar\nabla\sigma)(U,\xi,W)-(\bar\nabla\sigma)(R(\xi,Y)U,\xi,W)-(\bar\nabla\sigma)(U,R(\xi,Y)\xi,W)-(\bar\nabla\sigma)(U,\xi,R(\xi,Y)W)\\ &\quad = L_2\{-(\bar\nabla\sigma)((\xi\wedge_g Y)U,\xi,W)-(\bar\nabla\sigma)(U,(\xi\wedge_g Y)\xi,W)-(\bar\nabla\sigma)(U,\xi,(\xi\wedge_g Y)W)\}. \end{aligned} \tag{4.2}$$

Using (1.5), (2.2), (3.5) and (3.13) we have the following equalities:

$$\begin{aligned} (\bar\nabla\sigma)(U,\xi,W) &= (\bar\nabla_U\sigma)(\xi,W)\\ &= \nabla^{\perp}_U\sigma(\xi,W)-\sigma(\nabla_U\xi,W)-\sigma(\xi,\nabla_U W)\\ &= -\sigma(\nabla_U\xi,W), \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\bar\nabla\sigma)(R(\xi,Y)U,\xi,W) &= (\bar\nabla_{R(\xi,Y)U}\sigma)(\xi,W)\\ &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,W)-\sigma(\nabla_{R(\xi,Y)U}\xi,W)-\sigma(\xi,\nabla_{R(\xi,Y)U}W)\\ &= k\eta(U)\sigma(\nabla_Y\xi,W)+\mu\eta(U)\sigma(\nabla_{hY}\xi,W), \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\bar\nabla\sigma)(U,R(\xi,Y)\xi,W) &= (\bar\nabla_U\sigma)(R(\xi,Y)\xi,W)\\ &= \nabla^{\perp}_U\sigma(R(\xi,Y)\xi,W)-\sigma(\nabla_U R(\xi,Y)\xi,W)-\sigma(R(\xi,Y)\xi,\nabla_U W)\\ &= k\nabla^{\perp}_U\sigma(Y,W)-\mu\nabla^{\perp}_U\sigma(hY,W)-\sigma(\nabla_U k[\eta(Y)\xi-Y]-\mu hY,W)\\ &\quad +k\sigma(Y,\nabla_U W)+\mu\sigma(hY,\nabla_U W), \end{aligned} \tag{4.5}$$

$$\begin{aligned} (\bar\nabla\sigma)(U,\xi,R(\xi,Y)W) &= (\bar\nabla_U\sigma)(\xi,R(\xi,Y)W)\\ &= \nabla^{\perp}_U\sigma(\xi,R(\xi,Y)W)-\sigma(\nabla_U\xi,R(\xi,Y)W)-\sigma(\xi,\nabla_U R(\xi,Y)W)\\ &= k\eta(W)\sigma(\nabla_U\xi,Y)+\mu\eta(W)\sigma(\nabla_U\xi,hY), \end{aligned} \tag{4.6}$$

$$\begin{aligned} (\bar\nabla\sigma)((\xi\wedge_g Y)U,\xi,W) &= (\bar\nabla_{(\xi\wedge_g Y)U}\sigma)(\xi,W)\\ &= \nabla^{\perp}_{(\xi\wedge_g Y)U}(\sigma(\xi,W))-\sigma(\nabla_{(\xi\wedge_g Y)U})\xi,W)-\sigma(\xi,\nabla_{(\xi\wedge_g Y)U})W)\\ &= \eta(U)\sigma(\nabla_Y\xi,W), \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\bar\nabla\sigma)(U,(\xi\wedge_g Y)\xi,W) &= (\bar\nabla_U\sigma)((\xi\wedge_g Y)\xi,W)\\ &= \nabla^{\perp}_U(\sigma((\xi\wedge_g Y)\xi,W))-\sigma(\nabla_U(\xi\wedge_g Y)\xi,W)-\sigma((\xi\wedge_g Y)\xi,\nabla_U W)\\ &= -\nabla^{\perp}_U(\sigma(Y,W))-\sigma(\nabla_U(\eta(Y)\xi+Y),W)+\sigma(Y,\nabla_U W), \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\bar\nabla\sigma)(U,\xi,(\xi\wedge_g Y)W) &= (\bar\nabla_U\sigma)(\xi,(\xi\wedge_g Y)W)\\ &= \nabla^{\perp}_U(\sigma(\xi,(\xi\wedge_g Y)W))-\sigma(\nabla_U\xi,(\xi\wedge_g Y)W)-\sigma(\xi,\nabla_U(\xi\wedge_g Y)W)\\ &= \eta(W)\sigma(\nabla_U\xi,Y). \end{aligned} \tag{4.9}$$

Then substituting (4.3)-(4.9) in (4.2) we obtain

$$\begin{aligned} &-R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,W)-k\eta(U)\sigma(\nabla_Y\xi,W)-\mu\eta(U)\sigma(\nabla_{hY}\xi,W)-k\nabla^{\perp}_U\sigma(Y,W)+\mu\nabla^{\perp}_U\sigma(hY,W)\\ &+\sigma(\nabla_U(k[\eta(Y)\xi-Y]-\mu hY),W)-k\sigma(Y,\nabla_U W)-\mu\sigma(hY,\nabla_U W)-k\eta(W)\sigma(\nabla_U\xi,Y)-\mu\eta(W)\sigma(\nabla_U\xi,hY) \end{aligned}$$

$$= L_2\{-\eta(U)\sigma(\nabla_Y\xi,W)+\nabla_U^{\perp}\sigma(Y,W)+\sigma(\nabla_U(\eta(Y)\xi-Y),W)-\sigma(Y,\nabla_U W)-\eta(W)\sigma(\nabla_U\xi,Y). \quad (4.10)$$

Putting $W=\xi$ in (4.10) and using (3.13), we obtain

$$k\sigma(Y,\nabla_U\xi)+\mu\sigma(hY,\nabla_U\xi)=L_2\sigma(Y,\nabla_U\xi).$$

Applying (3.12), we get

$$k\sigma(Y,-\phi U-\phi hU)+\mu\sigma(hY,-\phi U-\phi hU)=L_2\sigma(Y,-\phi U-\phi hU).$$

Replace U by ϕU and using (3.1), (3.13) and in view of Lemma (4.1), the above equation is reduced to

$$(k\pm\mu\lambda)\sigma(U,Y)=L_2\sigma(U,Y),$$

which implies $\sigma(U,Y)=0$ provided $L_2\neq(k\pm\mu\lambda)$.

So analogous to the proof of the Theorem 4.1, we obtain $\sigma(U,Y)=0$ provided $L_2\neq(k\pm\mu\lambda)$.

The converse statement is trivial. This proves the theorem. □

5. Ricci-Generalized Pseudoparallel Invariant Submanifolds of (k,μ)-Contact Manifolds

In this section, we consider Ricci-generalized pseudoparallel and 2-Ricci generalized pseudoparallel invariant submanifolds of (k,μ)-contact manifolds. Now we prove the following theorems:

Theorem 5.1. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then M is Ricci-generalized pseudoparallel if and only if M is totally geodesic provided $L_3\neq\frac{(k\pm\mu\lambda)}{2nk}$.*

Proof. Let M be Ricci-generalized Pseudoparallel, then we have

$$(\bar{R}(X,Y)\cdot\sigma)(U,V)=L_3Q(S,\sigma)(X,Y,U,V).$$

Then making use of (1.4), (1.5) and (2.3) in above equation yields

$$R^{\perp}(X,Y)(\sigma(U,V))-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$=L_3\{-S(V,X)\sigma(U,Y)+S(U,X)\sigma(V,Y)-S(V,Y)\sigma(U,X)+S(U,Y)\sigma(V,X)\}. \quad (5.1)$$

Putting $V=\xi=Y$ in (5.1) and using (3.13), we get

$$\sigma(U,R(X,\xi)\xi)=L_3S(\xi,\xi)\sigma(X,U).$$

By the equations (3.6), (3.8) and (3.13), we obtain

$$\sigma(U,kX+\mu hX)=2nkL_3\sigma(X,U).$$

By virtue of Lemma 4.1, we have

$$(k\pm\mu\lambda-2nkL_3)\sigma(U,X)=0,$$

which implies $\sigma(U,X)=0$, provided $L_3\neq\frac{(k\pm\mu\lambda)}{2nk}$.

The converse part holds trivially. This completes the proof. □

Theorem 5.2. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then M is 2-Ricci-generalized pseudoparallel if and only if M is totally geodesic provided $L_4\neq\frac{(k\pm\mu\lambda)}{2nk}$.*

Proof. Let M be 2-Ricci-generalized pseudoparallel, then we have

$$(\bar{R}(X,Y)\cdot\bar{\nabla}\sigma)(U,V,W) = L_4 Q(S,\bar{\nabla}\sigma)(U,V,W,X,Y).$$

Setting $X = V = \xi$ and in view of (1.4) and (2.4), we obtain

$$R^{\perp}(\xi,Y)(\bar{\nabla}\sigma)(U,\xi,W) - (\bar{\nabla}\sigma)(R(\xi,Y)U,\xi,W) - (\bar{\nabla}\sigma)(U,R(\xi,Y)\xi,W) - (\bar{\nabla}\sigma)(U,\xi,R(\xi,Y)W)$$
$$= L_4\{-(\bar{\nabla}\sigma)((\xi\wedge_S Y)U,\xi,W) - (\bar{\nabla}\sigma)(U,(\xi\wedge_S Y)\xi,W) - (\bar{\nabla}\sigma)(U,\xi,(\xi\wedge_S Y)W)\}. \tag{5.2}$$

In view of equations (4.3)-(4.6) L.H.S of (5.2) can be written as

$$\begin{aligned}&-R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,W) - k\eta(U)\sigma(\nabla_Y\xi,W) - \mu\eta(U)\sigma(\nabla_{hY}\xi,W)\\ &-k\nabla^{\perp}_U\sigma(Y,W) + \mu\nabla^{\perp}_U\sigma(hY,W) + \sigma(\nabla_U(k[\eta(Y)\xi - Y] - \mu hY),W) - k\sigma(Y,\nabla_U W)\\ &-\mu\sigma(hY,\nabla_U W) - k\eta(W)\sigma(\nabla_U\xi,Y) - \mu\eta(W)\sigma(\nabla_U\xi,hY).\end{aligned} \tag{5.3}$$

Using (1.5), (2.2), (3.8) and (3.13) we have the following equalities:

$$\begin{aligned}(\bar{\nabla}\sigma)((\xi\wedge_S Y)U,\xi,W) &= (\bar{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W)\\ &= \nabla^{\perp}_{(\xi\wedge_S Y)U}(\sigma(\xi,W)) - \sigma(\nabla_{(\xi\wedge_S Y)U}\xi,W) - \sigma(\xi,\nabla_{(\xi\wedge_S Y)U}W)\\ &= 2nk\eta(U)\sigma(\nabla_Y\xi,W),\end{aligned} \tag{5.4}$$

$$\begin{aligned}(\bar{\nabla}\sigma)(U,(\xi\wedge_S Y)\xi,W) &= (\bar{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,W)\\ &= \nabla^{\perp}_U(\sigma((\xi\wedge_S Y)\xi,W)) - \sigma(\nabla_U(\xi\wedge_S Y)\xi,W) - \sigma((\xi\wedge_g Y)\xi,\nabla_U W)\\ &= -2nk\nabla^{\perp}_U(\sigma(Y,W)) - \sigma(\nabla_U 2nk(\eta(Y)\xi - Y),W)) + 2nk\sigma(Y,\nabla_U W),\end{aligned} \tag{5.5}$$

$$\begin{aligned}(\bar{\nabla}\sigma)(U,\xi,(\xi\wedge_S Y)W) &= (\bar{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W)\\ &= \nabla^{\perp}_U(\sigma(\xi,(\xi\wedge_S Y)W)) - \sigma(\nabla_U\xi,(\xi\wedge_S Y)W) - \sigma(\xi,\nabla_U(\xi\wedge_S Y)W)\\ &= 2nk\eta(W)\sigma(\nabla_U\xi,Y).\end{aligned} \tag{5.6}$$

Then substituting (5.3)-(5.6) in (5.2), we obtain

$$\begin{aligned}&-R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,W) - k\eta(U)\sigma(\nabla_Y\xi,W) - \mu\eta(U)\sigma(\nabla_{hY}\xi,W) - k\nabla^{\perp}_U\sigma(Y,W)\\ &+\mu\nabla^{\perp}_U\sigma(hY,W) + \sigma(\nabla_U(k[\eta(Y)\xi - Y] - \mu hY),W) - k\sigma(Y,\nabla_U W) - \mu\sigma(hY,\nabla_U W)\\ &-k\eta(W)\sigma(\nabla_U\xi,Y) - \mu\eta(W)\sigma(\nabla_U\xi,hY)\\ &\quad= L_4\{-2nk\eta(U)\sigma(\nabla_Y\xi,W) + 2nk\nabla^{\perp}_U(\sigma(Y,W)) + \sigma(\nabla_U 2nk(\eta(Y)\xi - Y),W)) - 2nk\sigma(Y,\nabla_U W)\\ &\qquad -2nk\eta(W)\sigma(\nabla_U\xi,Y)\}.\end{aligned} \tag{5.7}$$

Taking $W = \xi$, in (5.7) and using (3.13), we obtain

$$k\sigma(Y,\nabla_U\xi) + \mu\sigma(hY,\nabla_U\xi) = 2nkL_4\sigma(Y,\nabla_U\xi).$$

Using (3.12)

$$k\sigma(Y,-\phi U - \phi hU) + \mu\sigma(hY,-\phi U - \phi hU) = 2nkL_4\sigma(Y,-\phi U - \phi hU).$$

Replace U by ϕU and using (3.1), (3.13) and in view of Lemma 4.1, the above equation is reduced to

$$(k \pm \mu\lambda)\sigma(U,Y) = 2nkL_4\sigma(U,Y),$$

which implies $\sigma(U,Y) = 0$ provided $L_4 \neq \frac{(k\pm\mu\lambda)}{2nk}$.

So analogous to the proof of the theorem 5.5, we obtain $\sigma(U,Y) = 0$ provided $L_4 \neq \frac{(k\pm\mu\lambda)}{2nk}$.

The converse statement is trivial. This proves the theorem. □

6. Invariant Submanifolds of (k,μ)-Contact Manifolds satisfying $\mathcal{Z}(X,Y)\cdot\sigma = 0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$

This section deals with invariant submanifolds of (k,μ)-contact manifolds satisfying $\mathcal{Z}(X,Y)\cdot\sigma = 0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$.

Theorem 6.1. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then $\mathcal{Z}(X,Y)\cdot\sigma = 0$ holds on M if and only if M is totally geodesic provided $r \neq 2n(2n+1)(k\pm\mu\lambda)$.*

Proof. Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$ satisfying the condition $\mathcal{Z}(X,Y)\cdot\sigma = 0$. Then from (2.6), we have

$$R^{\perp}(X,Y)\sigma(Z,U) - \sigma(\mathcal{Z}(X,Y)Z,U) - \sigma(Z,\mathcal{Z}(X,Y)U) = 0. \tag{6.1}$$

Setting $X = U = \xi$ in (6.1) and using (3.10) and (3.13), we obtain

$$\sigma(Z,\mathcal{Z}(\xi,Y)\xi) = 0. \tag{6.2}$$

By virtue of (3.11) it follows from (6.2) that

$$\left(k - \frac{r}{2n(2n+1)}\right)\sigma(Z,\eta(Y)\xi - Y) - \mu\sigma(Z,hY). \tag{6.3}$$

Using (3.13) in (6.3) and in view of Lemma 4.1, we get

$$\left(k - \frac{r}{2n(2n+1)} \pm \mu\lambda\right)\sigma(Z,Y) = 0,$$

which gives $\sigma(Z,Y) = 0$, provided $r \neq 2n(2n+1)(k\pm\mu\lambda)$.

Hence the submanifold M is totally geodesic provided $r \neq 2n(2n+1)(k\pm\mu\lambda)$.

The converse statement is trivial and hence the theorem. □

Theorem 6.2. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$ holds on M if and only if M is totally geodesic provided $r \neq 2n(2n+1)(k\pm\mu\lambda)$.*

Proof. Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$ satisfying the condition $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma = 0$. Then from (2.7), we have

$$\begin{aligned} &R^{\perp}(X,Y)(\bar{\nabla}\sigma)(U,V,W) - (\bar{\nabla}\sigma)(\mathcal{Z}(X,Y)U,V,W) \\ &- (\bar{\nabla}\sigma)(U,\mathcal{Z}(X,Y)V,W) - (\bar{\nabla}\sigma)(U,V,\mathcal{Z}(X,Y)W) = 0. \end{aligned} \tag{6.4}$$

Setting $X = V = \xi$ in (6.4), we obtain

$$R^{\perp}(\xi,Y)(\bar{\nabla}\sigma)(U,\xi,W) - (\bar{\nabla}\sigma)(\mathcal{Z}(\xi,Y)U,\xi,W)$$

$$-(\bar{\nabla}\sigma)(U,\mathcal{Z}(\xi,Y)\xi,W)-(\bar{\nabla}\sigma)(U,\xi,\mathcal{Z}(\xi,Y)W)=0. \tag{6.5}$$

By virtue of (2.2), (3.10), (3.11) and (3.13), we get

$$\begin{aligned}
(\bar{\nabla}\sigma)(\mathcal{Z}(\xi,Y)U,\xi,W) &= (\bar{\nabla}_{\mathcal{Z}(\xi,Y)U}\sigma)(\xi,W)\\
&= \nabla^{\perp}_{\mathcal{Z}(\xi,Y)U}(\sigma(\xi,W))-\sigma(\nabla_{\mathcal{Z}(\xi,Y)U}\xi,W)-\sigma(\xi,\nabla_{\mathcal{Z}(\xi,Y)U}W)\\
&= \left(k-\frac{r}{2n(2n+1)}\right)\eta(U)\sigma(\nabla_Y\xi,W)+\mu\eta(U)\sigma(\nabla_{hY}\xi,W), \qquad (6.6)\\
(\bar{\nabla}\sigma)(U,\mathcal{Z}(\xi,Y)\xi,W) &= (\bar{\nabla}_U\sigma)(\mathcal{Z}(\xi,Y)\xi,W)\\
&= \nabla^{\perp}_U(\sigma(\mathcal{Z}(\xi,Y)\xi,W))-\sigma(\nabla_U\mathcal{Z}(\xi,Y)\xi,W)-\sigma(\mathcal{Z}(\xi,Y)\xi,\nabla_UW)\\
&= \left(k-\frac{r}{2n(2n+1)}\right)[-\nabla^{\perp}_U\sigma(Y,W)-\sigma(\nabla_U(\eta(Y)\xi-Y),W)+\sigma(Y,\nabla_UW)]\\
&\quad +\mu[-\nabla^{\perp}_U\sigma(hY,W)+\sigma(\nabla_UhY,W)+\sigma(hY,\nabla_UW)], \qquad (6.7)\\
(\bar{\nabla}\sigma)(U,\xi,\mathcal{Z}(\xi,Y)W) &= (\bar{\nabla}_U\sigma)(\xi,\mathcal{Z}(\xi,Y)W)\\
&= \nabla^{\perp}_U(\sigma(\xi,\mathcal{Z}(\xi,Y)W))-\sigma(\nabla_U\xi,\mathcal{Z}(\xi,Y)W)-\sigma(\xi,\nabla_U\mathcal{Z}(\xi,Y)W)\\
&= \left(k-\frac{r}{2n(2n+1)}\right)\eta(W)\sigma(\nabla_U\xi,Y)+\mu\eta(W)\sigma(\nabla_U\xi,hY). \qquad (6.8)
\end{aligned}$$

Taking account of (3.12), (4.3) and (6.6)-(6.8) in (6.5), we obtain

$$\begin{aligned}
&R^{\perp}(\xi,Y)\sigma(-\nabla_U\xi,W)-\left\{\left(k-\frac{r}{2n(2n+1)}\right)\eta(U)\sigma(\nabla_Y\xi,W)+\mu\eta(U)\sigma(\nabla_{hY}\xi,W)\right\}\\
&-\left\{\left(k-\frac{r}{2n(2n+1)}\right)[-\nabla^{\perp}_U\sigma(Y,W)-\sigma(\nabla_U(\eta(Y)\xi-Y),W)+\sigma(Y,\nabla_UW)]\right.\\
&\left.+\mu[-\nabla^{\perp}_U\sigma(hY,W)+\sigma(\nabla_UhY,W)+\sigma(hY,\nabla_UW)]\right\}\\
&-\left\{\left(k-\frac{r}{2n(2n+1)}\right)\eta(W)\sigma(\nabla_U\xi,Y)+\mu\eta(W)\sigma(\nabla_U\xi,hY)\right\}=0
\end{aligned} \tag{6.9}$$

Putting $W=\xi$ in (6.9) and using (3.12) and (3.13), we get

$$2\left(k-\frac{r}{2n(2n+1)}\right)\sigma(-\phi U-\phi hU,Y)-\mu\sigma(\phi U-\phi hU,hY). \tag{6.10}$$

Replace U by ϕU and in view of (3.1), we obtain

$$(1\pm\lambda)\left(k-\frac{r}{2n(2n+1)}\pm\mu\lambda\right)\sigma(Y,U)=0.$$

Since $(1\pm\lambda)\neq 0$, which implies that $\sigma(Y,U)=0$ provided $\left(k-\frac{r}{2n(2n+1)}\pm\mu\lambda\right)\neq 0$.
Hence M is totally geodesic provided $\left(k-\frac{r}{2n(2n+1)}\pm\mu\lambda\right)\neq 0$. This proves the theorem. □

In view of Theorems 4.1, 4.2, 5.1, 5.2, 6.1, 6.2 and Proposition 4.1.

Corollary 6.1. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\widetilde{M}$. Then the following conditions are equivalent:*

(1) *M is totally geodesic;*

(2) *M is semiparallel, if $(k\pm\mu\lambda)\neq 0$;*

(3) *M is Pseudoparallel, if $L_1\neq(k\pm\mu\lambda)$;*

(4) *M is 2-Pseudoparallel, if $L_2\neq(k\pm\mu\lambda)$;*

(5) *M is Ricci-generalized pseudoparallel, if $L_3\neq\frac{(k\pm\mu\lambda)}{2nk}$;*

(6) *M is 2-Ricci-generalized pseudoparallel, if $L_4\neq\frac{(k\pm\mu\lambda)}{2nk}$;*

(7) *M satisfies the condition $\mathcal{Z}(X,Y)\cdot\sigma=0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma=0$ with $r\neq 2n(2n+1)(k\pm\mu\lambda)$.*

Further, if $\mu=0$, then (k,μ)-contact manifolds are reduced to $N(k)$-contact manifolds [8]. Hence the above corollary can be restated as:

Corollary 6.2. *Let M be an invariant submanifold of a $N(k)$-contact manifold $\widetilde{M}$. Then the following conditions are equivalent:*

(1) *M is totally geodesic;*

(2) *M is semiparallel, if $k\neq 0$;*

(3) *M is Pseudoparallel, if $L_1\neq k$;*

(4) *M is 2-Pseudoparallel, if $L_2\neq k$;*

(5) *M is Ricci-generalized pseudoparallel, if $L_3\neq\frac{1}{2n}$;*

(6) *M is 2-Ricci-generalized pseudoparallel, if $L_4\neq\frac{1}{2n}$;*

(7) *M satisfies the condition $\mathcal{Z}(X,Y)\cdot\sigma=0$ and $\mathcal{Z}(X,Y)\cdot\bar{\nabla}\sigma=0$ with $r\neq 2n(2n+1)k$.*

7. Conclusion

If M is an invariant submanifolds of a (k,μ)-contact manifold then it is concluded that the conditions totally geodesicity, semi-parallelism, 2-semiparallelism, pseudo-parallelism, 2-pseudoparallelism, Ricci-generalized pseudoparallelism, 2-Ricci-generalized pseudoparallelism of M are equivalent under the suitable conditions. Also, the conditions semi-parallelism, 2-semiparallelism of M with respect to concircular curvature tensor are equivalent to the above conditions. Further, if $\mu=0$ then all the above results hold true for $N(k)$-contact manifold.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] B.S. Anitha and C.S. Bagewadi, Invariant submanifolds of Sasakian manifolds, *Proceedings of the National Conference on Differential Geometry*, 2013, 28–36.

[2] B.S. Anitha and C.S. Bagewadi, Invariant submanifolds of sasakian manifolds admitting semi-symmetric metric connection, *Communications in Mathematics and Applications* **4** (1) (2013), 29–38.

[3] A.C. Asperti, G.A. Lobos and F. Mercuri, Pseudo-paralel immersions in space forms, *Math. Contemp.* **17** (1999), 59–70.

[4] A.C. Asperti, G.A. Lobos and F. Mercuri, Pseudoparallel immersions of a space form, *Adv. Geom.* **2** (2002), 57–71.

[5] A. De, A note on invariant submanifolds of (k,μ)-contact manifolds, *Ukranian Mathematical J.* **62** (11) (2011), 1803–1809.

[6] D.E. Blair, Contact manifolds in Riemannian geometry, *Lecture Notes in Math., 509*, Springer-Verlag, Berlin (1976).

[7] D.E. Blair, Two remarks on contact metric structures, *Tohoku Math. J.* **29** (1977), 319–324.

[8] D.E. Blair, T. Koufogiorgos and B.J. Papantoniou, Contact metric manifolds satisfyng a nullity condition, *Israel J. Math.* **91** (1995), 189–214.

[9] E. Boeckx, A full classification of contact metric (k,μ)-spaces, *Illinois J. Math.* **44** (2000), 212–219.

[10] B.Y. Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, Tokyo (1981).

[11] F. Defever, J.M. Morvan, I. Van de Woestijne, L. Verstaelen and G. Zafindratafa, *Geometry and Topology of Submanifolds IX*, World Scientific Publishing (1999).

[12] J. Deprez, Semiparallel hypersurfaces, *Rend. Sem. Mat. Univ. Politechn. Torino* **45** (1986), 303–316.

[13] R. Deszcz, On pseudosymmetric spaces, *Bull. Soc. Belg. Math. Ser. A* **44** (1992), 1–34.

[14] F. Dillen, Semiparallel hypersurfaces of a real space form, *Israel J. Math.* **75** (1991), 193–202.

[15] H. Endo, Invariant submanifolds in contact metric manifolds, *Proc. Estonian Acad. Sci. Phys. Math.* **39** (1990), 1–8.

[16] M. Kon, Invariant submanifolds of normal contact metric manifolds, *Kodai Math. Sem. Rep.* **27** (1973), 330–336.

[17] M. Kon, Invariant submanifolds in Sasakian manifolds, *Math. Ann.* **219** (1976), 277–290.

[18] O. Kowalski, An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X,Y)\cdot R = 0$, *Czechoslov. Math. J.* **46** (1996), 427–474.

[19] U. Lumiste, *Semisymmetric submanifolds as second order envelope of a symmetric submanifolds.,* Tensor (N.S) **43(1)** (1986), 83-87.

[20] B.C. Montano, Di Terlizzi and M.M. Tripathi, Invariant submanifolds of contact (k,μ)-manifolds, *Glassgow Math J.* **50** (2008), 499–507.

[21] C. Murathan, K. Arslan and E. Ezentas, Ricci generalized pseudo-symmetric immersions, *Diff. Geom. Appl.* (2005), 99–108.

[22] C. Ozgur and C. Murathan, On invariant submanifolds of Lorenzian para-Sasakian manifolds, *The Arabian J. Sci. Engg.* **34** (2A) (2009), 177–185.

[23] P.A. Shirokov, Constant vector fields and tensor fields of second order in Riemannian spaces, *Izv. Kazan Fiz. Mat. Obshchestva Ser.* **25** (2) (1925), 86–114.

[24] M.S. Siddesha and C.S. Bagewadi, Totally geodesic submanifolds of (k,μ)-contact manifold, *IOSR Journal of Mathematics* **12** (6) (2016), 84–89.

[25] S. Sular and C. Ozgur, On some submanifolds of Kenmotsu manifolds, *Chaos, Solitons and Fractals* **42** (2009), 1990–1995.

[26] Z.I. Szabó, Structure theorems on Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$, I, *Local version. J. Differ. Geom.* **17** (1982), 531–582.

[27] Z.I. Szabó, Structure theorems on Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$, II, *Global version. Geom. Dedicata* **19** (1985), 65–108.

[28] S. Tanno, Ricci curvatures of contact Riemannian manifolds, *Tohoku Mathematical Journal* **40** (1988), 441.

[29] M.M. Tripathi, T. Sasahara and J.S. Kim, On invariant submanifolds of contact metric manifolds, *Tsukuba J. Math.* **29** (2) (2005), 495–510.

[30] L. Verstraelen,, Comments on pseudosymmetry in the sense of Ryszard Deszcz, In *Geometry and Topology of Submanifolds of World Scientific Publishing*, River Edge, Vol. **VI** (1994), 199–209.

[31] K. Yano, Concircular geometry I. Concircular transformations, in *Proceedings of the Imperial Academy Tokyo*, **16** (1940), 195–200.

[32] K. Yano and M. Kon, *Structures on Manifolds*, World Scintific Publishing (1984).

[33] A. Yildz and C. Murathan, Invariant submanifolds of Sasakian space forms, *J. Geom.* **95** (2009), 135–150.